Dynamic Mode Decomposition

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- Input:

  Consider an ensemble of snapshots $\boldsymbol{v}_i,\ i = 1, \cdots, N$ such that

$$V_1^N = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \cdots, \boldsymbol{v}_N\} \in \mathbb{R}^{m \times N}$$

- Hypothesis #1:

  Assume a linear mapping A between $\boldsymbol{v}_i$ and $\boldsymbol{v}_{i+1}$

$$\boldsymbol{v}_{i+1} = A\boldsymbol{v}_i \quad \text{with} \quad A \in \mathbb{R}^{m \times m}$$

  i.e. V_1^N is Krylov matrix of dimension $m \times N$

$$V_1^N = \left\{\boldsymbol{v}_1, A\boldsymbol{v}_1, A^2\boldsymbol{v}_1, \cdots, A^{N-1}\boldsymbol{v}_1\right\}$$

- Objective:

  Determine a good approximation of the eigen-elements of A without knowing A!!

- Hypothesis #2:

  If N is sufficiently large, we can express $\boldsymbol{v}_N$ as a linear combination of the previous $\boldsymbol{v}_i$, $(i = 1, \cdots, N-1)$ *i.e.*

$$\begin{aligned} \boldsymbol{v}_N &= c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_{N-1}\boldsymbol{v}_{N-1} + \boldsymbol{r} \\ &= V_1^{N-1}\boldsymbol{c} + \boldsymbol{r} \end{aligned}$$

  where $\boldsymbol{r} \in \mathbb{R}^m$ and $\boldsymbol{c} = (c_1, c_1, \cdots, c_{N-1})^T \in \mathbb{R}^{N-1}$

- Ruhe (1984) proved that Proof on blackboard

$$\boxed{AV_1^{N-1} = V_1^{N-1}S + \boldsymbol{r}\boldsymbol{e}_{\boldsymbol{N-1}}^T} \qquad (1)$$

  where $\boldsymbol{e_i}$ is the ith Euclidean unitary vector of length $(N-1)$ and S a Companion matrix

$$S = \begin{pmatrix} 0 & 0 & \dots & 0 & c_1 \\ 1 & 0 & \dots & 0 & c_2 \\ 0 & 1 & \dots & 0 & c_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_{N-1} \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}$$

- We will now show that if we already know eigen-elements of S then we can determine easily approximated eigen-elements of A. Indeed, we can demonstrate that:

$$\text{if} \quad S\boldsymbol{y_i} = \mu_i \boldsymbol{y_i} \qquad \text{then} \qquad A\boldsymbol{z_i} \simeq \mu_i \boldsymbol{z_i} \quad \text{with} \quad \boldsymbol{z_i} = V_1^{N-1}\boldsymbol{y_i}$$

Proof:

$$\begin{aligned} A\boldsymbol{z_i} - \mu_i \boldsymbol{z_i} &= AV_1^{N-1}\boldsymbol{y_i} - \mu_i V_1^{N-1}\boldsymbol{y_i} \\ &= AV_1^{N-1}\boldsymbol{y_i} - V_1^{N-1} S\boldsymbol{y_i} \\ &= \left(AV_1^{N-1} - V_1^{N-1}S\right)\boldsymbol{y_i} = \boldsymbol{r}\boldsymbol{e}_{\boldsymbol{N-1}}^T\boldsymbol{y_i} \longrightarrow 0 \quad \text{if} \quad \|\boldsymbol{r}\| \longrightarrow 0 \end{aligned}$$

- Next step: determination of S *i.e.* $\boldsymbol{c}$
- We can show (Bau and Trefethen, 1997) that

$$\boxed{\boldsymbol{c} = R^{-1} Q^H \boldsymbol{v}_N \quad \text{where} \quad V_1^{N-1} = QR}$$

- Difficulty: this algorithm is ill-conditioned *i.e.* it leads rapidly to non meaningful dynamic modes.

- Results from Bau and Trefethen (1997)
  Consider the linear system $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$. Its least-mean square solution is given by the QR algorithm:
  1. QR factorization of A: A=QR
  2. Determine Q^H
  3. Solve the upper triangular system $Rx = Q^H b$ or $x = R^{-1} Q^H b$

DMD algorithm

$[\boldsymbol{Z}, \boldsymbol{\mu}, \text{Res}] = \text{DMD}\left(V_1^N\right)$

Input: N sequence of snapshots $V_1^N = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \cdots, \boldsymbol{v}_N\}$

Output: $(N-1)$ empirical Ritz vectors $\boldsymbol{Z}$ and Ritz values $\boldsymbol{\mu}$; Res: residual.

1: $m = \text{size}\left(V_1^N, 1\right)$

2: $N = \text{size}\left(V_1^N, 2\right)$

3: $\boldsymbol{v}_N = V_1^N(:, N)$

4: $V_1^{N-1} = V_1^N(:, 1 : N-1)$

5: $V_2^N = V_1^N(:, 1 : N-1)$

6: $\boldsymbol{c} = V_1^{N-1} / \boldsymbol{v}_N$

7: $S = \text{companion}\,(\boldsymbol{c})$

8: $[Y, \boldsymbol{\mu}] = \text{eig}(S)$

9: $Z = V_1^{N-1} Y$

10: $\text{Res} = \text{norm}\left(V_2^N - V_1^{N-1} S, 1\right)$

with $Z = (\boldsymbol{z_1}, \cdots, \boldsymbol{z_N})$ and $Y = (\boldsymbol{y_1}, \cdots, \boldsymbol{y_N})$.

- Apply the SVD

$$V_1^{N-1} = U\Sigma W^H \qquad \text{with} \qquad UU^H = WW^H = I$$

<u>Remarks</u>:

- U contains the spatial POD eigenfunctions and,
- W contains the temporal POD eigenfunctions

so, we can claim that POD is a by-product of DMD!

- Starting from $AV_1^{N-1} = V_1^{N-1}S + \boldsymbol{r}\boldsymbol{e}_{\boldsymbol{N-1}}^T$ and first considering that $\boldsymbol{r} = \boldsymbol{0}$, we obtain after some manipulations:

$$U^H AU = U^H V_2^N W\Sigma^{-1} = S$$

Since $\boldsymbol{r} \neq \boldsymbol{0}$, we have:

$$\boxed{U^H AU = U^H V_2^N W\Sigma^{-1} = \tilde{S}} \qquad \text{where} \qquad \tilde{S} \quad \text{is a full matrix.}$$

- We will now show that if we already know eigen-elements of $\tilde{S}$ then we can determine easily approximated eigen-elements of A. Indeed, we can demonstrate that:

$$\boxed{\text{if} \quad \tilde{S}\boldsymbol{y_i} = \mu_i \boldsymbol{y_i} \qquad \text{then} \qquad A\boldsymbol{\Phi_i} = \mu_i \boldsymbol{\Phi_i} \quad \text{with} \quad \boldsymbol{\Phi_i} = U\boldsymbol{y_i}}$$

Proof:

$$\begin{aligned} A\boldsymbol{\Phi_i} = \mu_i \boldsymbol{\Phi_i} \quad &\Rightarrow AU\boldsymbol{y_i} = \mu_i U\boldsymbol{y_i} \\ &\Rightarrow U^H AU\boldsymbol{y_i} = \mu_i U^H U\boldsymbol{y_i} = \mu_i \boldsymbol{y_i} \\ &\Rightarrow \tilde{S}\boldsymbol{y_i} = \mu_i \boldsymbol{y_i} \end{aligned}$$

DMD:

- generalization of a Rayleigh-Ritz procedure to the case where the subspace of projection is not orthogonal.
- Direct link with the Arnoldi algorithm classically used when A is known.
- Determination this week using XAMC.

One possible conclusion

"without an inexpensive method for reducing the cost of flow computations, it is unlikely that the solution of optimization problems involving the three dimensional unsteady Navier-Stokes system will become routine"

M. Gunzburger, 2000

Questions ???