Coarse grid correction for domain decomposition methods for problems with high heterogeneities.

Victorita Dolean
dolean@unice.fr, www-math.unice.fr/~dolean
Laboratoire J.-A. Dieudonné, CNRS UMR 6621
Université de Nice-Sophia Antipolis, France

Joint work with F. Nataf (Universite Pierre et Marie Curie) and H. Xiang (Wuhan University).

Outline of the talk

1. Goal
2. Coarse Grid correction for smooth problems
3. Coarse Space for heterogeneous problems
4. Conclusion and perspectives

Goal

Consider the Darcy equation

$$-div(\kappa \nabla u) = f$$

discretized by a finite volume or a finite element method. We design a domain decomposition method for solving the corresponding linear system

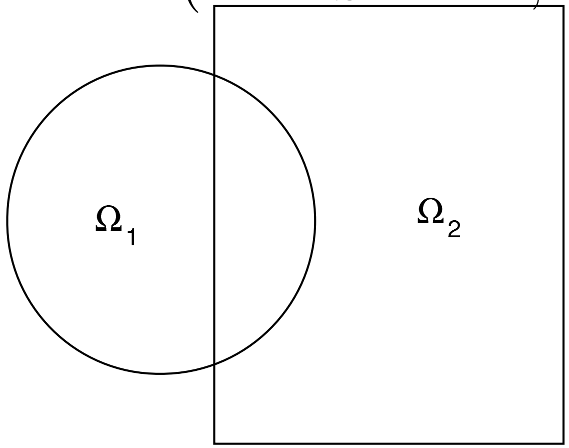
$$Ax = b\,.$$

For this problem, AMG works fine except for MPF (multi point flux volume schemes).
Our goal is to design design a two-level Schwarz domain decomposition method that is as algebraic as possible so that it could be applied to systems of PDEs with discontinuous coefficients. For such problems, AMG methods are not used.

Key issue: How to handle "along the interface" discontinuities in a simple and parallel manner?

Schwarz method – Two subdomain case

The original Schwarz Method (H.A. Schwarz, 1870)



$$-\Delta(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$\begin{aligned} -\Delta(u_1^{n+1}) &= f \quad \text{in } \Omega_1 \\ u_1^{n+1} &= 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \\ u_1^{n+1} &= u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}. \end{aligned} \qquad \begin{aligned} -\Delta(u_2^{n+1}) &= f \quad \text{in } \Omega_2 \\ u_2^{n+1} &= 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\ u_2^{n+1} &= u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}. \end{aligned}$$

Parallel algorithm, converges

Preconditioner for Krylov type methods

More subdomains

The method generalizes to an arbitrary number of subdomains, it is a matter of notations. But, performance may deteriorate with large number of subdomains. Plateaus appear in the convergence of the Krylov methods.

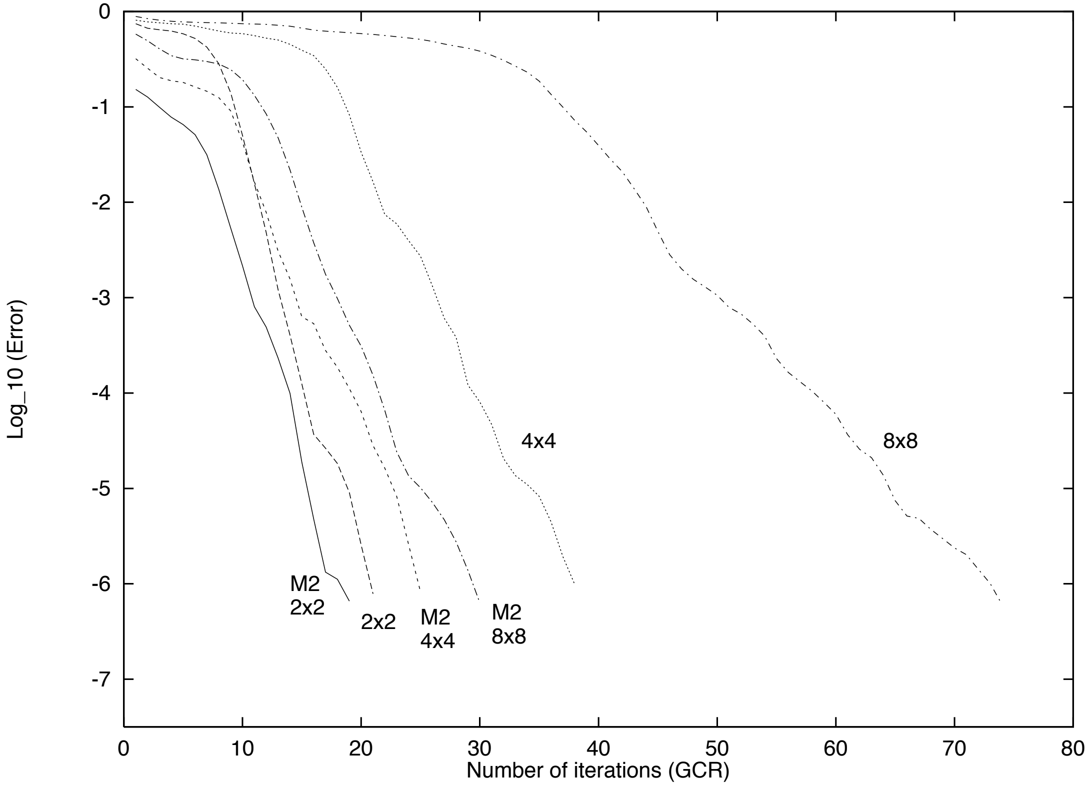


Figure 1: Japhet, Nataf, Roux (1998)

More subdomains

Iteration counts for a Poisson problem on a domain decomposed into strips.
The number of unknowns is proportional to the number of subdomains (scalability).

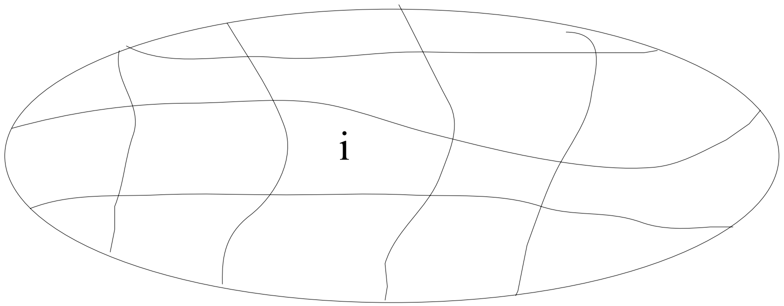
N subdomains	Schwarz	With coarse grid
4	18	25
8	37	22
16	54	24
32	84	25
64	144	25

Coarse grid correction for smooth problems

Stagnation corresponds to a few very low eigenvalues in the spectrum of the preconditioned problem. They are due to the lack of a global exchange of information in the preconditioner.

$$-\Delta u = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$



The mean value of the solution in domain i depends on the value of f on all subdomains.

A classical remedy consists in the introduction of a coarse grid problem that couples all subdomains. This is closely related to deflation technique classical in linear algebra (see Nabben and Vuik's papers in SIAM J. Sci. Comp, 200X).

Deflation and Coarse grid correction

Let A be a SPD matrix, we want to solve

$$Ax = b$$

with a preconditioner M (for example the Schwarz method).
Let Z be a rectangular matrix so that the "bad eigenvectors" belong to the space spanned by its columns. Define

$$P := I - AQ, \quad Q := ZE^{-1}Z^T, \quad E := Z^T AZ\,,$$

Examples of coarse grid preconditioners

$\mathcal{P}_{A-DEF2} := P^T M^{-1} + Q\,, \quad \mathcal{P}_{BNN} := P^T M^{-1} P + Q$ (Mandel, 1993)

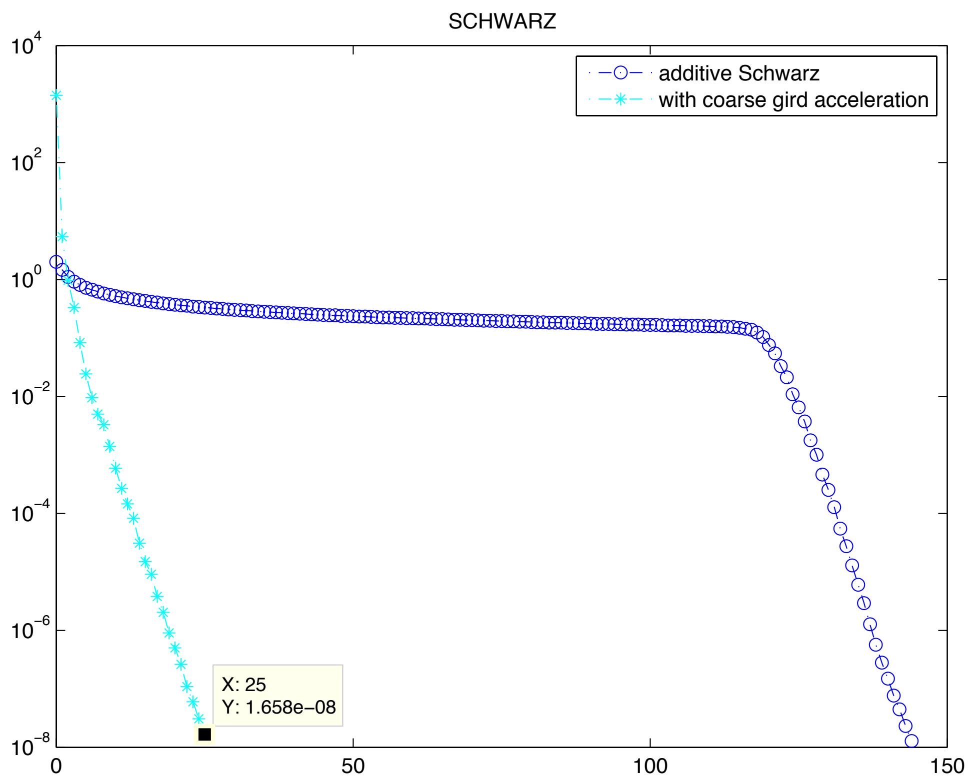
Some properties: $QAZ = Z$, $P^T Z = 0$ and $P^T Q = 0$.
Let r_n be the residual at step n of the algorithm: $Z^T r_n = 0$.

How to choose Z?

Coarse grid correction for smooth problems

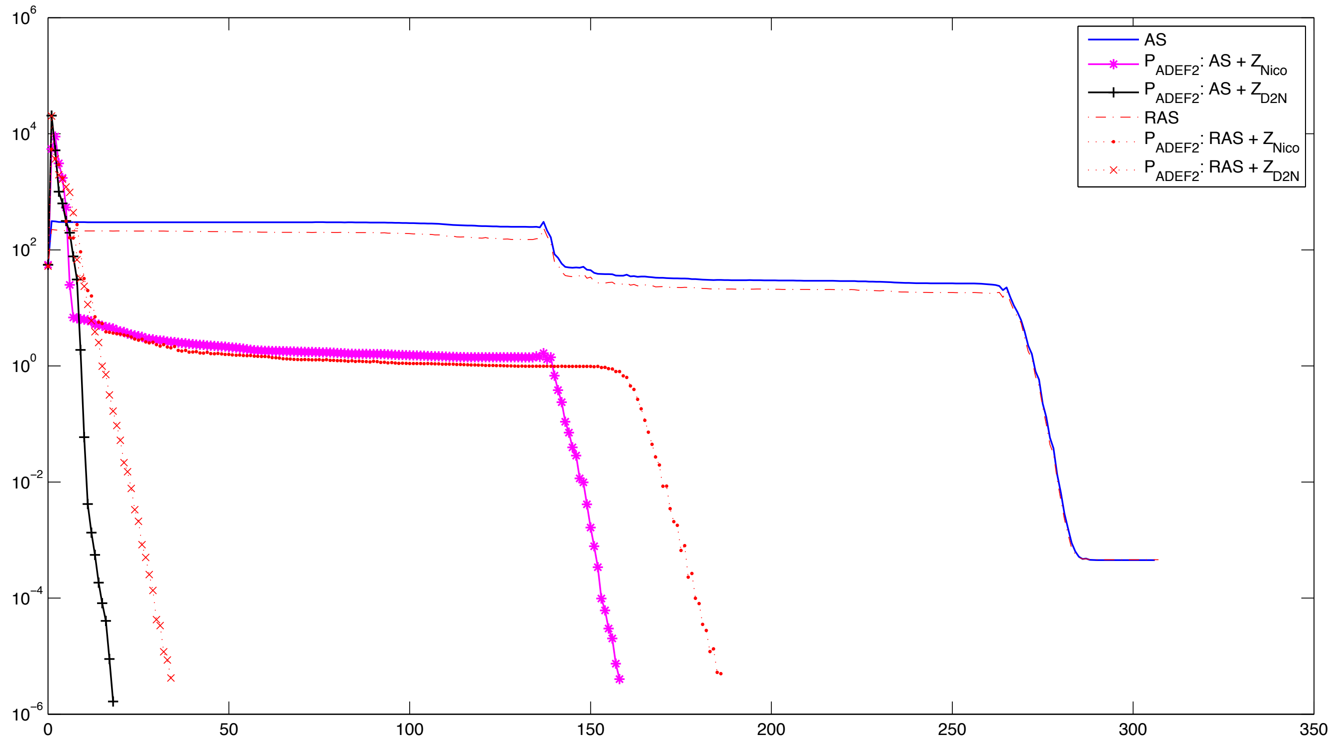
For a Poisson like problem, Nicolaides (1987) :

$$Z = \begin{bmatrix} 1_{\Omega_1} & 0 & \cdots & 0 \\ \vdots & 1_{\Omega_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1_{\Omega_d} \end{bmatrix}$$



Coarse Space for heterogeneous problems

Two layers with high heterogeneities, stripwise decomposition into 64 subdomains



Z_{Nico} (middle curve) does not work well, while taking two modes per subdomain Z_{D2N} (left curves) gives a fast convergence.

Goal

Design a Schwarz domain decomposition method for the Darcy equation

$$-div(\kappa\nabla u) = f$$

that is robust with respect to the permeability field κ and a many domains decomposition: either manually or via Metis.

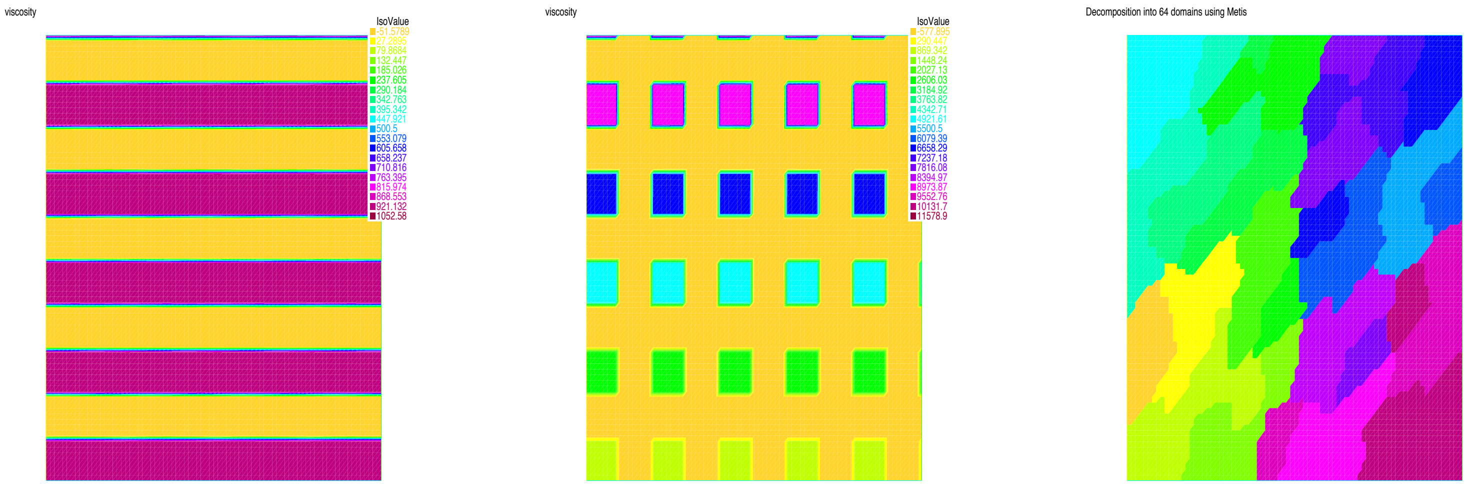


Figure 2: Permeability for multilayer and skyscraper cases – Metis

How to handle “along the interface” discontinuities in a simple and parallel manner?

Current alternatives

Avoid having discontinuities along the interface, discontinuities across the interface are "authorized".
Then there are efficient domain decomposition preconditioners: e.g. Dryja, Sarkis and Widlund (1996), J. Mandel and M. Brezina (1996), see Toselli and Widlund's book for further references.
But this puts a constraint on the partitioner and such partitions are not always feasible.

Sophisticated coarse space : Van lent, Scheichl and Graham (2009)

Here, our approach is different in that the construction is purely local (parallel) and adaptive and works for discontinuities along the interface.

Coarse Space for heterogeneous problems

In order to motivate the choice of the coarse space, consider the original Schwarz method for a domain Ω decomposed in a one-way partitioning. The errors e_i^n are harmonic in the subdomains and satisfy a Schwarz algorithm:

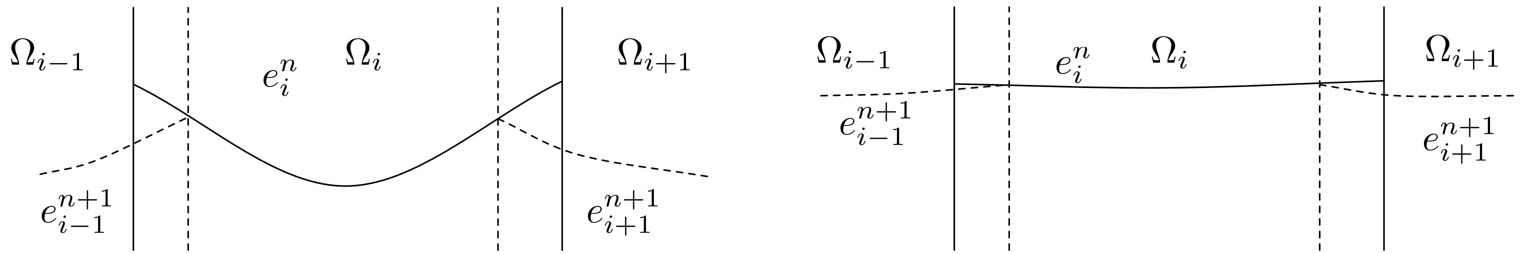


Figure 3: Fast or slow convergence of the Schwarz algorithm.

Coarse Space for heterogeneous problems

Let us consider at the continuous level the Dirichlet to Neumann map DtN_{Ω_i}. Let $g : \partial\Omega_i \mapsto \mathbb{R}$,

$$\mathrm{DtN}_{\Omega_i}(g) = \left.\frac{\partial v}{\partial n_i}\right|_{\Gamma_i},$$

where v satisfies

$$\begin{cases} \mathcal{L}(v) := (\eta - \mathrm{div}(k\nabla))v = 0, & \text{in } \ \Omega_i, \\ v = g, & \text{on } \partial\Omega_i, \end{cases} \tag{1}$$

To construct the coarse space, we use the low frequency modes associated with the DtN operator:

$$\mathrm{DtN}_{\Omega_i}(v_i^\lambda) = \lambda\, v_i^\lambda$$

with λ small. The functions v_i^λ are extended harmonically to the subdomains.

Coarse Space for heterogeneous problems

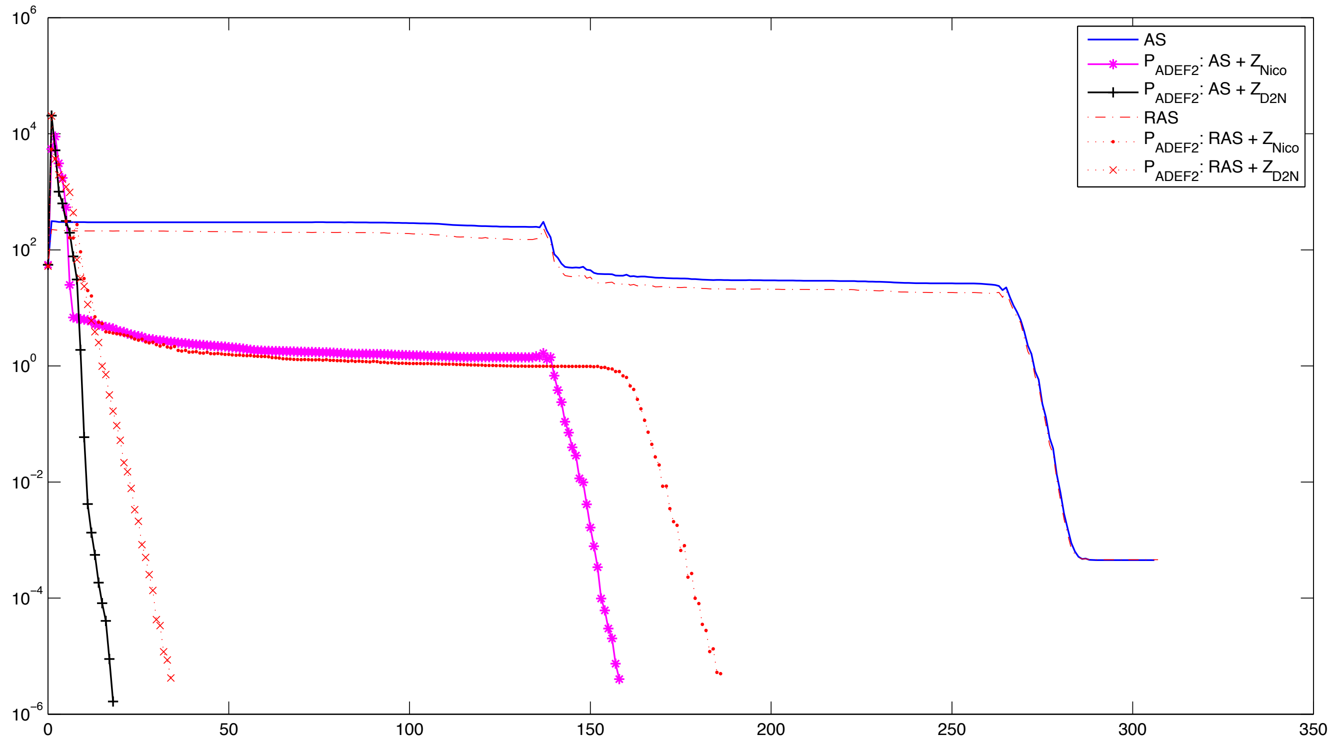
We still choose Z of the form

$$Z = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ \vdots & w_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & w_d \end{bmatrix}, \tag{2}$$

where d is the number of subdomains. The columns of w_i are the harmonic extensions of the eigenvectors associated with the smallest eigenvalues of the DtN map in subdomain Ω_i.
For a Poisson problem, the coarse space is made of piecewise constants as proposed by Nicolaides. It adapts automatically to heterogeneities.

Numerical Results

Two layers with high heterogeneities, stripwise decomposition into 64 subdomains



Z_{Nico} (middle curve) does not work well, while taking two modes per subdomain Z_{D2N} (left curves) gives a fast convergence.

Numerical results

Robustness w.r.t. the heterogeneities. Two layers with jumps in the coefficients ranging from 1 to 10^6. We have 32 subdomains.

Jump	1	10^3	10^5	10^6
Iteration counts	10	11	14	18

The iteration counts depend weakly on the size of the jump in the coefficients.

Numerical results

	uniform $N \times N$ decomposition			$N \times N$ decomp. using Metis		
N^2	RAS	RAS+Z_{Nico}	RAS+Z_{D2N}	RAS	RAS+Z_{Nico}	RAS+Z_{D2N}
4	30	29	11	46	41	19
16	80	58	26	116	84	25
64	180	61	31	220	110	31

Table 1: Layered case, N_{smeig}=4.

Numerical results

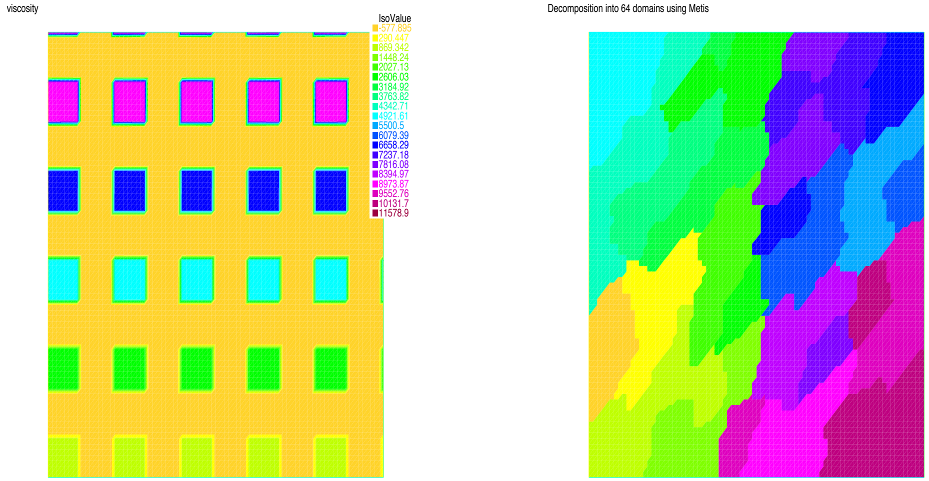


Figure 4: Permeability for the skyscraper case – Metis decomposition

	uniform $N \times N$ decomposition			$N \times N$ decomp. using Metis		
N^2	RAS	RAS+Z_{Nico}	RAS+Z_{D2N}	RAS	RAS+Z_{Nico}	RAS+Z_{D2N}
4	112	97	24	229	203	29
16	>400	299	18	>400	>400	32
64	>400	294	18	>400	>400	23

Numerical results

When the number of subdomains increases, the physical size of the overlap decreases. Thus, the iteration counts deteriorates. One way to keep the iteration counts fixed is to increase the size of the coarse space by taking more “bad” eigenvectors per subdomain.

	uniform $N \times N$ decomposition		$N \times N$ decomp. using Metis	
N^2	RAS+Z_{Nico}	RAS+$Z_{D2N}(N_{smeig})$	RAS+Z_{Nico}	RAS+$Z_{D2N}(N_{smeig})$
4	97	12(6)	203	12(6)
16	299	11(8)	>400	12(8)
64	294	13(10)	>400	14(10)

Table 2: The skyscraper, varying N_{smeig}.

Conclusion

- Robust, parallel and adaptive construction of a coarse space for problems with heterogeneities and jagged interfaces
- The method is as algebraic as possible which paves the way to extension to systems of equations e.g. multiphase flows, elasticity problems ...

Thanks !