Residual Distribution

basics,

recents developments,

relations with other techniques

PART II

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PART II OUTLINE

Higher (than second) order

Navier-Stokes and viscous problems

Time dependent problems

RD for Shallow Water simulations

Summary and perspectives

3
VERY HIGH ORDER
RD schemes

Why higher (than second) order schemes

Main motivation

Efficiency of kth order method

$$\eta_k = \frac{1}{\text{error} \times \text{CPU}} = \eta_k^{\text{scheme}} \frac{1}{n_{\text{DoF}-k}\, h^k}$$

with $\eta_k^{\text{scheme}} = 1/\text{characteristic cost}$. The bigger η_k^{scheme} the better the scheme.

Increasing η_k^{scheme} is hard.

But we can start by increasing k, thus boosting η_k

trying to minimize the $n_{\text{DoF-k}}$ required for a given error level

How to get higher (than second) order schemes

How do we get high order mesh convergence rates...

1. Polynomial approximations of arbitrary degree
2. Discretizations verifying some conditions for some error estimate to hold

CAVEAT

To get any asymptotic convergence rates, we need convergence

STABILITY plays a role

Higher (than second) order

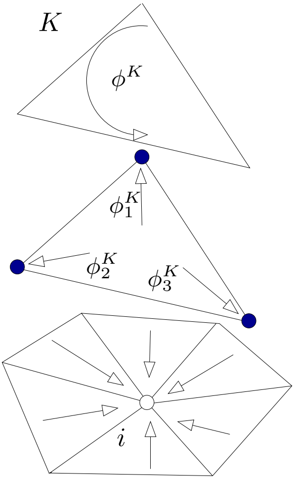
1. $\forall K \in \Omega_h$ compute : $\phi^K = \int_K \nabla \cdot \mathcal{F}_h(u_h)$

2. Distribution : $\phi^K = \sum_{i \in K} \phi_i^K$

   Distribution coeff.s : $\phi_i^K = \beta_i^K \phi^K$

3. Compute nodal values : solve algebraic system

$$\sum_{T|i \in T} \phi_i^K = 0, \quad \forall\, i \in \Omega_h \qquad (1)$$



Higher (than second) order

Accuracy condition

For a polynomial approximation of degree k,
a sufficient condition to have a $\|\epsilon_h\| \leq C\, h^{k+1}$ is (in 2d)

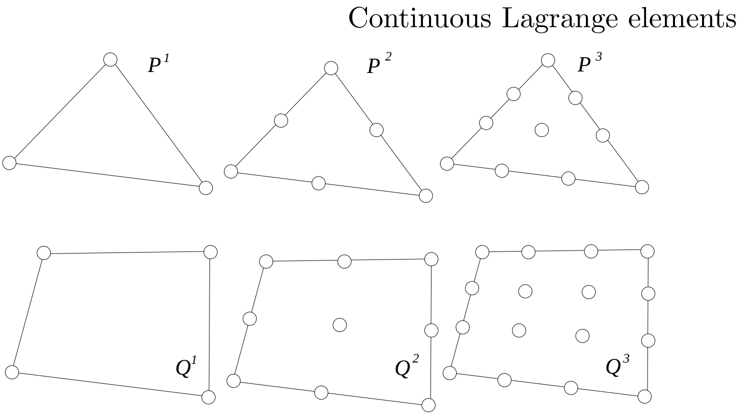
$$\phi_i^K(w_h) = \mathcal{O}(h^{k+2})\,, \quad \forall\, K_h\,,\, \forall\, i \in K$$

leading to the two high order prototypes

$$\phi_i^K = \int\limits_K \omega_i^K \nabla\cdot\mathcal{F}_h(u_h)\,, \quad \|\,\omega_i^K\| < C < \infty$$

$$\phi_i^K = \beta_i^K \phi^K\,, \qquad \|\beta_i^K\| < C < \infty$$

Higher (than second) order



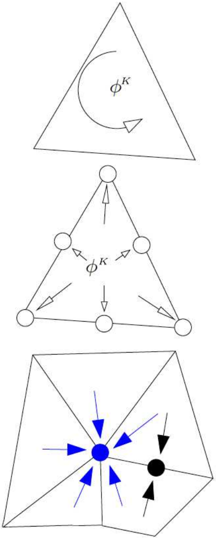
Higher (than second) order

RD on higher order elements

1. $\forall K$ compute : $\phi^K = \int_K \nabla \cdot \mathcal{F}_h(u_h)$
2. Distribution : $\phi^K = \sum_{j\in K} \phi_j^K$

   Distribution coeff.s : $\phi_i^K = \beta_i^K \phi^K$
3. Evolution : $\lim_{t\to\infty}$ of

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$



Higher (than second) order

A naive approach

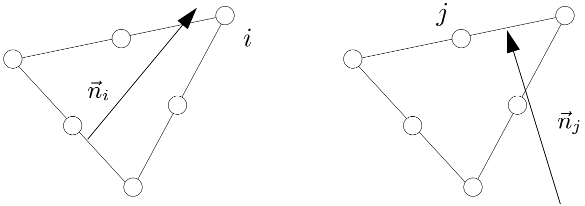
To solve the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

we write a scheme that "imitates" the P^1 LDA scheme

$$\phi_i^K = \beta_i^K \phi^K \,, \;\; \beta_i^K = k_i^+ \Big(\sum_{j \in K} k_j^+ \Big)^{-1}$$

where recall that $k_i = \vec{a} \cdot n_i$ defining the normals as



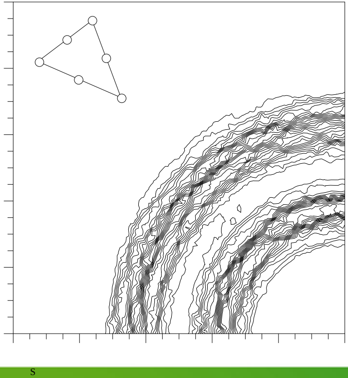
Higher (than second) order

A naive approach

To solve the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

We solve the scalar rotation problem we saw earlier and get



Higher (than second) order

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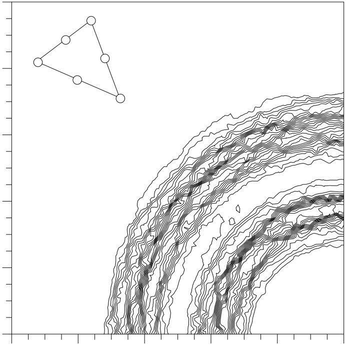
Monotonicity ?



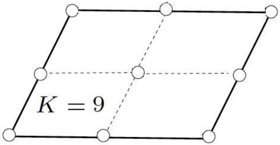
Higher (than second) order

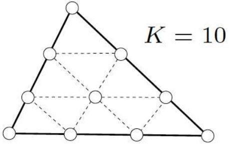
A second naive approach

To solve the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

1. Generalize the LF distribution





Lax-Friedrich's (Rusanov) :

$$\phi_i^{\mathrm{LF}} = \frac{1}{K}\phi^K + \alpha_{\mathrm{LF}} \sum_{j \in T} (u_i - u_j)$$

LED scheme for

$$\alpha_{\mathrm{LF}} \geq \frac{1}{2K} h \sup_{x \in K} \|\partial_u \mathcal{F}(u_h(x))\|$$

Higher (than second) order

A second naive approach

To solve the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

1. Generalize the LF distribution
2. Apply the limiter as done in the P^1 case

Higher (than second) order

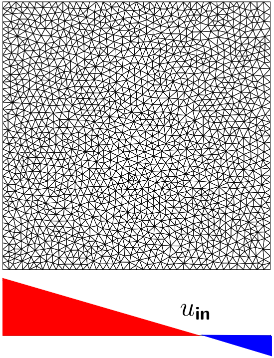
We proceed as in the P^1 case

1. Evaluation of $\phi^K = \oint\limits_{\partial K} \mathcal{F}_h(u_h) \cdot \vec{n}\, dl$
2. Evaluation of $\phi_i^{\mathrm{LF}} = \frac{1}{K}\phi^K + \alpha_{\mathrm{LF}} \sum\limits_{j \in T}(u_i - u_j)$
3. Limiting :
$$\beta_i^{\mathrm{LLF}} = \frac{\max(0, \beta_i^{\mathrm{LF}})}{\sum\limits_{j \in K} \max(0, \beta_j^{\mathrm{LF}})}$$
4. Distribution : $\phi_i^{\mathrm{LLF}} = \beta_i^{\mathrm{LLF}} \phi^K \big(= \gamma_i \phi_i^{\mathrm{LF}},\ \gamma_i \in [0, 1]\big)$
5. Evolve until steady state. Example :
$$u_i^{n+1} = u_i^n - \omega_i \sum_K \phi_i^{\mathrm{LLF}} \quad \xrightarrow{n \to \infty} \quad \sum_K \phi_i^{\mathrm{LLF}} = 0$$

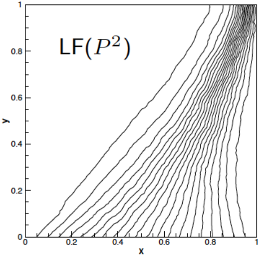
Higher (than second) order

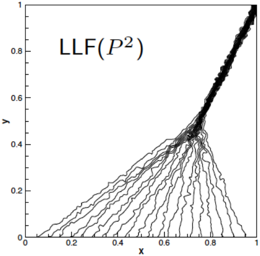
Scalar example : $\nabla \cdot \mathcal{F}(u) = 0$ with $\mathcal{F}(u) = (u, \frac{u^2}{2})$ and bcs

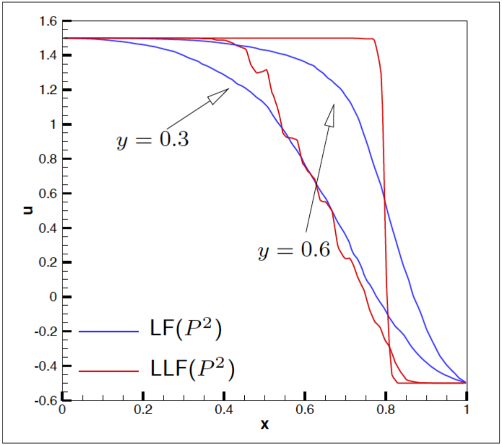
$$u(x, y = 0) = 1.5 - 2x$$



Higher (than second) order



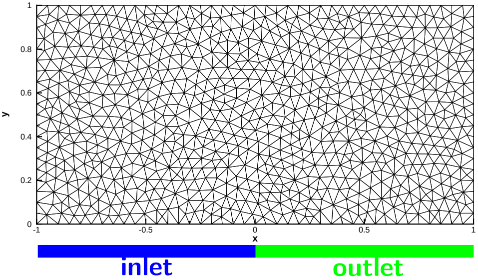




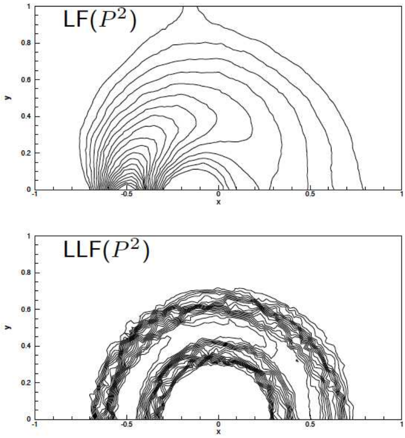
Higher (than second) order

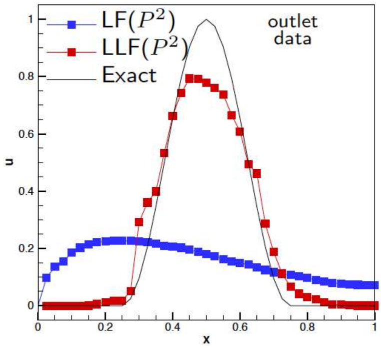
Scalar example : $\vec{a} \cdot \nabla u = 0$ with $\vec{a} = (y, 1-x)$ and bcs

$$u_{\text{in}} = \begin{cases} \cos(2\pi(x+0.5))^2 & \text{if} \quad x \in [-0.75, -0.25] \\ 0 & \text{otherwise} \end{cases}$$



Higher (than second) order





Higher (than second) order

Structural problems

We observe two problems

1. The first is really structural
2. The second is related to stability and can be cured

Higher (than second) order

Structural problems

We observe two problems

1. The first is really structural (Ricchiuto, 2011)

Proposition. *(Advection, spurious modes)* In 2d, any RD scheme for which $\phi_i^K = \beta_i^K \phi^K$ applied to P^k triangles with $k \geq 2$, and Q^k quads with $k \geq 1$ has spurious modes. These modes can be explicitly computed and are those for which $\forall\, f \in K$

$$\int_f u_h = 0$$

Higher (than second) order

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The paradigm of Roe and Deconinck
has to be modified

Higher (than second) order

Structural problems

We observe two problems

1. The first is really structural
2. The second is related to stability and can be cured :

   the limited LF scheme is entirely built on the algebraic preservation of the LED condition. No upwind bias is introduced.

Higher (than second) order

Solution to spurious modes 1 : modify the stencil

Enlarge the stencil to compute ϕ^K

1. Reconstruction operator R_{1k} that maps the P^1 approximation to a degree k edge continuous polynomial. This boils down to reconstructing gradients, hessians etc. in the nodes. Explored in (Caraeni *Computers & Fluids*, 2005 ; Chou, Shu *J.Comput.Phys*, 2006 ; Hubbard *J.Comput.Phys*, 2007)
2. Conformal sub-triangulation of the element, writing the scheme by sub-cells while using the macro-cell to define the polynomial. Explored in (Abgrall, Roe *J.Sci.Comp.*, 2003 ; Ricchiuto et al. *J.Comput.Appl.Math*, 2008 ; Vymazal et al. *J.Comput.Phys*, 2011). similarities with the spectral volume of Z.J. Wang (J.Comput.Phys) 2002.

Higher (than second) order

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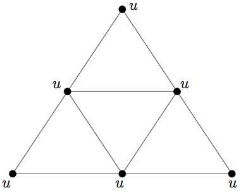
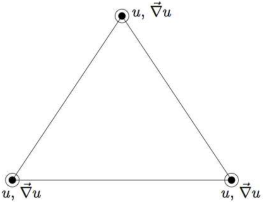


Submesh reconstruction



Gradient recovery

Higher (than second) order

Scalar example : $\vec{a} \cdot \nabla u = 0$ with $\vec{a} = (y, 1-x)$ and bcs

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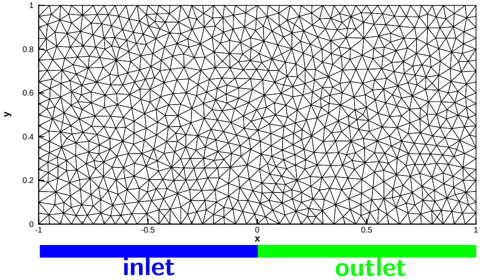
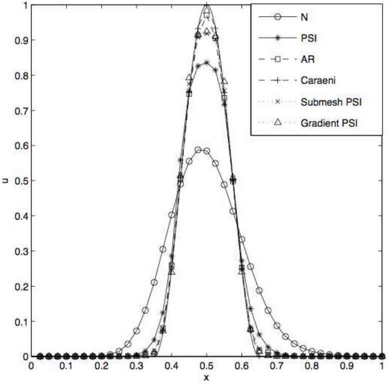
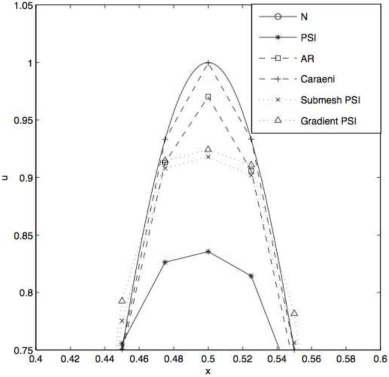


Higher (than second) order

Solution to spurious modes 1 : modify the stencil

From (Hubbard *J.Comput.Phys* 2007), outlet data



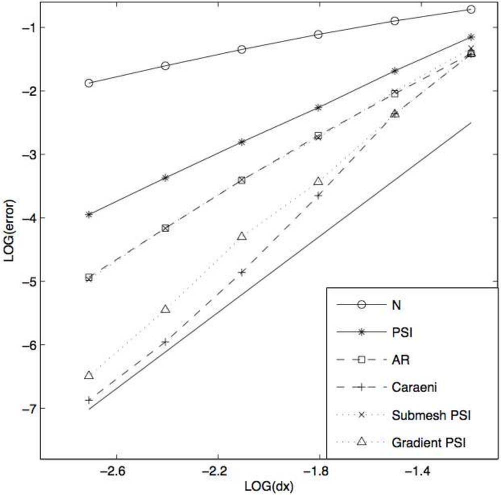


Higher (than second) order

Solution to spurious modes 1 : modify the stencil

L^1 convergence

(Hubbard *J.Comput.Phys* 2007)



Higher (than second) order

Solution to spurious modes 2

Distribute using variable weights. The scheme basically becomes a Petrov-Galerkin FE-like method reading

$$\sum_{K\in\Omega_h}\int_K \omega_i^K(x,y,u_h)\nabla\cdot\mathcal{F}_h(u_h)$$

How to define ω_i^K ?

Higher (than second) order

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How to define ω_i^K ?

One example is Hughes' SUPG scheme :

$$\omega_i^K = \psi_i + \vec{a}(u_h) \cdot \nabla \psi_i \, \tau$$

and all the limiters stuff ?

Higher (than second) order

Solution to spurious modes 2

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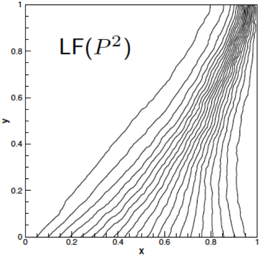
$$\sum_{K\in\Omega_h}\int_K \omega_i^K(x,y,u_h)\nabla\cdot\mathcal{F}_h(u_h)$$

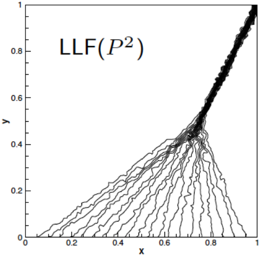
How to define ω_i^K ?

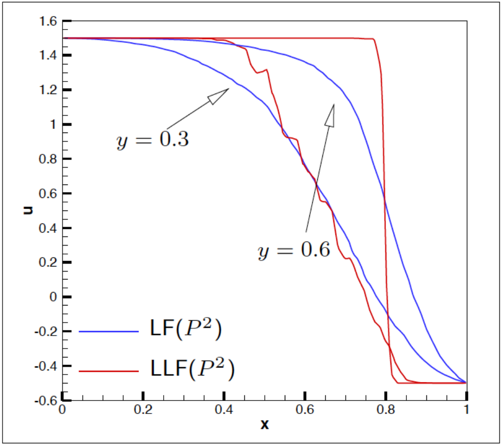
I will discuss one approach developed at Inria

(Abgrall, Larat, Ricchiuto *J.Comput.Phys.*) 2011

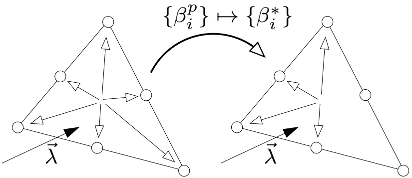
Higher (than second) order







Smooth solutions and spurious modes

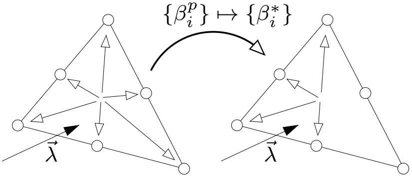


Think a lillte bit about it and ..

Things are OK in shocks. In smooth areas $\phi^K = \mathcal{O}(h^{k+2}) \ll 1$ (Abgrall, *J.Comput.Phys* 2006) :

- Linearize the nonlinear system $\sum_{K|i\in K} \phi_i^K = 0 : \quad \mathrm{M}_h^* = B_h^*$
- M_h^* does not have full range : infinite solutions, hence spurious modes

Smooth solutions and spurious modes



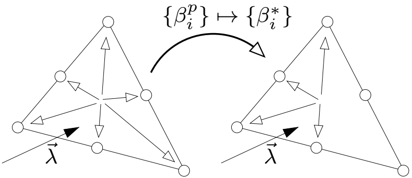
More simply (*Out the door, back through the window...*)

- The construction of the LLF scheme uses the algebraic constraint (for LED)

$$\phi_j^{\mathrm{LF}} \times \beta_j^{\mathrm{LLF}} \phi^K \geq 0 \rightarrow c_{ij}^{\mathrm{LLF}} = \gamma_i^K c_{ij}^{\mathrm{LF}} \geq 0$$

- Upwinding not included in the process
- Locally can have “down-winding” or zero entries in equation (as central scheme and advection)

Smooth solutions and spurious modes



More simply (*Out the door, back through the window...*)

Upwinding not included in the process ..

we want to "put it back in" in smooth regions .. how ?

Higher order nonlinear Lax Friedrich's scheme

The best we came up with so far

Add streamline diffusion (Abgrall *J.Comput.Phys* 2006 ; Abgrall, Larat, Ricchiuto *J.Comput.Phys* 2011)

$$\phi_i^{\mathrm{LLFs}} = \beta_i^{\mathrm{LLF}}\phi^K \quad + \quad \delta(u_h) \quad \int\limits_K \vec{a}(u_h)\cdot\nabla\psi_i\,\tau\,\vec{a}(u_h)\cdot\nabla u_h$$

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- As already seen

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- As already seen
- To identify smooth regions :

$$\delta(u_h) = \mathcal{O}(h) \quad \text{in discontinuities}$$

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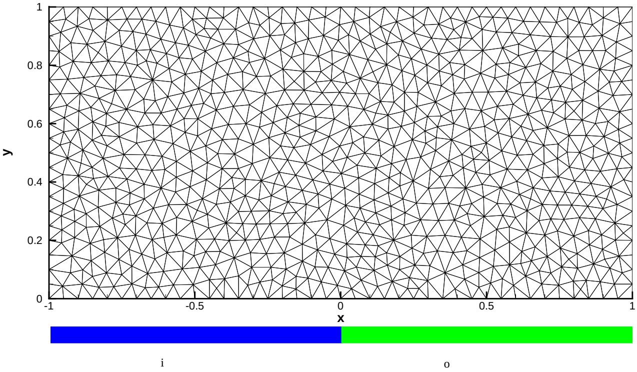
- As already seen
- To identify smooth regions :

$$\delta(u_h) = \mathcal{O}(h) \quad \text{in discontinuities}$$

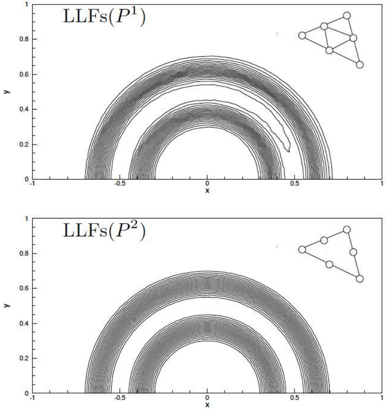
- Exactly as in the SUPG scheme

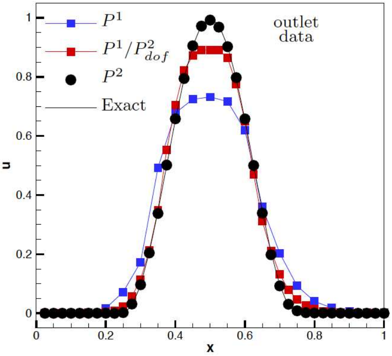
Higher order nonlinear Lax Friedrich's scheme

$$\vec{a} \cdot \nabla u = 0 \qquad \text{on} \quad [-1,\, 1] \times [0,\, 1]$$

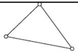
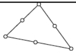
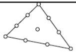


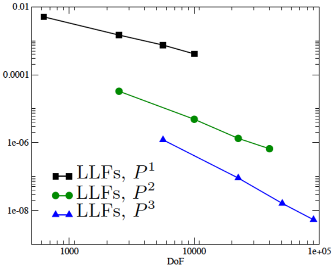
Higher order nonlinear Lax Friedrich's scheme





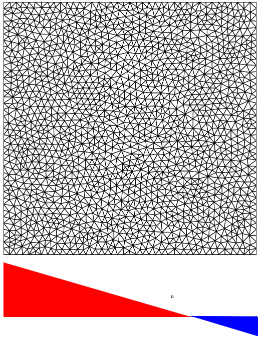
Higher order nonlinear Lax Friedrich's scheme

			
h	$\epsilon_{L^2}(P^1)$	$\epsilon_{L^2}(P^2)$	$\epsilon_{L^2}(P^3)$
1/25	0.50493E-02	0.32612E-04	0.12071E-05
1/50	0.14684E-02	0.48741E-05	0.90642E-07
1/75	0.74684E-03	0.13334E-05	0.16245E-07
1/100	0.41019E-03	0.66019E-06	0.53860E-08
	$\mathcal{O}^{\text{ls}}_{L^2} = \mathbf{1.790}$	$\mathcal{O}^{\text{ls}}_{L^2} = \mathbf{2.848}$	$\mathcal{O}^{\text{ls}}_{L^2} = \mathbf{3.920}$

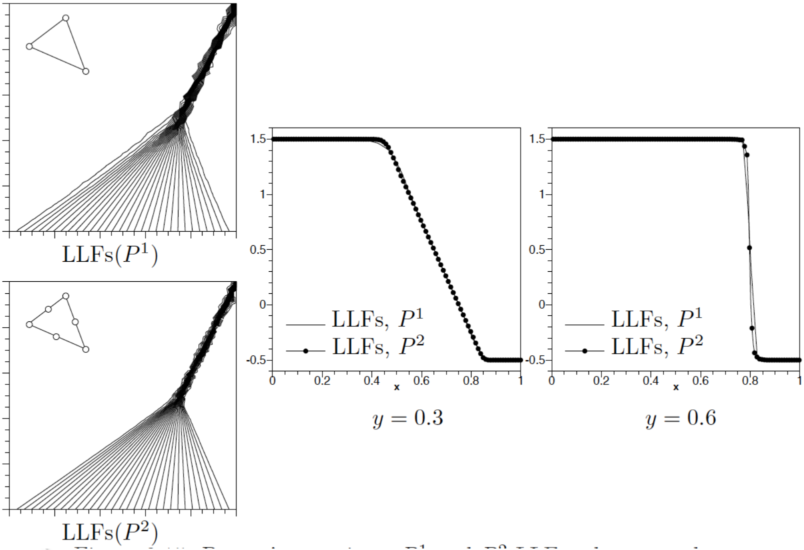


Higher order nonlinear Lax Friedrich's scheme

$$\nabla \cdot (u\,, \frac{u^2}{2}) = 0 \qquad \text{on} \quad [0,\,1]^2$$



Higher order nonlinear Lax Friedrich's scheme

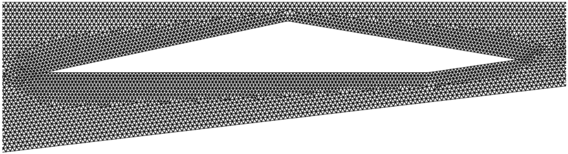


Extension to systems

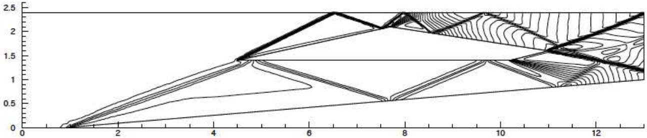
- All the steps extend formally
- Limiting step can either be done eq. by eq. or by a characteristic projection (as in FV schemes)

Example 1 : Mach 3.6 scramjet inlet (Euler, perfect gas)

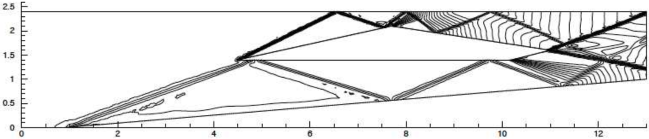
Mesh



LLFs scheme : P^1 on conformally refined mesh

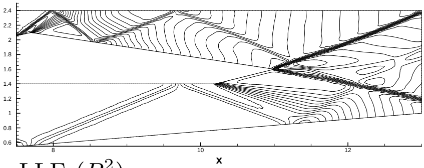


LLFs scheme : P^2

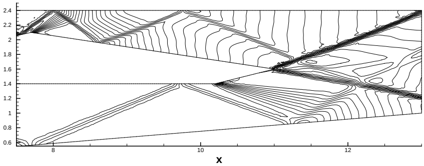


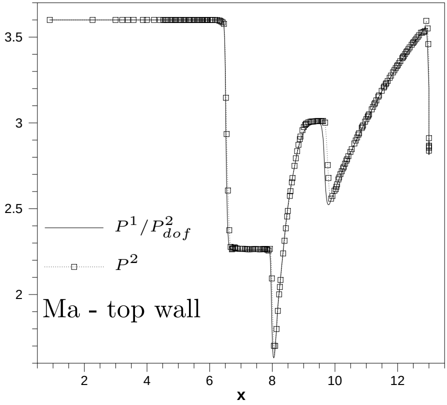
Scramjet inlet

$\mathrm{LLFs}(P^1/P^2_{dof})$

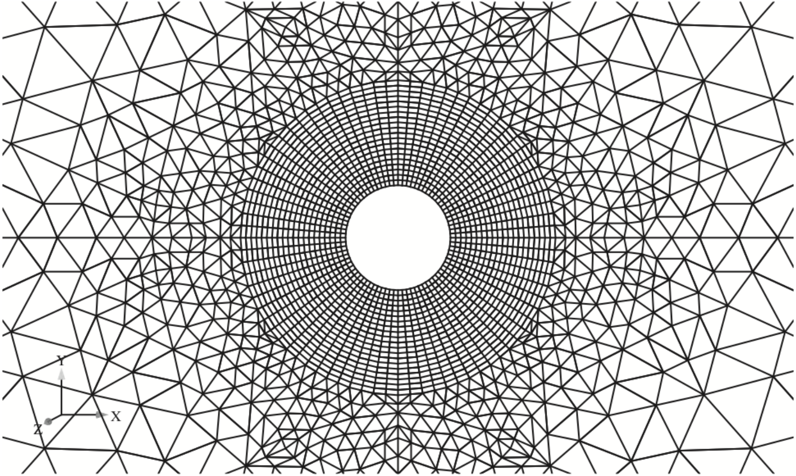


$\mathrm{LLFs}(P^2)$

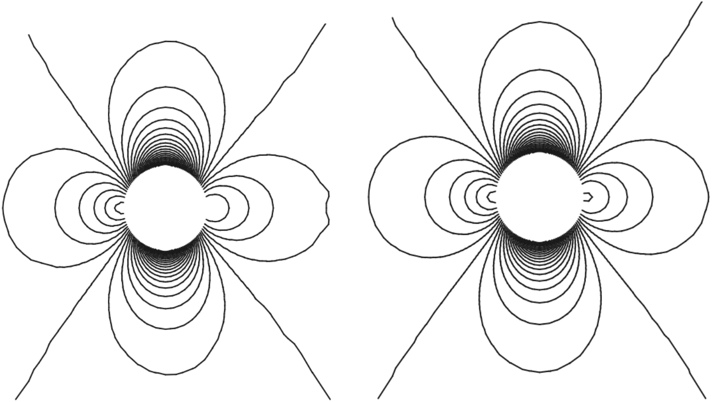




Euler equations : subsonic cylinder

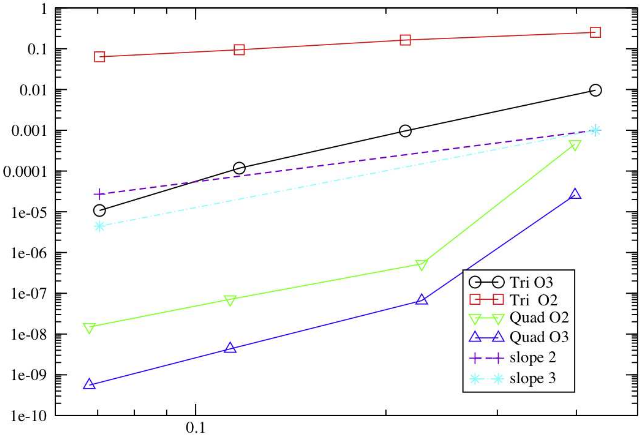


Euler equations : subsonic cylinder



Conformally refined $P^1 - Q^1$ (left) *vs* $P^2 - Q^2$ (right)

Grid convergence (entropy)



4
RD BASED SCHEMES
for viscous problems

Navier-Stokes and viscous problems

Consider now the problem

$$\nabla \cdot \mathcal{F}(u) = \nabla \cdot \mathcal{F}^{\nu}(u, \nabla u)$$

where most often

$$\mathcal{F}^{\nu}(u, \nabla u) = \mathcal{D}\nabla u \quad \text{with } \mathcal{D} \text{ a SPD matrix}$$

Navier-Stokes and viscous problems

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Early work on the extension of RD to this problem used a Galerkin approximation for the viscous term (Paillere et al *Int.J.Num.Meth.Fluids* 1996). :

$$\sum_{K|i\in K} \beta_i^K \phi^K + \int_{\Omega_h} \mathcal{D}\nabla u_h \cdot \nabla \psi_i = 0$$

This "decoupling" introduces a loss of accuracy in the so-called Pe=1 region (Ricchiuto et al *J.Comput.Appl.Math* 2008).

Navier-Stokes and viscous problems

Consider now the problem

$$\nabla \cdot \mathcal{F}(u) = \nabla \cdot \mathcal{F}^{\nu}(u, \nabla u)$$

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Early work on the extension of RD to this problem used a Galerkin approximation for the viscous term (Paillere et al *Int.J.Num.Meth.Fluids* 1996).

Attempts at improving this while keeping the viscous term in a variational form (Ricchiuto et al *J.Comput.Appl.Math* 2008 ; Villedieu et al *J.Comput.Phys* 2011) successful in the P^1 case and hard to justify in general (despite some interesting numerical results).

Navier-Stokes and viscous problems

$$\nabla \cdot \mathcal{F}(u) = \nabla \cdot \mathcal{F}^{\nu}(u, \nabla u)\,, \;\; \mathcal{F}^{\nu}(u, \nabla u) = \mathcal{D}\nabla u$$

A more sound approach : include the viscous flux in the element residual and carry it along together with the hyperbolic terms :

1. Evaluate

$$\phi^K(u_h) = \oint_{\partial K} (\mathcal{F}_h(u_h) - \mathcal{F}_h^{\nu}(u_h, \nabla u_h)) \cdot \hat{n}$$

2. Solve for $t \to \infty$

$$|C_i| \frac{du_i}{dt} + \sum_{K|i \in K} \beta_i^K \phi^K(u_h) = 0$$

Navier-Stokes and viscous problems

This approach introduces several issues which are only partially dealt with in

1. (Caraeni, Fuchs *Computes & Fluids* 2005)
2. (Chou, Shu *J.Comput.Phys* 2007)
3. (Nishikawa *J.Comput.Phys* 2007 ; Nishikawa *J.Comput.Phys* 2010 ; Nishikawa *Computers & Fluids* 2011)
4. (Abgrall et al *Int.J.Num.Meth.Fluids* 2012 ; Abgrall, De Santis, Ricchiuto *ICCFD7* 2012)

The objective is to give a little insight in these issues pointing out the common points with other numerical schemes

Navier-Stokes and viscous problems

Let us stick to the steady limit of (ν a constant viscosity)

$$\partial_t u + \vec{a} \cdot \nabla u = \nu \Delta u$$

Issue 1 : C^0 continuity

Navier-Stokes and viscous problems

Let us stick to the steady limit of (ν a constant viscosity)

$$\partial_t u + \vec{a} \cdot \nabla u = \nu \Delta u$$

Issue 1 : C^0 continuity

The underpinning approximation of the solution in RD methods is only C^0 continuous across the faces on which the hyperbolic flus is evaluated. But now (even in the P^1 case) :

$$\phi^K = \oint_{\partial K} u_h \vec{a} \cdot \hat{n} - \oint_{\partial K} \nu \nabla u_h \cdot \hat{n}$$

Navier-Stokes and viscous problems

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$$\phi^K = \oint_{\partial K} u_h \vec{a} \cdot \hat{n} - \oint_{\partial K} \underbrace{\nu \nabla u_h \cdot \hat{n}}_{\substack{\text{Not defined} \\ \text{on } \partial K}}$$

Navier-Stokes and viscous problems

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$$\partial_t u + \vec{a} \cdot \nabla u = \nu \Delta u$$

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$$\phi^K = \oint_{\partial K} u_h \vec{a} \cdot \hat{n} - \oint_{\partial K} \widehat{\mathcal{F}}_h^{\nu}(\nabla u_u\big|_K, \nabla u_u\big|_{K'}) \cdot \hat{n}$$

Conditions on $\widehat{\mathcal{F}}_h^{\nu}$

1. Consistency : $\widehat{\mathcal{F}}_h^{\nu} = \nu \nabla u$ if built using C^1 continuous data
2. Accuracy : $\widehat{\mathcal{F}}_h^{\nu} - \nu \nabla u = \mathcal{O}(h^{k+1})$ on P^k is u is smooth

Navier-Stokes and viscous problems

Issue 1 : C^0 continuity

$$\phi^K = \oint_{\partial K} u_h \vec{a} \cdot \hat{n} - \oint_{\partial K} \widehat{\mathcal{F}}_h^{\nu}(\nabla u_u\big|_K, \nabla u_u\big|_{K'}) \cdot \hat{n}$$

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Solutions proposed

- Gradient reconstruction in the degrees of freedom
- First order formulation of the problem

Navier-Stokes and viscous problems

Issue 1 : C^0 continuity and gradient reconstruction

- Reconstruct in every degree of freedom $i \in \Omega_h$ an accurate value of ∇u_i using the set of values

$$\left\{ u_j \,,\ \forall\, j \in K \text{ and } \forall\, K \in \Omega_h \middle| i \in K \right\}$$

- Set $\widehat{\nabla u}_h = \sum_j \psi_j \nabla u_j$
- Set $\widehat{\mathcal{F}}_h^\nu = \nu \widehat{\nabla u}_h$

Navier-Stokes and viscous problems

Reconstruction procedures

- Green-Gauss reconstruction
- Least squares reconstruction
- L^2 projection

Gradient reconstruction : pros & cons

- Well known procedures in the FV framework +
- Simple enough and can be coded efficiently +
- Non local -
- Accuracy condition : $\nabla u_j - \nabla u^{\mathrm{ex}}(x_j) = \mathcal{O}(h^{k+1})$ -

Navier-Stokes and viscous problems

Gradient Reconstruction procedures

- Simple and efficient approaches limited to second and third order accuracy
- More complex WENO reconstructions up to fourth order in (Chou, Shu *J.Comput.Phys* 2007) but limited to structured meshes

Navier-Stokes and viscous problems

Issue 1 : C^0 continuity and FOS

Recast the problem as the limit of the First Order System

$$\begin{aligned} \partial_t u + \vec{a}\cdot\nabla u - \nu\partial_x p - \nu\partial_y q = 0 \\ T_R\,\partial_t p + p - \partial_x u = 0 \\ T_R\,\partial_t q + q - \partial_y u = 0 \end{aligned}$$

Write a scheme for the coupled system until steady state.

Navier-Stokes and viscous problems

FOS pros & cons

- Works very well +
- Relaxation time T_R can be optimized to achieve fast convergence +
- Very memory demanding -
- Extension to Navier-Stokes difficult (see next) -

- A different way to look at the mixed problem
- Due to the coupling, equivalent at steady state to an (expensive) "implicit reconstruction" technique

$$\widehat{\nabla u_h} = F(u_h, \nabla u_h, \widehat{\nabla u_h})$$

Navier-Stokes and viscous problems

Let us stick to the steady limit of (ν a constant viscosity)

$$\partial_t u + \vec{a} \cdot \nabla u = \nu \Delta u$$

Issue 2 : what is the correct "distribution direction" ?

How do we distribute the element residual

$$\phi^K = \oint_{\partial K} u_h \vec{a} \cdot \hat{n} - \oint_{\partial K} \widehat{\mathcal{F}}_h^{\nu} \cdot \hat{n}$$

In 1d reduces to the a problem very similar to finding a viscous numerical flux with the right stability properties faced in the DG community

Navier-Stokes and viscous problems

Issue 2 : what is the correct "distribution direction" ?

How do we distribute the element residual

$$\phi^K = \oint_{\partial K} u_h \vec{a} \cdot \hat{n} - \oint_{\partial K} \widehat{\mathcal{F}}_h^{\nu} \cdot \hat{n}$$

Three approaches

1. Blend the $\nu = 0$ scheme with a central scheme. Blending parameter written as a function of the Reynolds number (Peclet)

$$\mathrm{Re}_K = \frac{\|\vec{a}\|_K \, h_K}{\nu}$$

Stability problems (weak iterative convergence)

Navier-Stokes and viscous problems

Issue 2 : what is the correct "distribution direction" ?

$$\begin{aligned} \partial_t u + \vec{a} \cdot \nabla u - \nu \partial_x p - \nu \partial_y q &= 0 \\ T_R\, \partial_t p + p - \partial_x u &= 0 \\ T_R\, \partial_t q + q - \partial_y u &= 0 \end{aligned}$$

Three approaches

2. Discretize the coupled First Order System
   - The system is hyperbolic : use standard RD techniques
   - Optimization of T_R : fast convergence with explicit time stepping. $\Delta t = \mathcal{O}(h)$ offsets the cost of extra equations
   - Convergence rates of $O(h^{k+1})$ in the H^1 norm attainable
   - Memory demanding
   - Navier-Stokes remains a challenge

Navier-Stokes and viscous problems

Issue 2 : what is the correct "distribution direction" ?

How do we distribute the element residual

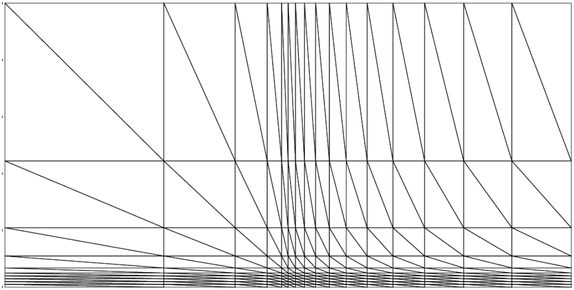
$$\phi^K = \oint_{\partial K} u_h \vec{a} \cdot \hat{n} - \oint_{\partial K} \widehat{\mathcal{F}}_h^{\nu} \cdot \hat{n}$$

Three approaches

3. Use the FOS to derive an equation for u. In the coupling terms containing the FOS gradients, replace these by reconstructed gradients.
   - Allows to define viscous numerical fluxes for DG, FV , etc. (Nishikawa *Computers & Fluids* 2011)
   - Works very well with the central (or nonlinear) + streamline dissipation developed at Inria
   - With simple schemes allows Navier-Stokes
   - Limited by the accuracy of the gradient reconstruction

Laminar flat plate

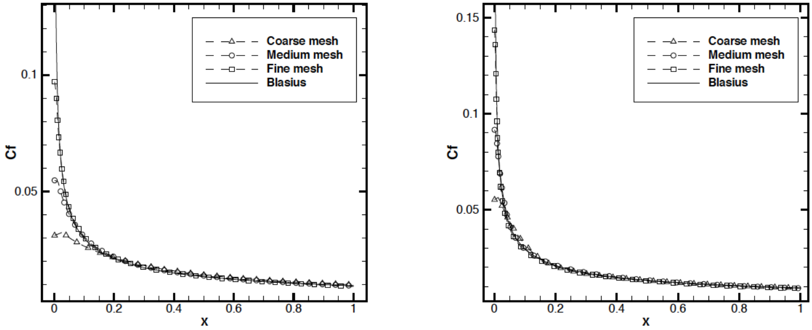
From (Abgrall, De Santis, Ricchiuto *ICCFD7* 2012)



$$\mathrm{Ma}_\infty = 0.3\,, \quad \mathrm{Re}_\infty = 5000$$

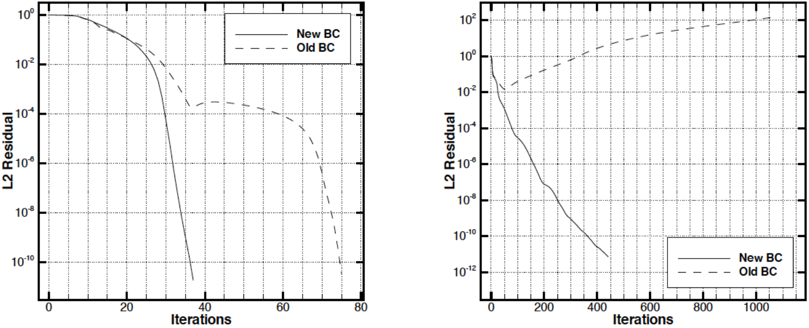
Coarse grid

Laminar flat plate



Skin friction coefficient *vs* Blasius' laminar boundary layer theory
Left : P^1, Right : P^2

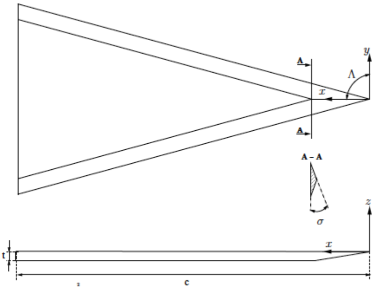
Laminar flat plate

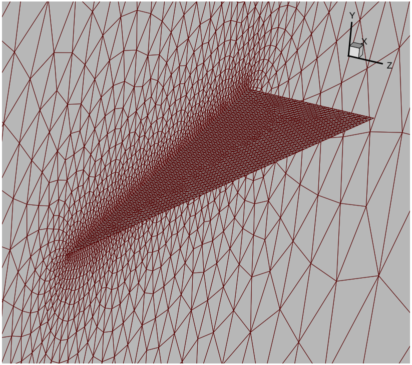


Typical iterative convergence (matrix free GMRES, LU-SGS prec.)
Left : P^1, Right : P^2

Laminar delta wing computations

From (Abgrall, De Santis, Ricchiuto *ICCFD7* 2012)





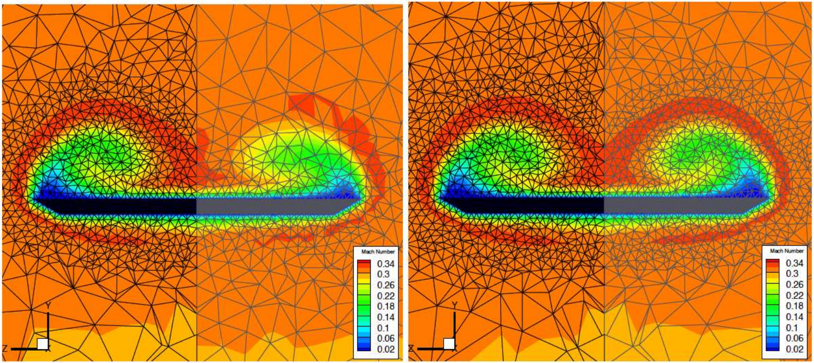
$$\mathrm{Ma}_\infty = 0.3\,, \quad \mathrm{Re}_\infty = 2000\,, \quad \mathrm{AoA} = 12.5^\circ$$

Laminar delta wing computations

2 cycles of refinement based on vorticity magnitude

(MMG3D generator by C.Dobrzynski available under GNU GPL license at

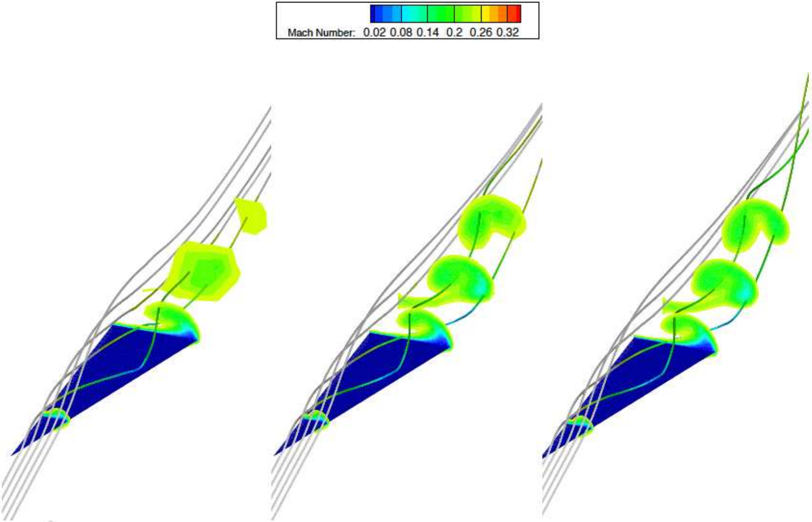
`http://www.math.u-bordeaux1.fr/ cdobrzyn/logiciels/mmg3d.php`)



Fine grid 600k tets

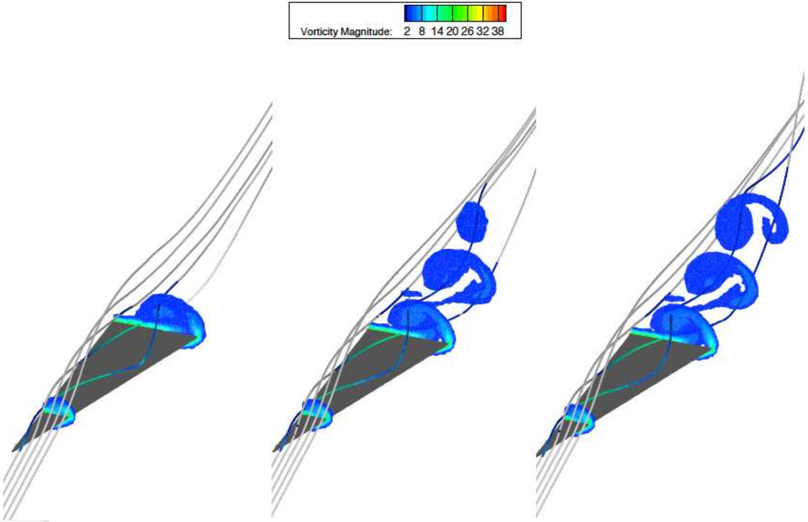
Laminar delta wing computations

Flow separation and Mach number (P^1 scheme)



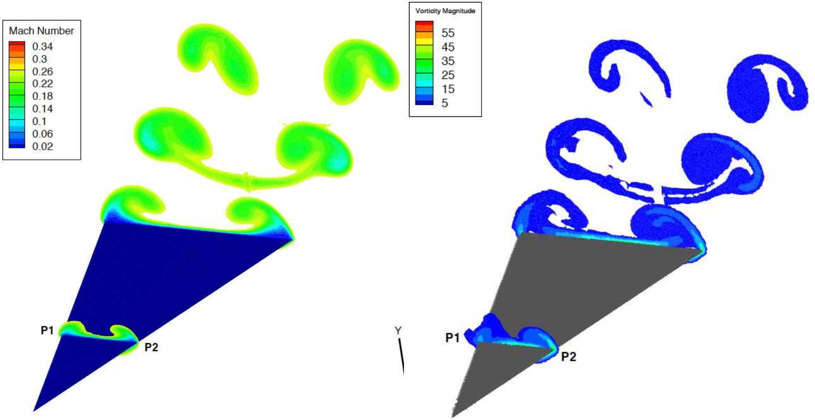
Laminar delta wing computations

Flow separation and vorticity magnitude (P^1 scheme)



Laminar delta wing computations

Flow separation, Mach number and vorticity (P^1 *vs* P^2)



Laminar delta wing computations. Drag coefficient

Reference value from (Leicht, Hartmann J.Comput.Phys. 2010) using a second order DG scheme with adjoint based error estimation and grid adaptation (finest meshes 2.5M elements) :

$$C_D = 0.1658$$

Initial grid P^1	0.145
Adapted grid 1, P^1	0.146
Adapted grid 2, P^1	0.147
Adapted grid 2, P^2	0.162

2
TIME DEPENDENT

... problems ...

Time dependent problems

We now consider the time dependent advection equation

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

Time dependent problems

We now consider the time dependent advection equation

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

The accuracy problem

The prototype

$$|C_i| \frac{du_i}{dt} + \sum_{K|i \in K} \phi_i^K = 0$$

with

$$\sum_{j \in K} \phi_j^K = \int_K \vec{a} \cdot \nabla u_h \; \forall K \in \Omega_h$$

is in general first order accurate <u>in space</u>.

Time dependent problems

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K}\beta_i^K\phi^K = 0\,, \quad \phi^K = \int\limits_K \vec{a}\cdot\nabla u_h$$

Time continuous error analysis

P^1 triangles to fix ideas (Deconinck, Ricchiuto *Enc.Comput.Mech.* 2007)

Time dependent problems

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K}\beta_i^K\phi^K = 0\,,\quad \phi^K = \int_K \vec{a}\cdot\nabla u_h$$

Time continuous error analysis

P^1 triangles to fix ideas (Deconinck, Ricchiuto *Enc.Comput.Mech.* 2007).

- (i) Let w be a smooth exact solution : $\partial_t w + \vec{a}\cdot\nabla w = 0$
- (ii) Set $w_i(t) = w(t, x_i, y_i)$
- (iii) Let $\phi^K(w_h)$ the quantity obtained when formally replacing the nodal values of the numerical solution by the w_is
- (iii) $\psi \in C_0^1$ compactly supported smooth function, $\psi_i = \psi(x_i, y_i)$
- (iv) define the integral truncation error

$$\epsilon(w,\psi) := \Big| \sum_{i\in\Omega_h}\sum_{K|i\in K}\psi_i\big(|C_i|\frac{dw_i}{dt} + \beta_i^K\phi^K(w_h)\big)\Big|$$

Time dependent problems

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \beta_i^K \phi^K = 0\,, \quad \phi^K = \int\limits_K \vec{a}\cdot\nabla u_h$$

Time continuous error analysis

Proceeding as in the steady case :

$$\epsilon(w,\psi) = \Big| \sum_{K\in\Omega_h} \sum_{j\in K} \psi_j \big(|C_j|\frac{dw_j}{dt} + \beta_j^K \phi^K(w_h)\big)\Big|$$

Time dependent problems

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \beta_i^K \phi^K = 0\,, \quad \phi^K = \int\limits_K \vec{a}\cdot\nabla u_h$$

Time continuous error analysis

Proceeding as in the steady case :

$$\begin{aligned}\epsilon(w,\psi) \leq & \Big|\int\limits_\Omega \psi_h\big(\partial_t(w_h - w) + \vec{a}\cdot\nabla(w_h - w)\big)\Big| \\ & + \sum_{K\in\Omega_h}\sum_{i,j\in K}|\psi_j - \psi_i|\Big(\Big||C_j|\frac{dw_j}{dt} + \beta_j^K\phi^K(w_h)\Big| \\ & + \Big|\int\limits_K \varphi_i\big(\partial_t(w_h - w) + \vec{a}\cdot\nabla(w_h - w)\big)\Big|\Big)\end{aligned}$$

Time dependent problems

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \beta_i^K \phi^K = 0\,, \quad \phi^K = \int\limits_K \vec{a}\cdot\nabla u_h$$

Time continuous error analysis

Estimating terms (approximation theory on P^1 triangles)

$$\epsilon(w,\psi) \leq C_1 h^2 + C_2 h^{-1} \sup_{\substack{K\in\Omega_h\\ j\in K}} \left| |C_j|\frac{dw_j}{dt} + \beta_j^K \phi^K(w_h)\right|$$

Second order local truncation error condition :

$$\sup_{\substack{K\in\Omega_h\\ j\in K}} \left| |C_j|\frac{dw_j}{dt} + \beta_j^K \phi^K(w_h)\right| = \sup_{\substack{K\in\Omega_h\\ j\in K}} \epsilon_j^K \leq Ch^3$$

Time dependent problems

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K}\beta_i^K\phi^K = 0\,, \quad \phi^K = \int\limits_K \vec{a}\cdot\nabla u_h$$

Time continuous error analysis

Pushing it a bit more :

$$\begin{aligned}
\epsilon_j^K| =& \left| |C_j|\frac{dw_j}{dt} + \beta_j^K \int\limits_K \vec{a}\cdot\nabla w_h \right| \\
=& \left| |C_j|\frac{dw_j}{dt} - \beta_j^K \int\limits_K \partial_t w_h + \beta_j^K \int\limits_K \big(\partial_t(w_h - w) + \vec{a}\cdot\nabla(w_h - w)\big) \right| \\
\leq& \left| |C_j|\frac{dw_j}{dt} - \beta_j^K \int\limits_K \partial_t w_h \right| + Ch^3
\end{aligned}$$

Time dependent problems

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \beta_i^K \phi^K = 0\,, \quad \phi^K = \int\limits_K \vec{a}\cdot\nabla u_h$$

Time continuous error analysis

Second order accuracy constraint :

$$\left| |C_j|\frac{dw_j}{dt} - \beta_j^K \int\limits_K \partial_t w_h \right| \leq Ch^3$$

Only true for : centered scheme (mass lumping stuff)

Time dependent problems

High order schemes : time dependent case

There is a number of different ways to do it right discussed in

- (Hubbard, Roe *Iint.J.Num.Meth.Fluids* 2000)
- (Csik, Ricchiuto, Deconinck *AIAA CP* 2001)
- (Abgrall, Mezine *J.Comput.Phys.* 2003)
- (Abgrall, Andrianov, Mezine *J.Comput.Phys.* 2003)
- (Caraeni, Fuchs *Computers & Fluids* 2005)
- (Ricchiuto, Csik, Deconinck *J.Comput.Phys.* 2005)
- (Dobes, Deconinck *J.Comput.Appl.Math.* 2008)
- (Ricchiuto, Bollermann *J.Comput.Phys.* 2009)
- (Ricchiuto, Abgrall *J.Comput.Phys.* 2010)
- (Hubbard, Ricchiuto *Computers & Fluids* 2011)

.. and references therein ...

Time dependent problems

The main idea is to recover second (to fix ideas) order accuracy in space by modifying the (semi-)discrete equations as follows :

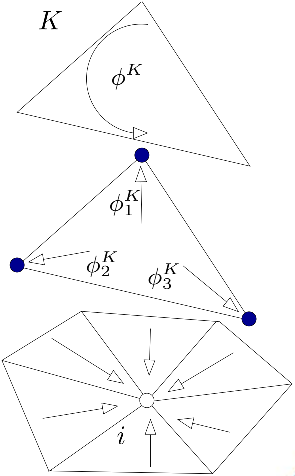
1. $\forall K \in \Omega_h$ compute :

$$\phi^K = \int_K \Big(\partial_t u_h + \nabla \cdot \mathcal{F}_h(u_h) \Big)$$

2. Distribution : $\phi_i^K = \beta_i^K \phi^K$

3. Integrate ODE system $\sum_{T|i \in T} \beta_i^K \phi^K = 0$:

$$\sum_{T|i \in T} \beta_i^K \int_K \partial_t u_h = - \sum_{T|i \in T} \beta_i^K \int_K \nabla \cdot \mathcal{F}_h(u_h)$$



Time dependent problems

Remarks

- After time discretization, independently on the explicit or implicit nature of the time stepping scheme, we end with a nonlinear system of equations of the type

$$M(\mathrm{u}^{n+1})\mathrm{u}^{n+1} + \Delta t F(\mathrm{u}^{n+1}) = \Delta t G(\mathrm{u}^{n}, tu^{n-1}, \ldots)$$

where M depends on u^{n+1} via the β_i^K

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- As in all weighted residual methods we have a mass matrix
- Even for simple wave propagation problems, almost all of these schemes do not allow simple high order explicit (*e.g.* Runge-Kutta) time stepping
- A lot of effort has gone into understanding how and if to modify the distribution strategy w.r.t the steady state case

Time dependent problems : genuinely explicit schemes

Explicit stabilization operators for stabilized FEM

Exception to the rule : (Ricchiuto, Abgrall *J.Comput.Phys* 2010).

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Idea explained for SUPG

Time dependent problems : genuinely explicit schemes

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Exception to the rule : (Ricchiuto, Abgrall *J.Comput.Phys* 2010).
Idea explained for SUPG

- Consider an explicit (single- or multi-step) high order time integration scheme

$$u' + f(u) = 0 \rightarrow u^{n+1} - u^n + \Delta t E(u^n, u^{n-1}, \ldots) = 0$$

- Assume that, given a smooth exact solution w, the scheme has the local truncation error

$$\epsilon^n = |w^{n+1} - w^n + \Delta t E(w^n, w^{n-1}, \ldots)| = C\,\Delta t^{p+1}$$

- Now set set $f(u) = \nabla \cdot \mathcal{F}(u)$ in our ODE

Time dependent problems : genuinely explicit schemes

Explicit stabilization operators for stabilized FEM

The SUPG finite element scheme reads

$$\int_{\Omega_h} \psi_i \big(u_h^{n+1} - u_h^n + \Delta t E_h(u_h^n,\, u_h^{n-1},\, \ldots) \big)$$
$$+ \sum_{K|i\in K} \int_K \vec{a}(u_h^n) \cdot \nabla \psi_i \, \tau \left(u_h^{n+1} - u_h^n + \Delta t E_h(u_h^n,\, u_h^{n-1},\, \ldots) \right) = 0$$

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To remain consistent, the streamline dissipation term contains the whole unsteady residual leading to a skew algebraic system despite of the explicit time integration

Time dependent problems : genuinely explicit schemes

Explicit stabilization operators for stabilized FEM

In (Ricchiuto, Abgrall *J.Comput.Phys* 2010) : without any formal loss of accuracy, the time dependent residual in the stabilization term can be replaced by a weakly consistent shifted one

$$\sum_{l=0}^{L} \alpha_l u_h^{n-l} + \Delta t E_h(u_h^n, u_h^{n-1}, \ldots)$$

that for a smooth solution w satisfies a lower order consistency estimate

$$|\sum_{l=0}^{L} \alpha_l w^{n-l} + \Delta t E(w^n, w^{n-1}, \ldots)| = C' \Delta t^p$$

Time dependent problems : genuinely explicit schemes

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The SUPG finite element scheme is modified as

$$\int_{\Omega_h} \psi_i \big(u_h^{n+1} - u_h^n + \Delta t E_h(u_h^n,\, u_h^{n-1},\, \ldots) \big)$$

$$+ \sum_{K|i\in K} \int_K \vec{a}(u_h^n) \cdot \nabla\psi_i \, \tau \, \big(\sum_{l=0}^{L} \alpha_l u_h^{n-l} + \Delta t E_h(u_h^n,\, u_h^{n-1},\, \ldots) \big) = 0$$

It remains to invert the SPD Galerkin mass matrix
or further simplify via mass lumping

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It remains to invert the SPD Galerkin mass matrix
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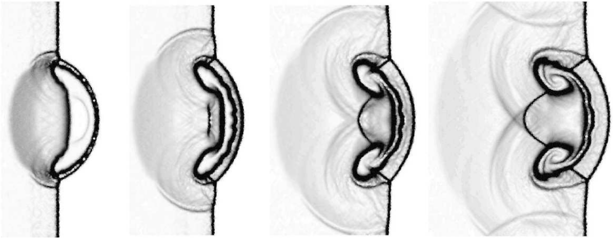
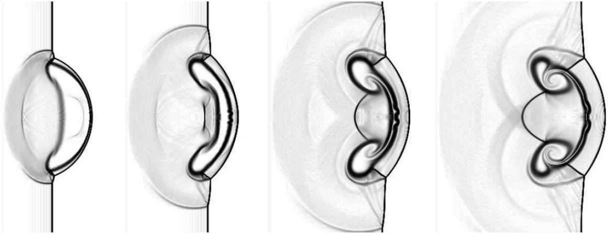
Same construction allows to obtain genuinely explicit second
order nonlinear RD
Higher order in the works

Time dependent problems : shock "bubble" interaction

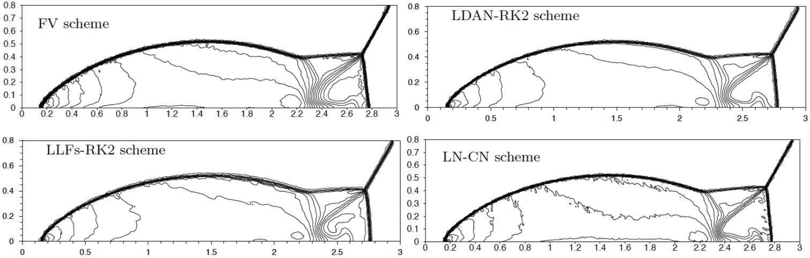


RD (CN in time, unstructured $h = 1/200$)



(Holden et al *J.Comput.Phys* 1999, cartesian $h = 1/400$)

Time dependent problems : double Mach reflection



Reflection of a Ma=10 moving shock on a 30 ramp
Comparison on the same grid with cell centered FV + limiter of Barth and Jespersen + RK2

5

SHALLOW WATER SIMULATIONS

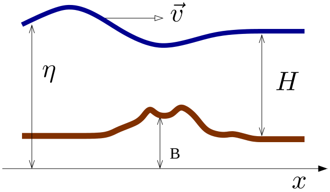
with RD based schemes

RD for Shallow Water simulations

$$\partial_t H + \nabla \cdot (H\vec{v}) = 0$$

$$\partial_t (H\vec{v}) + \nabla \cdot (H\vec{v} \otimes \vec{v} + g\frac{H^2}{2}\mathbf{I}) + gH(\nabla b + c_{\mathrm{f}}\vec{v}) = 0$$

- good in deep water
- not very good in the surf region and before wave break up (need non-hydrostatic corrections)
- quite good at predicting runup on sloping shores and flooding



RD for Shallow Water simulations

$$
\begin{aligned}
\partial_t H + \nabla \cdot (H\vec{v}) = 0 \\
\partial_t (H\vec{v}) + \nabla \cdot (H\vec{v} \otimes \vec{v} + g\frac{H^2}{2}\mathbf{I}) + gH(\nabla b + c_{\mathrm{f}}\vec{v}) = 0
\end{aligned}
$$

Numerical challenges

- Std stuff of hyperbolic conservation laws (shocks, contacts, expansions, etc) ;
- Dry are areas ($H = 0$) way more common than zero density in gas dynamics ;
- Source terms dominated flows ;
- A large number of simple and non-trivial equilibria flux div-source term ;

RD for Shallow Water simulations

$$\partial_t H + \nabla \cdot (H\vec{v}) = 0$$

$$\partial_t (H\vec{v}) + \nabla \cdot (H\vec{v} \otimes \vec{v} + g\frac{H^2}{2}\mathbf{I}) + gH(\nabla b + c_{\mathrm{f}}\vec{v}) = 0$$

Interesting topics

- Preservation of equilibria with RD (well balancedness or C-property)
- Construction of well balanced FV fluxes using RD
- Long wave run up on complex bathymetries

RD for Shallow Water simulations

Equilibria with invariants I : homoenergetic frictionless flows

Consider the following set of derived quantities :

$$\mathcal{E} = g\eta + \frac{\|\vec{v}\|^2}{2} \qquad \text{(total energy)}$$
$$\vec{q} = H\vec{v} \qquad \text{(discharge)}$$

Under the compatibility condition $\vec{v}^\perp \cdot \nabla b = 0$, the shallow water equations admit the family of steady solutions

$$\mathcal{E} = g\eta + \frac{\|\vec{v}\|^2}{2} = \mathcal{E}_0$$
$$\vec{q} = H\vec{v} = \vec{q}_0$$

The condition $\vec{v}^\perp \cdot \nabla b = 0$ only allows pseudo-one dimensional flows, with no cross-wind bathymetry variations.

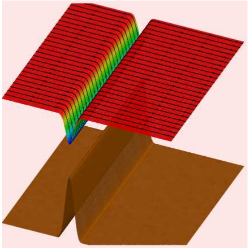
RD for Shallow Water simulations

Equilibria with invariants I & 1/2 : lake at rest

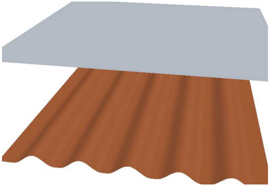
For $\vec{v} = 0$ we recover the well known lake at rest state

$$\eta = \eta_0$$
$$\vec{v} = 0$$

The velocity being null, the bathymetry can be arbitrary without violating the compatibility condition.



Constant energy



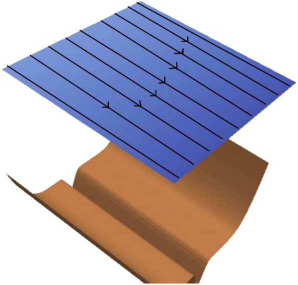
Lake at rest

RD for Shallow Water simulations

Equilibria with invariants II : sloping channels with friction and transverse bed variations

$$
\begin{aligned}
b(x,y) &= b_0 - \xi_0 x + \beta(y) \\
\eta &= \eta_0 - \xi_0 x \\
H &= H_0 - \beta(y) \\
c_{\mathrm{f}}(u(y),H(y))u(y) &= \xi_0 = -\partial_x b
\end{aligned}
$$

Pseudo 1d flow with transverse bathymetry, depth, velocity variations.



RD for Shallow Water simulations

Equilibria with invariants III : C-property

A scheme is said to verify the C-property (Conservation) for a certain steady equilibrium if it is able to preserve ir exactly and indefinitely (Bermudez, Vazquez *Computers & Fluids*)

One speaks of approximate C-property, if the equilibrium is preserved within a certain error, smaller than the truncation of the scheme. In this case, here we say that the discretization is super-consistent with the given equilibrium.

Schemes verifying the C-property are often referred to as well-balanced (Greenberg, Leroux *SISC* 1996)

RD for Shallow Water simulations

$$\partial_t u + \nabla \cdot \mathcal{F}(u) + \mathcal{S}(u,x,y) = 0 \qquad \text{on} \quad \Omega \times [0, T_f] \subset \mathbb{R}^2 \times \mathbb{R}^+$$

Super consistency results forRD schemes on P^1 triangles

Consider now the RD schemes obtained as some form of

$$\sum_{K|i\in K} \beta_i^K \int_K \big(\partial_t u_h + \nabla \cdot \mathcal{F}_h + \mathcal{S}_h\big) = 0$$

RD for Shallow Water simulations

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Since the whole equation is “carried along” in the distribution it is natural to expect that equilibria between different terms should be resolved accurately

RD for Shallow Water simulations

Main result (Ricchiuto, 2011)

On P^1 meshes, high order RD schemes preserve *exactly* steady equilibria with a set of invariants v provided that

1. exact integration is used
2. the approximation of the flux and of the source term is written as $\mathcal{F}_h = \mathcal{F}(\mathrm{v}_h)$, $\mathcal{S}_h = \mathcal{S}(\mathrm{v}_h)$

For approximate quadrature and for a smooth enough bathymetry the super consistency estimate holds

$$|\epsilon_h| \leq C\, h^l\,, \quad l = \min(p_f + 1,\, p_v + 2)$$

with p_f and p_v the degrees of the polynomials exactly integrated by the quadrature formulae used.

RD for Shallow Water simulations

Meaning and remarks

- Equilibria described by v=const :
  1. homoenergetic flow : $\mathrm{v}=[\mathcal{E},\, \vec{q}]$
  2. lake ate rest : $\mathrm{v}=[\eta,\, \vec{q}]$
  3. channel flows with friction (no transverse b) : $\mathrm{v}=[H,\, \vec{v}]$
- v is interpolated and everything else derived from its values
- perturbation = quadrature error in computing $\phi^K(\mathrm{v}_h)$
- Similar to FV (Gallouet, Hérard, Seguin *Computers & Fluids* 2003 ; Noelle, Xing, Shu *J.Comput.Phys.* 2007) but here unstructured triangulations

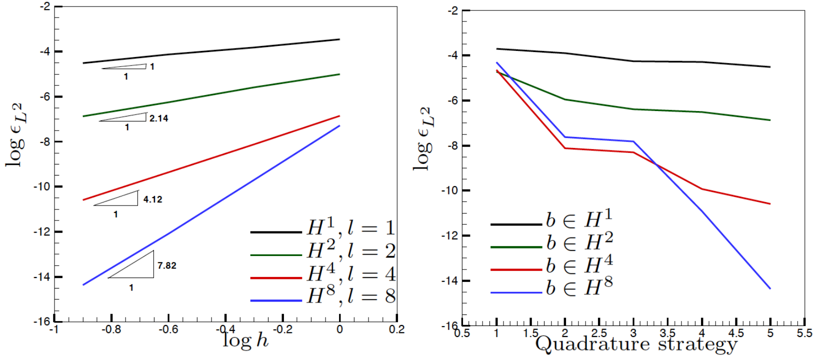
RD for Shallow Water simulations

Example : homoenergetic flow, unstructured triangular grids

... video ...

RD for Shallow Water simulations

Example : homoenergetic flow, unstructured triangular grids



The scheme is second order accurate

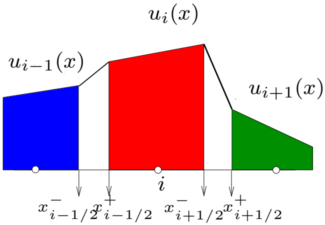
Dependence by data regularity well described by theory

RD for Shallow Water : construction of FV fluxes

$$\partial_t \begin{bmatrix} h \\ hu \end{bmatrix} + \partial_x \begin{bmatrix} hu \\ hu^2 + g\dfrac{h^2}{2} \end{bmatrix} + gh\partial_x \begin{bmatrix} 0 \\ b(x) \end{bmatrix} = 0$$

Integrate over cell i plus RD scheme on ghost cells

$$\Delta x_i \frac{u_i^{n+1} - u_i^n}{\Delta t} + f(u_i(x^-_{i+1/2})) - f(u_i(x^+_{i-1/2})) + \mathcal{S}_i +$$
$$\beta_i^{i-1/2}\Big(f(u_i(x^+_{i-1/2})) - f(u_{i-1}(x^-_{i-1/2})) + \mathcal{S}_{i-1/2}\Big) +$$
$$\beta_i^{i+1/2}\Big(f(u_{i+1}(x^+_{i+1/2})) - f(u_i(x^-_{i+1/2}) + \mathcal{S}_{i+1/2}\Big) = 0$$



RD for Shallow Water : construction of FV fluxes

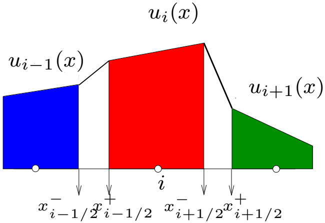
For a linear approximation of $b(x)$

$$\mathcal{S}_i = g\tilde{h}_i(b_i(x^-_{i+1/2}) - b_i(x^+_{i-1/2}))$$

$$\mathcal{S}_{i-1/2} = g\tilde{h}_{i-1/2}(b_i(x^+_{i-1/2}) - b_{i-1}(x^-_{i-1/2}))$$

$$\mathcal{S}_{i-1/2} = g\tilde{h}_{i+1/2}(b_{i+1}(x^+_{i+1/2}) - b_i(x^-_{i+1/2}))$$

The actual value of the average height $\tilde{h}$ only depends on the interpolation within the cells.



RD for Shallow Water : construction of FV fluxes

Using the well balanced RD approach we get in general

$$\tilde{h}_{i-1/2} = \frac{1}{x^+_{i-1/2} - x^-_{i-1/2}} \int_{x^-_{i-1/2}}^{x^+_{i-1/2}} h\left(\begin{bmatrix} h + b + u^2/2g \\ hu \end{bmatrix}\right) = \sum_{gp} \omega_{gp} h\left(\begin{bmatrix} \mathcal{E}_{gp} \\ q_{gp} \end{bmatrix}\right)$$

in each quadrature point need to solve the nonlinear system

$$\begin{cases} h_{gp} + u^2_{gp}/2g & = \mathcal{E}_{gp} - b_{gp} \\ h_{gp} u_{gp} = q_{gp} \end{cases}$$

Where the total energy $\mathcal{E}$ and the flux $q = hu$ are interpolated linearly.

RD for Shallow Water : construction of FV fluxes

The final well balanced FV discretization reads :

$$\Delta x_i \frac{u_i^{n+1} - u_i^n}{\Delta t} + S_i + f_{i+1/2} - f_{i-1/2} + S_{i+1/2} + S_{i-1/2} = 0$$

$$S_i = \left[\begin{array}{c} 0 \\ g\tilde{h}_i(b_L^{i+1/2} - b_R^{i-1/2}) \end{array} \right]$$

$$f_{i+1/2}(u_L, u_R) = f(u_L) + \beta_i^{i+1/2}\,(f(u_R) - f(u_L))$$

$$S_{i+1/2} = \beta_i^{i+1/2} \left[\begin{array}{c} 0 \\ g\tilde{h}_{i+1/2}(b_R - b_L) \end{array} \right]$$

Generalized form of

well balanced quadrature of (Noelle, Xing, Shu, *JCP* 226, 2007)

Extra degree of freedom in the actual splitting

viz the definition of the β_i coefficients

Run up on complex bathymetries

Studied in (Ricchiuto, Bollermann *JCP* 2009 ; Ricchiuto AIP Proc. 1389 2011)

1. Adapted nonlinear variants of the Lax-Friedrich's distribution guaranteeing in some form

$$H_i^{n+1} \geq 0 \ \text{ whenever } \ H_h^n \geq 0$$

2. Several time-stepping strategies allow the preservation of this constraint :
   - Implicit Crank-Nicholson. Positivity preserved under a CFL=2 constraint ;
   - Space-time schemes (discontinuous in time). Unconditional positivity ;
   - Genuinely explicit RK-RD schemes. Positivity preserved under a CFL=1 constraint.
3. A strategy to maintain the lake at rest near dry regions

see examples

6
SUMMARY AND PERSPECTIVES

Summary and perspectives

Acknowledge contributions of my collaborators/friends

- R. Abgrall, G. Baurin, P. Congedo, D. De Santis, C. Dobrzynski, D. Genet, P. Jacq +++ (Inria)
- H. Deconinck, S. D'angelo, N. Villedieu, M. Vymazal ++ (VKI)
- M. Hubbard, D. Sarmany, A. Warzynski ++ (Leeds University)
- A. Larat (Ecole Central Paris)
- E. Vazquez-Cendon (University of Santiago de Compostela)
- and all those I have inevitably forgotten

Summary and perspectives

- RD as a general framework to study non-oscillatory higher order schemes
- On on hand "true RD schemes"
- On the other a means of improving other techniques via several "bridges" allowing to recast one as the other

Current work related to RD

- Turbulence modeling (PhD D. De Santis)
- GPU implementation (PhD D. Genet)
- Adjoint error estimation for RD (PhD S. D'angelo)
- Higher order time dependent (with R. Abgrall, A. Larat and M. Hubbard - PhD A. Warzynski, Leeds)
- Other polynomial approximations (bridge with DG, Bezier)
- Non-hydrostatic free surface modeling with UQ (with R. Abgrall, P. Congedo, A.I. Delis, F. Marche)
- Local adaptation for unsteady problems (with R. Abgrall, C. Dobrzynski)
- Mass consistent coupling with transport equations (with E. Vazquez)
- etc. etc.

THANK YOU

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BACCHUS Inria team :
http://bacchus.bordeaux.inria.fr