Fast techniques for the incompressible variable density Navier-Stokes equations

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Acknowledgments

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1. Introduction

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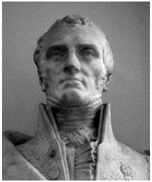
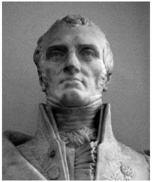
Navier-Stokes equations



Claude L. M. H. Navier

George G. Stokes

Navier-Stokes equations



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$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho \mathrm{u}) = 0 \quad \text{in } \Omega \times [0, T], \\ \rho(\partial_t \mathrm{u} + \mathrm{u}\cdot\nabla \mathrm{u}) - \mu \nabla^2 \mathrm{u} + \nabla \mathrm{p} = f \quad \text{in } \Omega \times [0, T], \\ \nabla\cdot \mathrm{u} = 0 \quad \text{in } \Omega \times [0, T], \\ \mathrm{u}|_\Gamma = 0 \quad \text{in } [0,\mathrm{T}], \qquad \text{and } \mathrm{u}|_{t=0} = u_0 \quad \text{in } \Omega, \end{cases}$$

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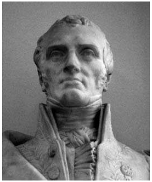

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George G. Stokes

- Ω fluid domain
- T some time
- f smooth source term
- u_0 smooth solenoidal data

Objectives

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Strategy: Fractional time-stepping, Chorin–Temam idea (1968-1969).

Non-incremental pressure-correction schemes

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$$p^{k+1} = \phi^{k+1}$$

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- Step 2 amounts to

$$\tilde{u}^{k+1} = u^{k+1} + \nabla\left(\frac{\Delta t}{\rho}\phi^{k+1}\right), \qquad u^{k+1} \in H,\ \phi^{k+1} \in H^1(\Omega)$$

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$$\text{(i)} \qquad \nabla^2\phi^{k+1} = \frac{\rho}{\Delta t}\nabla\cdot\tilde{u}^{k+1}; \quad \partial_n\phi^{k+1}|_\Gamma = 0$$

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- $\boxed{\text{Very simple algorithm}}$

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- $\boxed{\text{Very simple algorithm}}$ $\Rightarrow$ Very popular

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Theorem (Rannacher (1991), Shen (1992))

$$\|\mathrm{u}_{\Delta t} - u_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} + \|\mathrm{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} \leq c(\mathrm{u}, \mathrm{p}, T)\,\Delta t,$$

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- Irreducible splitting error of order $\mathcal{O}(\Delta t)$ $\Rightarrow$ using higher-order time stepping does not improve the overall accuracy.

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- Simple idea: use the old pressure p^k in the viscous step and correct the pressure appropriately afterwards (Goda (1979) Van Kan (1986)).

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- Time stepping can be replaced by any 2nd order A-stable stepping.

Rotational incremental pressure-correction schemes

- A new simple idea: use $\nabla^2 u = \nabla\nabla\cdot u - \nabla\times\nabla\times u$ (Timmermans, Minev and Van De Vosse (1996)).

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- Where is the catch?
  The tangent component of u^{k+1} is still not correct! $\Rightarrow$ sub-optimality

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Theorem (Guermond-Shen (2006))

With appropriate initialization,

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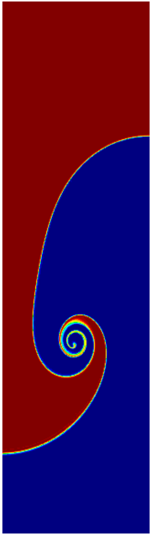
- OPEN QUESTION: can we regain the missing $\Delta t^{\frac{1}{2}}$?

Variable density flows



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- All The current splitting algorithms are based on this model!
- Only two proofs of stability available (Guermond-Quartapelle (2000), Pyo-Shen (2007)).

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$$\begin{cases} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p - \mu\nabla^2\mathbf{u} = \mathbf{f}, & \mathbf{u}|_\Gamma = 0, \\ \nabla\cdot\mathbf{u} - \epsilon\nabla^2\phi = 0, \qquad \partial_n\phi|_\Gamma = 0, & p = \phi \end{cases}$$

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A new (old) idea

- Ex 3: Incremental rotational pressure-correction equivalent to

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- The new idea: Adopt the penalty point of view instead of the Helmholtz decomposition.

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Non-incremental version

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Incremental version

Proposition

The algorithm is stable provided $\chi \leq \inf_{\mathbf{x}\in\Omega} \rho^0(\mathbf{x})$, *for all* $n \in \overline{0, N}$.

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$$+ \nabla\cdot(\frac{1}{2}\rho^\star\mathbf{u}^n)\mathbf{u}^{n+1} + \nabla p^\star = \mathbf{f}^{n+1}, \qquad \tilde{\mathbf{u}}^{n+1}|_\Gamma = 0,$$

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- Error $\mathcal{O}(\Delta t)$ on velocity in $\mathbf{H}^1$-norm and pressure in L^2-norm.
- $p^n + \phi^{=} 2p^n - p^{n-1}$: second-order extrapolation on pressure $\Rightarrow$ second-order accuracy reachable.

Rotational incremental version + BDF2

- Velocity extrapolation $\mathbf{u}^\star = 2\mathbf{u}^n - \mathbf{u}^{n-1}$.

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- Velocity

$$\frac{3\rho^\star\mathbf{u}^{n+1} - 4\rho^{n+1}\mathbf{u}^n + \rho^{n+1}\mathbf{u}^{n-1}}{2\Delta t} + \rho^{n+1}\left(\mathbf{u}^\star\cdot\nabla\right)\mathbf{u}^{n+1} - \mu\nabla^2\mathbf{u}^{n+1} + \left(\nabla\cdot\frac{\rho^\star}{2}\mathbf{u}^\star\right)\mathbf{u}^{n+1} + \nabla p^\star = \mathbf{f}^{n+1}.$$

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The non-rotational version of the algorithm is stable provided $\chi \leq \inf_{\mathbf{x}\in\Omega} \rho^n(\mathbf{x})$*, for all* $n \in \overline{0, N}$.

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- Requires only to solve a Poisson problem!
- Algorithm formally $\mathcal{O}(\Delta t^2)$.

Numerical illustrations

Convergence tests

- Velocity $\mathbb{P}_2$, pressure $\mathbb{P}_1$, density $\mathbb{P}_2$

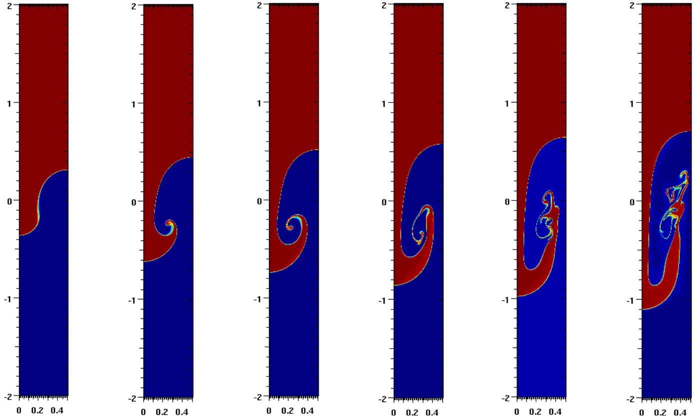
Δt	Vel. L^2	rate	Vel. H^1	rate	Pre. L^2	rate	Den. L^2	rate
0.100000	3.90E-3		1.63E-2		1.25E-2		1.25E-2	
0.050000	1.18E-3	1.73	5.03E-3	1.70	3.61E-3	1.79	2.93E-3	2.09
0.025000	3.35E-4	1.82	1.47E-3	1.77	1.00E-3	1.85	7.60E-4	1.95
0.012500	9.04E-5	1.89	4.13E-4	1.83	2.70E-4	1.89	2.08E-4	1.87
0.006250	2.37E-5	1.93	1.15E-4	1.84	7.10E-5	1.93	5.85E-5	1.83
0.003125	6.12E-6	1.95	3.17E-5	1.86	1.87E-5	1.93	1.67E-5	1.81

- Rotational incremental version + BDF2; Smooth domains: $(\mathcal{O}(\Delta t)^2)$ on velocity in $\mathbf{L}^2$-norm, a little les in $\mathbf{H}^1$-norm and pressure in L^2-norm.

Convergence tests

- Rotational incremental version + BDF2; Non-smooth domains:
  $(\mathcal{O}(\Delta t)^2)$ on velocity in $\mathbf{L}^2$-norm, $(\mathcal{O}(\Delta t)^{\frac{3}{2}})$ on velocity in $\mathbf{H}^1$-norm and pressure in L^2-norm.

Numerical illustrations. RT, $\rho_{max}/\rho_{min} = 3,\ R_e = 1000$



- $t = 1$ $t = 1.5$ $t = 1.75$ $t = 2$ $t = 2.25$ $t = 2.5$

movie

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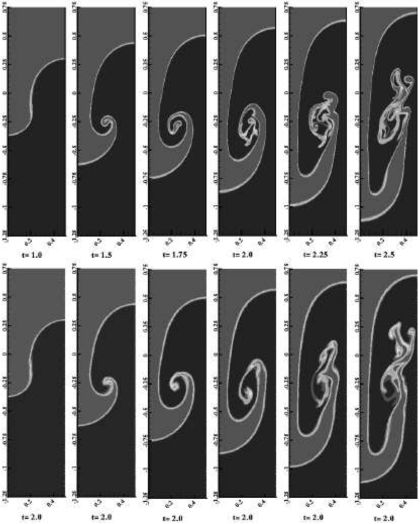

- Guermond, Fraigneau (2001)
  Standard algorithm
  Finite volume 256×512

- Guermond, Fraigneau (2001)
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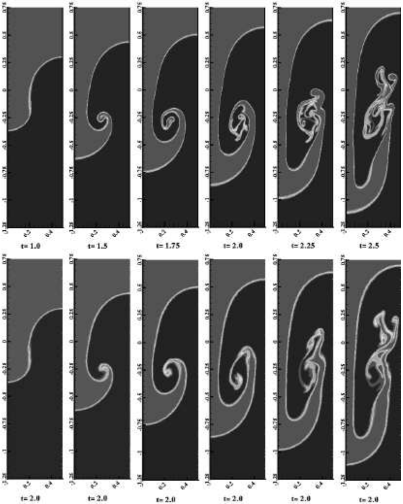
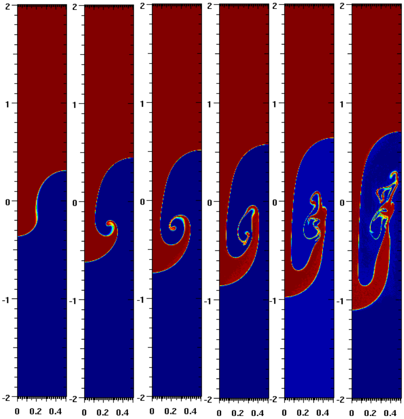
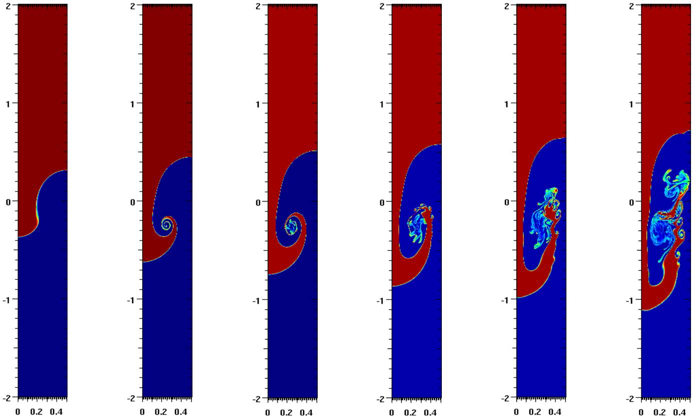
  Finite elements 30189
  $\mathbb{P}_2$ nodes

Numerical illustrations. RT, $\rho_{max}/\rho_{min} = 3,\ R_e = 1000$





Numerical illustrations. RT, $\rho_{max}/\rho_{min} = 3,\ R_e = 5000$



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movie

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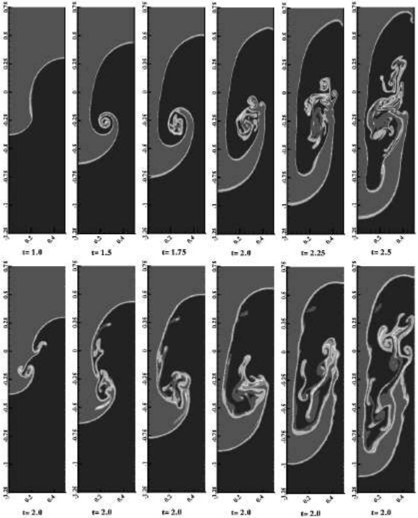

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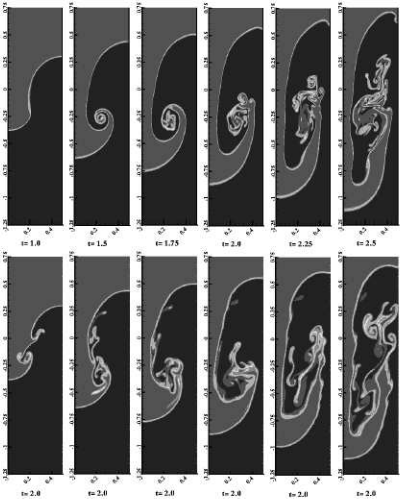
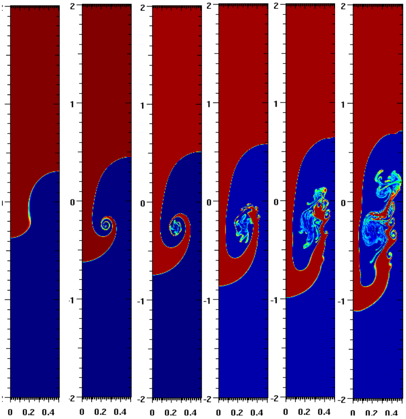
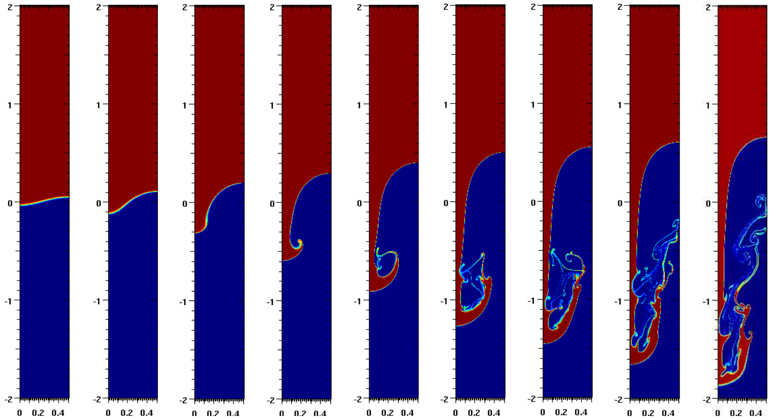




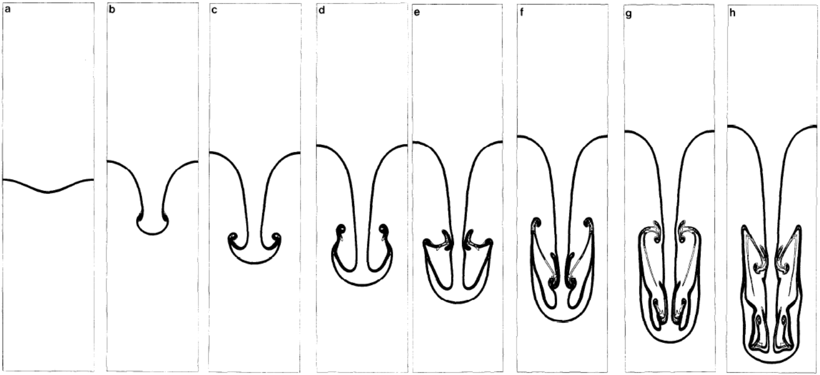
Numerical illustration. RT, $\rho_{\max}/\rho_{\min} = 7,\ R_e = 1000$



- $t = 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5,\ 3.75,\ 4,\ 4.25$

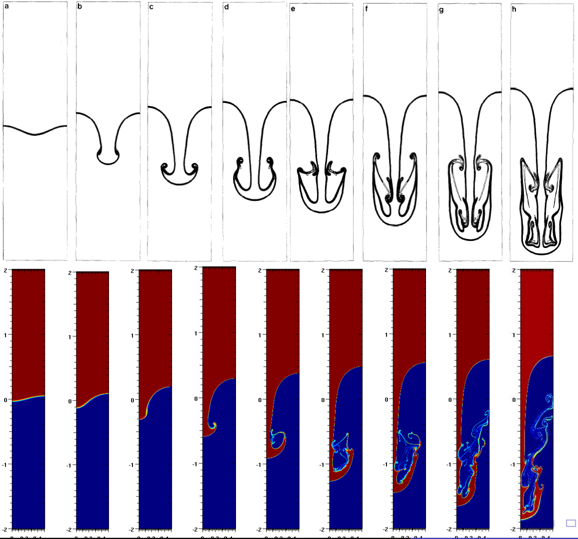
movie

Numerical illustration. RT, $\rho_{max}/\rho_{min} = 7,\ R_e = 1000$



- $t = 1.5$ $t = 2$ $t = 2.5$ $t = 3$ $t = 3.5$ $t = 3.75$ $t = 4$ $t = 4.25$
- Bell, Marcus (1992), Standard algorithm, Finite volume 200×800

Numerical illustration. RT, $\rho_{max}/\rho_{min} = 7,\ R_e = 1000$



Concluding remarks

- Splitting algorithms are fast and easy to implement.

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- New fast splitting algorithm for solving variable density flows: solve $\nabla^2\phi = \psi$ instead of $\nabla\cdot(\frac{1}{\rho}\nabla\phi) = \psi$.
- Stability proven up to second-order time stepping. Convergence analysis coming soon.

Open issues

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- Does there exist a splitting scheme that is fully $\mathcal{O}(\Delta t^2)$ in non-smooth domains?
- Be suspicious about any splitting method that claims convergence order (in H^1-norm) $> \mathcal{O}(\Delta t^{\frac{3}{2}})$.