Méthodes numériques d'homogénéisation pour des problèmes elliptiques non-linéaires non-monotones.

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Plan of the talk

Problem.
$$\begin{aligned} -\nabla \cdot (a^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) &= f \quad \text{in } \Omega, \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

1. Introduction
2. One-scale nonlinear problems
3. Homogenization nonlinear problems (two scales)

Analytical framework: homogenization

Macroscopic behaviour of multiple scale problems (Bakhvalov, Babuska, Bensoussan, Lions, Papanicolaou, Tartar, Sanchez-Palencia, Jikov, Kozlov, Oleinik, Nguetseng, Fusco, Moscariello, Boccardo, Murat,...)

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Elliptic example. $-\nabla \cdot (\underbrace{a^\varepsilon \nabla u_\varepsilon}_{\xi_\varepsilon}) = f$, on Ω, $u_\varepsilon = 0$ on $\partial\Omega$.

where the tensor $a^\varepsilon(x)$ varies rapidly in space (at the scale ε).
Question: $u_\varepsilon \to u_0$ for $\varepsilon \to 0$? equation for u_0 ?
Assuming a^ε uniformly elliptic and bounded, we have:

$$u_\varepsilon \overset{H^1}{\rightharpoonup} u_0, \qquad \xi_\varepsilon \overset{L^2}{\rightharpoonup} \xi_0, \qquad \text{for } \varepsilon \to 0.$$

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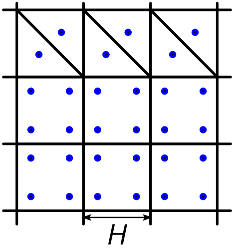
Homogenization problem: find $a^0 \in L^\infty(\Omega)^{d\times d}$ such that

$$-\nabla\cdot(\underbrace{a^0\nabla u_0}_{\xi_0}) = f, \text{ on } \Omega,\ u_0 = 0 \text{ on } \partial\Omega.$$

Remark: In general a^0 is not a “simple average” (no explicit formula).

Finite Element Heterogeneous Multiscale Method (E, Engquist 2003)

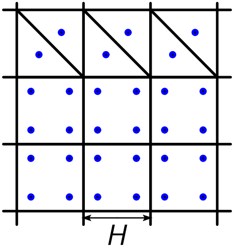
$$A(v^H, w^H) = \int_K a(x)\nabla v^H(x) \cdot \nabla w^H(x) dx, \quad \forall v^H, w^H$$



standard FEM
with quadrature formulas

Finite Element Heterogeneous Multiscale Method (E, Engquist 2003)

$$A_H(v^H, w^H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \omega_{K_j} a(x_{K_j}) \nabla v^H(x_{K_j}) \cdot \nabla w^H(x_{K_j}), \quad \forall v^H, w^H$$



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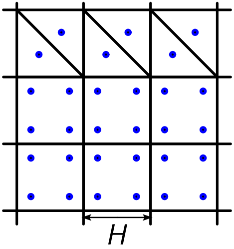
Finite Element Heterogeneous Multiscale Method (E, Engquist 2003)

$$B_H(v^H, w^H) = \sum_{K\in\mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a^\varepsilon(x)\nabla v^h_{K_j}(x)\cdot w^h_{K_j}(x)dx, \quad \forall v^H, w^H \in S_0^\ell(\Omega, \mathcal{T}_H),$$

where $w^h_{K_j}$ is the solution of the micro problem $w^h_{K_j} - w^H_{lin} \in S^q(K_{\delta_j}, \mathcal{T}_h)$,

$$\int_{K_{\delta_j}} a^\varepsilon(x)\nabla w^h_{K_j}(x)\cdot\nabla z^h(x)dx = 0, \quad \forall z^h \in S^q(K_{\delta_j}, \mathcal{T}_h).$$



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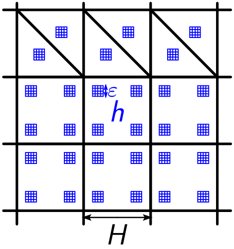
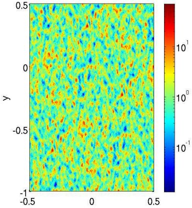


Heterogeneous Multiscale Method (HMM)
(micro meshes: FEs of size h)

Case of linear parabolic problems (Abdulle and V., 2011)

random tensor a^ε:



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\partial_t u_\varepsilon - \nabla \cdot (a^\varepsilon \nabla u_\varepsilon) &= f \quad \text{in } \Omega \times (0,1) \\
u_\varepsilon(0) &= 0 \quad \text{in } \Omega,
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+boundary conditions

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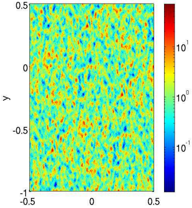

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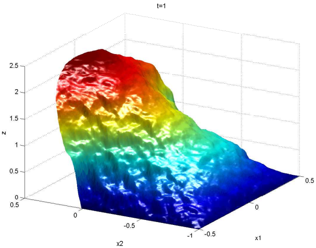

finescale solution u_ε at $t = 1$
(FE standard, 10^6 degrees of freedom)

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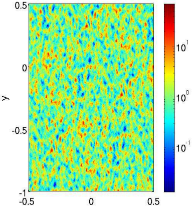

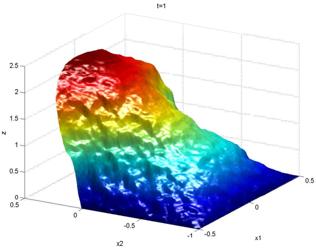

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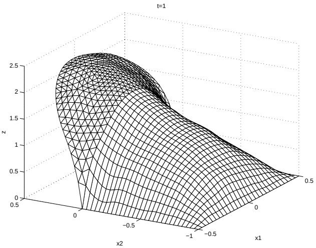

Homogenized solution u_0 at $t = 1$
(FE-HMM, 10^3 degrees of freedom)

A priori error analysis for parabolic homogenization problems

A priori error analysis with convergence rates as a function of the macro and micro mesh sizes H and h:

$$\begin{aligned}
\|u_0 - u^H\|_{L^2(0,T;H^1(\Omega))} &\leq C\Big(H^\ell + \Big(\frac{h}{\varepsilon}\Big)^{2q} + r_{MOD}\Big), \\
\|u_0 - u^H\|_{\mathcal{C}^0([0,T],L^2(\Omega))} &\leq C\Big(H^{\ell+1} + \Big(\frac{h}{\varepsilon}\Big)^{2q} + r_{MOD}\Big).
\end{aligned}$$

where C is a constant independent of H, h, ε.
A key ingredient is the convergence estimates for FEM with numerical quadrature (Raviart, 1973).

For the time discretization, we consider Runge-Kutta methods of implicit type (e.g. Radau) and of stabilized explicit type (Chebyshev) (semigroups techniques in a Hilbert space framework).

A. Abdulle & G. Vilmart, *Coupling heterogeneous multiscale FEM with Runge-Kutta methods for parabolic homogenization problems: a fully discrete space-time analysis*, submitted for publication, 2011.

Nonlinear problems: numerical homogenization

Problem.

$$\begin{aligned} -\nabla \cdot (a^{\varepsilon}(x, u_{\varepsilon}) \nabla u_{\varepsilon}) &= f \quad \text{in } \Omega, \\ u_{\varepsilon} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

First results for the analysis of a numerical homogenization method (HMM) by E,Ming,Zhang (2005); Chen, Savchuk 2007 (MsFEM):

- Use ideas from *Two-grid discretization techniques...* (Xu 1996).

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Questions:

- analysis in 3D? their arguments rely on bounds for 2D discrete Green functions.
- Fully discrete analysis (macro and micro errors?)
- L^2 convergence rates
- quadrilateral elements
- uniqueness of the (fully discrete) numerical solution (depends on $H, h, \ldots$)

One-scale nonlinear problems

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A. Abdulle & G. Vilmart, *A priori error estimates for finite element methods with numerical quadrature for nonmonotone nonlinear elliptic problems*, *submitted for publication*, 2011, 32 pages.

Nonlinear nonmonotone problem

Problem. $\nabla \cdot (a(x,u)\nabla u) = f$ in Ω, $\qquad u = 0$ on $\partial\Omega$.

Important problems: thermal diffusion in materials, water infiltration in porous medium (Richards), ...

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- Feistauer, Ženíšek (1987) FEM with numerical integration, for monotone problems where the weak formulation form satisfies $A(v;v,v-w) - A(w;w,v-w) \geq C\|\nabla v - \nabla w\|^2_{L^2(\Omega)}, \forall v,w \in H^1_0$.
- Feistauer, Křížek, Sobotíková (1993) FEM with numerical quadrature. For nonmonotone problems, but no convergence rates).
- Korotov, Křížek (2000) FEM with numerical integration (domain approx. in 3D), no convergence rates

Basic assumptions on the tensor

Define $a(x,s) = (a_{mn}(x,s))_{1\leq m,n\leq d}$.

Assume

1. a_{mn} continuous on $\overline{\Omega} \times \mathbb{R}$,
2. $|a_{mn}(x,s_1) - a_{mn}(x,s_2)| \leq \Lambda_1 |s_1 - s_2|, \forall x \in \overline{\Omega}, s_1, s_2 \in \mathbb{R}$.
3. $\lambda \|\xi\|^2 \leq a(x,s)\xi \cdot \xi, \quad \|a(x,s)\xi\| \leq \Lambda_0 \|\xi\|, \forall \xi \in \mathbb{R}^d$.

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1.,2.,3.$\Rightarrow$ nonlinear elliptic problem has one and only one solution, classical result, Douglas, Dupont, Serrin (1971) see Chipot (2009)

1.,3.,$(\mathbf{Q}) \Rightarrow$ for all $h > 0$ the nonlinear problem with numerical quadrature has at least one solution $u^h \in S_0^\ell(\Omega, \mathcal{T}_h)$ (Brouwer fixed point argument).

A priori error analysis for nonlinear FEM with numerical quadrature

Theorem

u sol. of nonlinear pb., u^h sol. of nonlinear FEM with quadrature. Assume 1.,2.,3.,(**Q**) and $h/h_K \leq C, \forall K \in \mathcal{T}_h$. Let $\ell \geq 1$ and

$$u \in H^{\ell+1}(\Omega) \cap W^{1,\infty}(\Omega),$$
$$a_{mn} \in W^{\ell+1,\infty}(\Omega \times \mathbb{R}), \qquad \forall m, n = 1 \dots d.$$

Assume that $\partial_u a_{mn} \in W^{1,\infty}(\Omega \times \mathbb{R})$, and that $\partial_u a_{mn}(x,s)$ and $\partial_{uu} a_{mn}(x,s)$ are continuous and bounded on $\overline{\Omega} \times \mathbb{R}$.
Then, there exists $h_0 > 0$ s.t. for all $h \leq h_0$, u^h is unique and

$$\|u - u^h\|_{H^1(\Omega)} \leq Ch^\ell, \qquad \|u - u^h\|_{L^2(\Omega)} \leq Ch^{\ell+1}.$$

$\Rightarrow$ For linear problems, we recover estimates of Ciarlet, Raviart (1972) with the same assumptions (excepted $u \in W^{1,\infty}(\Omega)$ and the inverse assumption $h/h_K \leq C, \forall K \in \mathcal{T}_h$).

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$$\|u - u^h\|_{H^1(\Omega)} \leq Ch^\ell, \qquad \|u - u^h\|_{L^2(\Omega)} \leq Ch^{\ell+1}.$$

Ingredients of the proof: Gagliardo-Niremberg inequality, compactness argument, Aubin-Nitche's duality argument. Study of FEM with numerical quadrature for the linearized differential operator (non-coercive, but satisfying the Gårding inequality), Schatz's compactness argument.

Idea of the analysis: convergences rates

Step 1. Using the boundedness of u^h in $H^1(\Omega)$, the compact injection $H^1(\Omega) \subset L^2(\Omega)$ and the uniqueness of u, we show

$$\|u - u^h\|_{L^2(\Omega)} \to 0 \quad \text{for } h \to 0.$$

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Step 2. Using estimates for $A - A_H$ (using the Bramble-Hilbert lemma) and the Gagliardo-Niremberg inequality $\|v\|^2_{L^3(\Omega)} \leq C\|v\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$ ($\dim \Omega \leq 3$), we derive

$$\|u - u^h\|_{H^1(\Omega)} \leq C\big(h^\ell + \|u - u^h\|_{L^2(\Omega)}\big)$$

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Step 3. Using the Aubin-Nitche duality argument, we show

$$\begin{aligned} \|u - u^h\|_{L^2(\Omega)} &\leq C\big(h^\ell + \|u - u^h\|^2_{H^1(\Omega)}\big) \\ \|u - u^h\|_{L^2(\Omega)} &\leq C\big(h^{\ell+1} + \|u - u^h\|^2_{H^1(\Omega)}\big) \end{aligned}$$

The idea is to consider the adjoint L^* of the linearized differential operator, i.e. $L\varphi := -\nabla\big(a(\cdot, u)\nabla\varphi + \varphi\partial_u a(\cdot, u)\nabla u\big)$.

The Newton method and the uniqueness of u^h

Newton method for the non-linear FEM. Initial guess $z_0^h \approx u^h$.

$$N_h(z_k^h; z_{k+1}^h - z_k^h, v^h) = F_h(v^h) - (a(z_k^h)\nabla z_k^h, \nabla v^h)_h, \quad \forall v^h \in S_0^\ell(\Omega, \mathcal{T}_h),$$

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Theorem (Convergence of the Newton method)

Under assumptions of theorem, there exist $h_0, \delta > 0$ s.t. if $h \leq h_0$ and $\sigma_h \|z_0^h - u^h\|_{H^1(\Omega)} \leq \delta$, then $\{z_k^h\}$ is well defined, and

$$\|z_{k+1}^h - u^h\|_{H^1(\Omega)} \leq C\sigma_h \|z_k^h - u^h\|^2_{H^1(\Omega)}.$$

$(\sigma_h \leq C(1 + |\log h|)^{1/2}$ for $d = 2$, $\sigma_h \leq Ch^{-1/2}$ for $d = 3)$.

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$$\|z_{k+1}^h - u^h\|_{H^1(\Omega)} \leq C\sigma_h \|z_k^h - u^h\|_{H^1(\Omega)}^2.$$

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Proof of the uniqueness of u^h. Given two solutions $u^h, \widetilde{u}^h$, apply the Newton method convergence theorem with initial guess $z_0^h := \widetilde{u}^h$.

Homogenization nonlinear problems (two scales)

$$\begin{aligned} -\nabla \cdot (a^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) &= f \quad \text{in } \Omega, \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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Fully discrete error analysis

Theorem

u_0 solution of homogenized problem. Assume (**H**). Then there exist $r_0 > 0$ and $H_0 > 0$ such that if $H \leq H_0$, $r_{HMM} \leq r_0$, any solution u^H of the FE-HMM satisfies

$$\begin{aligned} \|u_0 - u^H\|_{H^1(\Omega)} &\leq C(H^\ell + r_{HMM}) \\ \|u_0 - u^H\|_{L^2(\Omega)} &\leq C(H^{\ell+1} + r_{HMM}) \end{aligned}$$

r_{HMM} is analyzed similarly as for linear problems:

Fully discrete error analysis

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r_{HMM} is analyzed similarly as for linear problems:

$$r_{HMM} \leq C\left(\frac{h}{\varepsilon}\right)^{2q} + r_{MOD}$$

Remark. r_{MOD} modeling error
If a^ε locally periodic and $\delta = \varepsilon \Rightarrow r_{MOD} = 0$.
Otherwise: a^ε locally periodic with $\delta/\varepsilon \notin \mathbb{N}^*$ with Dirichlet bound. cond. $\Rightarrow$ resonance errors e.g. $r_{HMM} \leq \delta + \varepsilon/\delta$.

The Newton method and the uniqueness of the solution

Theorem (Convergence of the Newton method)

If in addition, $\|u^H\|_{W^{1,6}} \leq C$ and $r_{HMM} + r'_{HMM} \leq r'_0$ and $H \leq H_0$, then the Newton method to compute u^H is well defined and converges (in a neighbour of u^H).

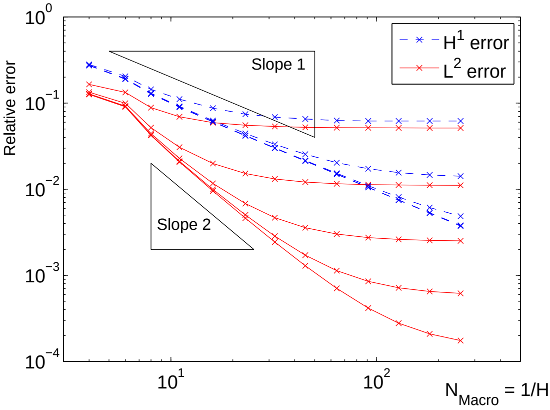
Theorem (Uniqueness of the FE-HMM solution)

For a (non-uniformly) periodic tensor with periodic coupling FE-HMM has a unique solution u^H for all H, h satisfying

$$\left(\frac{h}{\varepsilon}\right)^{2q} \leq H, \quad h \leq h_0, \quad H \leq H_0.$$

Numerical experiment: convergence rates

Example (2D multiscale problem, $\ell = 1$, $q = 1$)
Macro mesh refinement, fixed micro mesh $(4 \times 4, 8 \times 8, 16 \times 16, \ldots)$



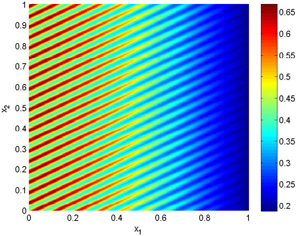
$$a^\varepsilon(x,s) = \mathrm{diag}\Big((2+\sin(2\pi x_1/\varepsilon))(1+x_1\sin(\pi s)),\ (2+\sin(2\pi x_2/\varepsilon))(2+\arctan(s)\Big).$$

$$\|u_0 - u^H\|_{H^1(\Omega)} \leq C(H + (h/\varepsilon)^2) \quad \|u_0 - u^H\|_{L^2(\Omega)} \leq C(H^2 + (h/\varepsilon)^2).$$

Numerical example: Richards equation (stationary state)

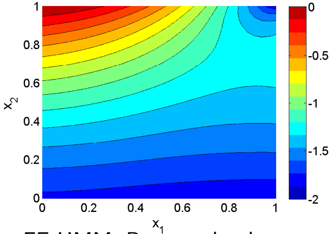
Model for water infiltration in unsaturated porous media.

Porous media

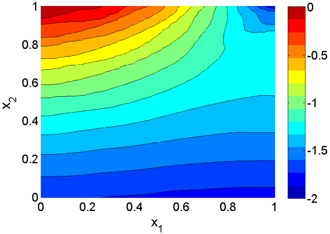


Permeability tensor (exponential model)

$$a^{\varepsilon}(s) = \alpha^{\varepsilon}(x)e^{\alpha^{\varepsilon}(x)s}$$



FE-HMM. Pressure level curves.
(macro and micro mesh sizes 32×32).



FEM. mesh size 32×32 (unresolved).

Post-processing procedure

Question: How can we reconstruct the oscillatory solution u_ε from the homogenized solution u_0 ?

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- For linear problems $\nabla \cdot (a(x, x/\varepsilon)\nabla u_\varepsilon) = f$, it is well known that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}, \quad \text{(provided smoothness assumptions)}$$

but $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$. For a strong H^1 convergence we need a corrector:

$$\|u_\varepsilon - u_0 - u_1^\varepsilon\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}$$

where $u_1^\varepsilon(x) := \varepsilon \sum_{j=1}^d \chi^j(x, x/\varepsilon)\frac{\partial u^0(x)}{\partial x_j}$.

It is known that $u_\varepsilon \approx u_0 + u_1^\varepsilon$ can be approximated using the FE-HMM by extending periodically the micro problem solutions.

Post-processing procedure

Question: How can we reconstruct the oscillatory solution u_ε from the homogenized solution u_0 ?

- Theorem (Boccardo, Murat, 1981) Consider nonlinear problems

$$\nabla \cdot (a(x, x/\varepsilon, u_\varepsilon)\nabla u_\varepsilon) = f,$$

then, any corrector u_1^ε for the linear problem

$$\nabla \cdot (a(x, x/\varepsilon, u_0)\nabla \overline{u}_\varepsilon) = f$$

is also a corrector for the nonlinear problem:

$$u_\varepsilon - u_0 - u_1^\varepsilon \to 0 \text{ strongly in } L^1_{loc}(\Omega)^d.$$

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$\Rightarrow$ For the considered class of nonlinear problems, we can apply the standard FE-HMM post-processing procedure to the linear problem

$$\nabla \cdot (a(x, x/\varepsilon, u^H)\nabla \widetilde{u}_\varepsilon) = f.$$

This yields an approximation of u_ε in $H^1(\Omega)$.

Summary

We studied FEMs with numerial quadrature for nonmonotone nonlinear elliptic problems.

One-scale problems:

- Optimal a priori H^1 and L^2 estimates on FEM.
- Newton method convergence and uniqueness of FEM solution (for a sufficiently fine mesh).

Homogenization problems (two-scales):

- Optimal fully discrete error analysis (H^1 and L^2 norms) where both the macro and micro errors are take into account (mesh sizes H, h have to be refined simultaneously).
- Newton method convergence and uniqueness of FEM solution (for sufficiently fine macro and micro meshes).