Latency hiding of global reductions in pipelined Krylov methods

Wim Vanroose\textsuperscript{1}, Pieter Ghysels\textsuperscript{2} & Bram Reps\textsuperscript{1}

wim.vanroose@uantwerp.be pghysels@lbl.gov bram.reps@uantwerp.be

\textsuperscript{1} University of Antwerp - Dept Math & Computer Science, Belgium
\textsuperscript{2} LBNL - Future Technologies Group, Berkeley, CA, USA

CANUM 2014
March 31 - April 4, 2014
Introduction

What are we working on?

Figure: Latency hiding of global *drying* in pipelined *Laundry* methods
Introduction

What are we working on?

Figure: Latency hiding of global *drying* in pipelined *Laundry* methods
Introduction

What EXA2CT-ly are we working on?

Increasing gap between computation and communication costs

- Floating point performance steadily increases
- Network latencies only go down marginally
- Memory latencies decline slowly
- Avoid communication by trading communication for computation
- Hide latency of communications

EXascale Algorithms and Advanced Computational Techniques

https://projects.imec.be/exa2ct/
Latency hiding of global reductions in pipelined Krylov methods

Outline of the talk

Krylov subspace methods
(cf. Laundry methods)

Hiding global reductions
(cf. hiding drying time)

Increasing arithmetic intensity
(cf. piling up laundry)

Conclusions & future work
(cf. washing instructions and ecological detergents)
Latency hiding of global reductions in pipelined Krylov methods

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(cf. Laundry methods)

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Increasing arithmetic intensity
(cf. piling up laundry)

Conclusions & future work
(cf. washing instructions and ecological detergents)
Iteratively improve an approximate solution of linear system $Ax = b$,

$$x_i \in x_0 + \mathcal{K}_i(A, r_0) = x_0 + \text{span}\{r_0, Ar_0, A^2r_0, \ldots, A^{i-1}r_0\}$$

- minimize an error measure over expanding Krylov subspace $\mathcal{K}_i(A, r_0)$
- usually in combination with sparse linear algebra
- three building blocks
  i. axpy
  ii. SpMVM
  iii. dot-product

E.g.: Conjugate Gradients

1: $r^{(0)} \leftarrow b - Ax^{(0)}$
2: $p^{(0)} \leftarrow r^{(0)}$
3: for $i = 0, \ldots$ do
4: $w \leftarrow Ap^{(i)}$
5: $\alpha_i \leftarrow (r^{(i)}, r^{(i)}) / (w, p^{(i)})$
6: $x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)}$
7: $r^{(i+1)} \leftarrow r^{(i)} - \alpha_i w$
8: $\beta_i \leftarrow (r^{(i+1)}, r^{(i+1)}) / (r^{(i)}, r^{(i)})$
9: $p^{(i+1)} \leftarrow r^{(i+1)} + \beta_i p^{(i)}$
10: end for
Iteratively improve an approximate solution of linear system $Ax = b$,

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3: for $i = 0, \ldots$ do
4: $w \leftarrow Ap^{(i)}$
5: $\alpha_i \leftarrow (r^{(i)}, r^{(i)})/(w, p^{(i)})$
6: $x^{(i+1)} \leftarrow x^{(i)} + \alpha_ip^{(i)}$
7: $r^{(i+1)} \leftarrow r^{(i)} - \alpha_iw$
8: $\beta_i \leftarrow (r^{(i+1)}, r^{(i+1)})/(r^{(i)}, r^{(i)})$
9: $p^{(i+1)} \leftarrow r^{(i+1)} + \beta_ip^{(i)}$
10: end for
Krylov subspace methods
Communication patterns in the building blocks

i. axpy
   ▶ no dependencies on other vector elements
      (no communication)
   ▶ scales well

ii. SpMVM
   ▶ dependencies given by matrix/vector
      partition (one-to-one communication)
   ▶ bandwidth limited
   ▶ scales

iii. dot-product
   ▶ dependency on all vector elements (global reduction)
   ▶ latency dominated
   ▶ scales as $\log_2(\#\text{partitions})$

E.g.: Conjugate Gradients

1: $r^{(0)} \leftarrow b - Ax^{(0)}$
2: $p^{(0)} \leftarrow r^{(0)}$
3: for $i = 0, \ldots$ do
4: \hspace{1em} $w \leftarrow Ap^{(i)}$
5: \hspace{1em} $\alpha_i \leftarrow (r^{(i)}, r^{(i)})/(w, p^{(i)})$
6: \hspace{1em} $x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)}$
7: \hspace{1em} $r^{(i+1)} \leftarrow r^{(i)} - \alpha_i w$
8: \hspace{1em} $\beta_i \leftarrow (r^{(i+1)}, r^{(i+1)})/(r^{(i)}, r^{(i)})$
9: \hspace{1em} $p^{(i+1)} \leftarrow r^{(i+1)} + \beta_i p^{(i)}$
10: end for
Hestenes and Stiefel (1952)

```
1: \( r^{(0)} \leftarrow b - Ax^{(0)} \)
2: \( p^{(0)} \leftarrow r^{(0)} \)
3: for \( i = 0, \ldots \) do
4: \( w \leftarrow Ap^{(i)} \)
5: \( \alpha_i \leftarrow (r^{(i)}, r^{(i)})/(w, p^{(i)}) \)
6: \( x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)} \)
7: \( r^{(i+1)} \leftarrow r^{(i)} - \alpha_i w \)
8: \( \beta_i \leftarrow (r^{(i+1)}, r^{(i+1)})/(r^{(i)}, r^{(i)}) \)
9: \( p^{(i+1)} \leftarrow r^{(i+1)} + \beta_i p^{(i)} \)
10: end for
```
Krylov subspace methods
Case study: Conjugate Gradients

Chronopoulos and Gear (1989)

1: \( r(0) \leftarrow b - Ax^{(0)} \)
2: \( \ldots \) (loop-unrolling)
3: \( \text{for } i = 1, \ldots \text{ do} \)
4: \( p^{(i)} \leftarrow r^{(i)} + \beta_i p^{(i-1)} \)
5: \( s^{(i)} \leftarrow w^{(i)} + \beta_i s^{(i-1)} \)
6: \( x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)} \)
7: \( r^{(i+1)} \leftarrow r^{(i)} - \alpha_i s^{(i)} \)
8: \( w^{(i+1)} \leftarrow Ar^{(i+1)} \)
9: \( \gamma_{i+1} \leftarrow (r^{(i+1)}, r^{(i+1)}) \)
10: \( \delta \leftarrow (w^{(i+1)}, r^{(i+1)}) \)
11: \( \beta_{i+1} \leftarrow \gamma_{i+1} / \gamma_i \)
12: \( \alpha_{i+1} \leftarrow \gamma_{i+1} / (\delta - \beta_{i+1} \gamma_{i+1} / \alpha_i) \)
13: end for
Chronopoulos and Gear (1989)

- Equivalent to CG (in infinite precision)
- Extra recurrence relation for $s^{(i)} = Ap^{(i)}$
- Two dot-products are grouped in one global reduction
- Communication avoiding

1: $r^{(0)} \leftarrow b - Ax^{(0)}$
2: ... (loop-unrolling)
3: for $i = 1, \ldots$ do
4: $p^{(i)} \leftarrow r^{(i)} + \beta_i p^{(i-1)}$
5: $s^{(i)} \leftarrow w^{(i)} + \beta_i s^{(i-1)}$
6: $x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)}$
7: $r^{(i+1)} \leftarrow r^{(i)} - \alpha_i s^{(i)}$
8: $w^{(i+1)} \leftarrow Ar^{(i+1)}$
9: $\gamma_{i+1} \leftarrow (r^{(i+1)}, r^{(i+1)})$
10: $\delta \leftarrow (w^{(i+1)}, r^{(i+1)})$
11: $\beta_{i+1} \leftarrow \gamma_{i+1} / \gamma_i$
12: $\alpha_{i+1} \leftarrow \gamma_{i+1} / (\delta - \beta_{i+1} \gamma_{i+1} / \alpha_i)$
13: end for
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(cf. hiding drying time)

Increasing arithmetic intensity
(cf. piling up laundry)

Conclusions & future work
(cf. washing instructions and ecological detergents)
Hiding global reductions

Objective

- Dot-products are latency dominated
- Dot-products block all other (local) work
- Other (local) operations (SpMVM/axpy) scale well

Objective

Rewrite Krylov solvers such that latency of *dot-products* (global reductions) can be overlapped with application of the SpMVM and/or the preconditioner.

- Use non-blocking asynchronous global communication
- MPI-3 standard introduces MPI_Iallreduce()
- GPI-2 introduces gaspi_allreduce() + uses PGAS (partitioned global address space)
Hiding global reductions
Pipelined Conjugate Gradients

Ghysels and Vanroose (2013)

► Equivalent to CG (in infinite precision)
► Extra recurrence relations for \( s^{(i)} = Ap^{(i)} \) and \( z = As^{(i)} \)
► Two dot-products are grouped in one global reduction
► Communication avoiding
► Overlap global communication with local computations: line 4 + 5 + 6
► Communication avoiding + communication hiding

1: \( r(0) \leftarrow b - Ax^{(0)} \)
2: \( \ldots \) (loop-unrolling)
3: for \( i = 1, \ldots \) do
4: \( \gamma_i \leftarrow (r^{(i)}, r^{(i)}) \)
5: \( \delta \leftarrow (w^{(i)}, r^{(i)}) \)
6: \( q^{(i)} \leftarrow Aw^{(i)} \)
7: \( \beta_i \leftarrow \gamma_i / \gamma_{i-1} \)
8: \( \alpha_i \leftarrow \gamma_i / (\delta - \beta_i \gamma_i / \alpha_{i-1}) \)
9: \( z^{(i)} \leftarrow q^{(i)} + \beta_i z^{(i-1)} \)
10: \( s^{(i)} \leftarrow w^{(i)} + \beta_i s^{(i-1)} \)
11: \( p^{(i)} \leftarrow r^{(i)} + \beta_i p^{(i-1)} \)
12: \( x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)} \)
13: \( r^{(i+1)} \leftarrow r^{(i)} - \alpha_i s^{(i)} \)
14: \( w^{(i+1)} \leftarrow w^{(i)} - \alpha_i z^{(i)} \)
15: end for
Hiding global reductions

Ghysels and Vanroose (2013)

1: \( r(0) \leftarrow b - Ax(0) \)
2: \( \ldots \) (loop-unrolling)
3: \begin{align*}
   & \text{for } i = 1, \ldots \text{ do} \\
   & 4: \quad \gamma_i \leftarrow (r^{(i)}, r^{(i)}) \\
   & 5: \quad \delta \leftarrow (w^{(i)}, r^{(i)}) \\
   & 6: \quad q^{(i)} \leftarrow Aw^{(i)} \\
   & 7: \quad \beta_i \leftarrow \gamma_i / \gamma_{i-1} \\
   & 8: \quad \alpha_i \leftarrow \gamma_i / (\delta - \beta_i \gamma_i / \alpha_{i-1}) \\
   & 9: \quad z^{(i)} \leftarrow q^{(i)} + \beta_i z^{(i-1)} \\
   & 10: \quad s^{(i)} \leftarrow w^{(i)} + \beta_i s^{(i-1)} \\
   & 11: \quad p^{(i)} \leftarrow r^{(i)} + \beta_i p^{(i-1)} \\
   & 12: \quad x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)} \\
   & 13: \quad r^{(i+1)} \leftarrow r^{(i)} - \alpha_i s^{(i)} \\
   & 14: \quad w^{(i+1)} \leftarrow w^{(i)} - \alpha_i z^{(i)} \\
   & 15: \text{end for}\end{align*}
Hiding global reductions
Preconditioned pipelined Conjugate Gradients

Ghysels and Vanroose (2013)

- Equivalent to CG (in infinite precision)
- Extra recurrence relations for $w^{(i)} = Au^{(i)}$, $s^{(i)} = Ap^{(i)}$ and $z = Aq^{(i)}$
- Two dot-products are grouped in one global reduction
- Overlap global communication with extra local computations: line 4 + 5 + 7 + 6
- Communication avoiding + communication hiding

1: $r^{(0)} \leftarrow b - Ax^{(0)}$
2: \( \ldots \) (loop-unrolling)
3: \textbf{for} $i = 1, \ldots$ \textbf{do}
4: \quad $\gamma_i \leftarrow (r^{(i)}, u^{(i)})$
5: \quad $\delta \leftarrow (w^{(i)}, u^{(i)})$
6: \quad $m^{(i)} \leftarrow M^{-1}w^{(i)}$
7: \quad $n^{(i)} \leftarrow Am^{(i)}$
8: \quad $\beta_i \leftarrow \gamma_i / \gamma_{i-1}$
9: \quad $\alpha_i \leftarrow \gamma_i / (\delta - \beta_i \gamma_i / \alpha_{i-1})$
10: \quad $z^{(i)} \leftarrow n^{(i)} + \beta_i z^{(i-1)}$
11: \quad $q^{(i)} \leftarrow m^{(i)} + \beta_i q^{(i-1)}$
12: \quad $s^{(i)} \leftarrow w^{(i)} + \beta_i s^{(i-1)}$
13: \quad $p^{(i)} \leftarrow u^{(i)} + \beta_i p^{(i-1)}$
14: \quad $x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)}$
15: \quad $r^{(i+1)} \leftarrow r^{(i)} - \alpha_is^{(i)}$
16: \quad $u^{(i+1)} \leftarrow u^{(i)} - \alpha_i q^{(i)}$
17: \quad $w^{(i+1)} \leftarrow w^{(i)} - \alpha_i z^{(i)}$
18: \textbf{end for}
Hiding global reductions

Preconditioned pipelined Conjugate Gradients

Ghysels and Vanroose (2013)

1: \( r^{(0)} \leftarrow b - Ax^{(0)} \)
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9: \( \alpha_i \leftarrow \gamma_i/(\delta - \beta_i\gamma_i/\alpha_{i-1}) \)
10: \( z^{(i)} \leftarrow n^{(i)} + \beta_i z^{(i-1)} \)
11: \( q^{(i)} \leftarrow m^{(i)} + \beta_i q^{(i-1)} \)
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16: \( u^{(i+1)} \leftarrow u^{(i)} - \alpha_i q^{(i)} \)
17: \( w^{(i+1)} \leftarrow w^{(i)} - \alpha_i z^{(i)} \)
18: \( \text{end for} \)
Hiding global reductions

Preconditioned pipelined Conjugate Residuals

Ghysels and Vanroose (2013)

- Equivalent to CR (in infinite precision)
- Based on $(\cdot, \cdot)_A$-inner product
- Two dot-products are grouped in one global reduction
- Overlap global communication with local computations: line 5 + 6 + 7
- No overlap with preconditioner
- Only 3 additional axpy’s save memory

```plaintext
1: \( r^{(0)} \leftarrow b - Ax^{(0)} \)
2: \ldots \) (loop-unrolling)
3: \textbf{for} \( i = 1, \ldots \) \textbf{do}
4: \( m^{(i)} \leftarrow M^{-1}w^{(i)} \)
5: \( \gamma_i \leftarrow (w^{(i)}, u^{(i)}) \)
6: \( \delta \leftarrow (m^{(i)}, w^{(i)}) \)
7: \( n^{(i)} \leftarrow Am^{(i)} \)
8: \( \beta_i \leftarrow \gamma_i / \gamma_{i-1} \)
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11: \( q^{(i)} \leftarrow m^{(i)} + \beta_i q^{(i-1)} \)
12: \( p^{(i)} \leftarrow u^{(i)} + \beta_i p^{(i-1)} \)
13: \( x^{(i+1)} \leftarrow x^{(i)} + \alpha_i p^{(i)} \)
14: \( u^{(i+1)} \leftarrow u^{(i)} - \alpha_i q^{(i)} \)
15: \( w^{(i+1)} \leftarrow w^{(i)} - \alpha_i z^{(i)} \)
16: \textbf{end for}
```
## Hiding global reductions

### Comparison of CG variants

<table>
<thead>
<tr>
<th></th>
<th>flops</th>
<th>time (excl axpy’s, dot’s)</th>
<th># syncs</th>
<th>mem</th>
</tr>
</thead>
<tbody>
<tr>
<td>CG</td>
<td>10</td>
<td>$2G + \text{SpMVM} + \text{PC}$</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Chron/Gear-CG</td>
<td>12</td>
<td>$G + \text{SpMVM} + \text{PC}$</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Gropp-CG</td>
<td>14</td>
<td>$\max(G,\text{SpMVM}) + \max(G,\text{PC})$</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>pipe-CG</td>
<td>20</td>
<td>$\max(G,\text{SpMVM}+\text{PC})$</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>CR</td>
<td>12</td>
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<td>2</td>
<td>5</td>
</tr>
<tr>
<td>pipe-CR</td>
<td>16</td>
<td>$\max(G,\text{SpMVM}) + \text{PC}$</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

- **G**: latency of global reduction
- **SpMVM**: sparse matrix-vector time
- **PC**: application of preconditioner
Hiding global reductions
Strong scaling experiment

- Hydrostatic ice sheet flow, $100 \times 100 \times 50$ Q1 finite elements
- Line search Newton method ($\text{rtol}=10^{-8}, \text{atol}=10^{-15}$)
- CG preconditioned with block Jacobi with ICC(0) ($\text{rtol}=10^{-5}, \text{atol}=10^{-50}$)

- max pipe-CG/CG speedup: $2.14 \times$
- max pipe-CG/CG1 speedup: $1.43 \times$
- max pipe-CR/CR speedup: $2.09 \times$

(CG1 = Chrono/Gear CG)
Hiding global reductions
Other pipelined Krylov methods

- Preconditioned pipelined GMRES
  Ghysels, Ashby, Meerbergen and Vanroose (2012)

\[ V_{i-\ell+1} = [v_0, v_1, \ldots, v_{i-\ell}] \]
\[ Z_{i+1} = [z_0, z_1, \ldots, z_{i-\ell}, z_{i-\ell+1}, \ldots, z_i] \]

- Compute \( \ell \) new basis vectors for Krylov subspace (SpMVMs) during global communication (dot-products).
- Orthogonalization step when previous global reduction has finished
- More technical, but deeper and variable pipelining possible \( p(\ell) \)-GMRES

- Augmented and deflated Krylov subspace methods
Latency hiding of global reductions in pipelined Krylov methods

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(cf. hiding drying time)

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(cf. piling up laundry)

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(cf. washing instructions and ecological detergents)
Hiding global reductions

Roofline Model

- Arithmetic intensity: \( q = \frac{\text{floating-point operations}}{\text{byte off-chip memory traffic}} \)
- High \( q \) → compute bound (dense algebra, fft, ...)
- Low \( q \) → bandwidth bound (sparse algebra, stencils, ...)
- Roofline gives upperbound for performance for given \( q \)

Williams, Waterman, Patterson (2008)
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Williams, Waterman, Patterson (2008)
Hiding global reductions

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- Roofline gives upperbound for performance for given \( q \)

\[
\begin{array}{c|c|c|c}
\text{Attainable GFlop/s (DP)} & \text{Arithmetic intensity (flop/byte)} & \text{peak GFlop/s} & \text{peak BW no SIMD} \\
\hline
1 & 10 & 100 & 1000 \\
\hline
10 & 1000 & 10000 & 10000 \\
\hline
100 & 10000 & 100000 & 100000 \\
\hline
1000 & 100000 & 1000000 & 1000000 \\
\hline
10000 & 1000000 & 10000000 & 10000000 \\
\hline
\end{array}
\]

Williams, Waterman, Patterson (2008)
Increasing arithmetic intensity

Arithmetic intensity of $s$ dependent SpMVMs

<table>
<thead>
<tr>
<th></th>
<th>1 SpMVM</th>
<th>$s \times$ SpMVM</th>
<th>$s \times$ SpMVM in place</th>
</tr>
</thead>
<tbody>
<tr>
<td>flops</td>
<td>$2n_{nz}$</td>
<td>$2s \cdot n_{nz}$</td>
<td>$2s \cdot n_{nz}$</td>
</tr>
<tr>
<td>words moved</td>
<td>$n_{nz} + 2n$</td>
<td>$sn_{nz} + 2sn$</td>
<td>$n_{nz} + 2n$</td>
</tr>
<tr>
<td>$q$</td>
<td>2</td>
<td>2</td>
<td><strong>2s</strong></td>
</tr>
</tbody>
</table>

See J. Demmel’s course: CS 294-76 on Communication-Avoiding algorithms
Increasing arithmetic intensity

\[ V(\nu_1, \nu_2) \text{-cycle multigrid} \]

\[ \text{while } \| r^h \| > \text{tol} \| f^h \| \text{ do} \]
  \[ V\text{-cycle}(v^h, f^h) \]
\[ \text{end while} \]

\[ V\text{-cycle}(v^h, f^h) \]

\[ \text{if Coarsest level then} \]
  \[ v^h \leftarrow (A^h)^{-1} f^h \]
\[ \text{else} \]
  \[ \text{for } k = 1, \ldots, \nu_1 \text{ do} \]
    \[ v^h \leftarrow (1 - \omega D^{-1} A^h) v^h + \omega D^{-1} f^h \]
  \[ \text{end for} \]
  \[ r^h \leftarrow f^h - A^h v^h \]
  \[ r^{2h} \leftarrow I^{2h}_h r^h \]
  \[ e^{2h} \leftarrow V\text{-cycle}^{2h}(0, r^{2h}) \]
  \[ e^h \leftarrow I^{2h}_h e^{2h} \]
  \[ v^h \leftarrow v^h + e^h \]
  \[ \text{for } k = 1, \ldots, \nu_2 \text{ do} \]
    \[ v^h \leftarrow (1 - \omega D^{-1} A^h) v^h + \omega D^{-1} f^h \]
  \[ \text{end for} \]
\[ \text{end if} \]

- \( I^{2h}_h \) Full weighting
- \( I^h_{2h} \) Linear interpolation
Increasing arithmetic intensity

Consecutive smoothing steps

- A smoother is an SpMVM kernel with dependent vectors where only the last vector is required
  - Possibility to increase arithmetic intensity
  - Tiling over different smoother iterations
  - \( q(\nu \times \omega-\text{Jac}) = \nu q_1(\omega-\text{Jac}) \)
- Divide the domain in tiles which fit in the cache
- Ground surface is loaded in cache and reused \( s (= \nu) \) times
- Redundant work at the tile edges
Increasing arithmetic intensity

Cost of $\nu$ smoothing steps

Since the arithmetic intensity increases for more smoothing steps

\[ q(\nu \times \omega-\text{Jac}) = \nu q_1(\omega-\text{Jac}) \]

going according to the roofline:

performance increases & the average cost decreases
Increasing arithmetic intensity

Work Unit Cost model

- Classical Work Unit cost model ignores memory bandwidth

\[ 1 \text{ WU} = \text{smoother cost} = \mathcal{O}(n) \]

- Cost of multigrid to reach tolerance

\[ = (9\nu + 19)(1 + \frac{1}{4} + \frac{1}{16} + \ldots) \left\lceil \frac{\log(\text{tol})}{\log(\rho(\nu))} \right\rceil \text{WU} \leq (9\nu + 19) \frac{4}{3} \left\lceil \frac{\log(\text{tol})}{\log(\rho(\nu))} \right\rceil \text{WU} \]

- Optimum for low \( \nu \) because computational cost increases with \( \nu \)

- \ldots but communication overhead decreases!

![Graphs showing MG iterations and Work Units vs. smoothing steps]
In contrast to naive model, the modified cost model suggests to repeat application of the smoother.

By tiling the smoother
  ▶ the optimal number of smoothing steps shifts to the right
  ▶ vectorization can be exploited
In contrast to naive model, the modified cost model suggests to repeat application of the smoother.

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- the optimal number of smoothing steps shifts to the right
- vectorization can be exploited
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Krylov subspace methods
- 3 building blocks: axpy, SpMVM, dot-product
- CG variants that group building blocks
- Reduce global reduction steps
- Communication avoiding

Hiding global reductions
- Pipelined CG and pipelined CR
- Preconditioned versions
- Overlap global reduction steps with other computational steps
- Communication hiding (+ communication avoiding)

Increasing arithmetic intensity
- Tiling of smoother improves data locality and scalability
- Trade-off between better convergence and increasing cost of smoother
- Optimal number of smoothing steps increases
- This allows exploiting of vector units
- Still to be combined with an improved interpolation and restriction
Conclusions & future work

References


Q: What’s the difference between pipelined and s-step Krylov methods?
   A: Global communication is hidden vs avoided
   A: Off-the-shelf preconditioning possible vs specialized preconditioning

Q: Is the code available online?
   A: Yes, pipe-CG, Gropp-CG, pipe-CR and p(ℓ)-GMRES are in the PETSc library