Hybrid High-Order (HHO) methods on general meshes

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Paris, 19th June, 2015
Mimetic Finite Differences
- Extension to polyhedral meshes [Kuznetsov et al., 2004]
- Convergence analysis [Brezzi et al., 2005]

Mixed/Hybrid Finite Volumes
- Pure diffusion (mixed) [Droniou and Eymard, 2006]
- Pure diffusion (primal) [Eymard et al., 2010]
- Link with MFD [Droniou et al., 2010]

More recently
- Compatible Discrete Operators [Bonelle and Ern, 2014]
- Generalized Crouzeix–Raviart [DP and Lemaire, 2015]
Bibliography: High-order polyhedral methods

- **Discontinuous Galerkin**
  - General meshes [DP and Ern, 2012]
  - Adaptive coarsening [Bassi et al., 2012, Antonietti et al., 2013]

- **Hybridizable Discontinuous Galerkin**
  - Pure diffusion [Cockburn et al., 2009]

- **Virtual elements**
  - Pure diffusion [Beirão da Veiga et al., 2013a]
  - Nonconforming VEM [Ayuso de Dios et al., 2014]
  - Linear elasticity [Beirão da Veiga et al., 2013b]

- **Hybrid High-Order**
  - Pure diffusion [DP and Ern, 2014b]
  - Linear elasticity [DP and Ern, 2015]
  - Bridge between HHO and HDG [Cockburn, DP and Ern, 2015]
Features of HHO

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Reproduction of desirable continuum properties
  - Integration by parts formulas
  - Kernels of operators
  - Symmetries
- Reduced computational cost after hybridization

\[
N_{\text{dof}}^{\text{hho}} \approx \frac{1}{2} k^2 \text{card}(\mathcal{F}_h) \quad N_{\text{dof}}^{\text{dg}} \approx \frac{1}{6} k^3 \text{card}(\mathcal{T}_h)
\]
1. Poisson

2. Variable diffusion and local conservation

3. Linear elasticity
1. Poisson

2. Variable diffusion and local conservation

3. Linear elasticity
Definition (Mesh regularity)

We consider a sequence \((\mathcal{T}_h)_{h \in \mathcal{H}}\) of polyhedral meshes s.t., for all \(h \in \mathcal{H}\), \(\mathcal{T}_h\) admits a simplicial submesh \(\mathcal{S}_h\) and \((\mathcal{S}_h)_{h \in \mathcal{H}}\) is

- **shape-regular** in the sense of Ciarlet;
- **contact-regular**: every simplex \(S \subset T\) is s.t. \(h_S \approx h_T\).

Main consequences:

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces
Mesh regularity II

**Figure:** Admissible meshes in 2d and 3d: [Herbin and Hubert, 2008, FVCA5] and [Di Pietro and Lemaire, 2015] (above) and [Eymard et al., 2011, FVCA6] (below)
Let $\Omega$ denote a bounded, connected polyhedral domain.

For $f \in L^2(\Omega)$, we consider the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

In weak form: Find $u \in H^1_0(\Omega)$ s.t.

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega)$$
Key ideas

- **DOFs**: polynomials of degree $k \geq 0$ at elements and faces
- **Differential operators reconstructions** tailored to the problem:

$$a_T(u, v) \approx (\nabla p_T^{k+1} u_T, \nabla p_T^{k+1} v_T) + \text{stab.}$$

with

- high-order reconstruction $p_T^{k+1}$ from local Neumann solves
- stabilization via face-based penalty
- **Construction yielding superconvergence** on general meshes
For $k \geq 0$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$U^k_T := \mathbb{P}^k_d(T) \times \left\{ \prod_{F \in \mathcal{F}_T} \mathbb{P}^k_{d-1}(F) \right\}$$

The global space has single-valued interface DOFs

$$U^k_h := \left\{ \prod_{T \in \mathcal{T}_h} \mathbb{P}^k_d(T) \right\} \times \left\{ \prod_{F \in \mathcal{F}_h} \mathbb{P}^k_{d-1}(F) \right\}$$
Local potential reconstruction

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$p_{T}^{k+1} : U_{T}^k \rightarrow \mathbb{P}^k(T)$$

is s.t. $\forall v_T \in U_{T}^k$, $(p_{T}^{k+1}v_T, 1)_T = (v_T, 1)_T$ and $\forall w \in \mathbb{P}^k(T)$,

$$ (\nabla p_{T}^{k+1}v_T, \nabla w)_T := -(v_T, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (v_F, \nabla w \cdot n_{TF})_F $$

- To compute $p_{T}^{k+1}$, we solve a small SPD linear system of size

$N_{k,d} := \binom{k+1 + d}{k+1}$

- Perfectly suited to GPU computing!
Lemma (Approximation properties for $p_T^{k+1} I_T^k$)

Define the local reduction map $I_T^k : H^1(T) \to U_T^k$ s.t.

$$I_T^k : v \mapsto \left( \pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T} \right).$$

Then, for all $T \in \mathcal{T}_h$ and all $v \in H^{k+2}(T)$,

$$\| v - p_T^{k+1} I_T^k v \|_T + h_T \| \nabla (v - p_T^{k+1} I_T^k v) \|_T \lesssim h_T^{k+2} \| v \|_{k+2,T}.$$
Since \( \triangle w \in \mathbb{P}^{k-1}_{d}(T) \) and \( \nabla w|_{F} \cdot \mathbf{n}_{TF} \in \mathbb{P}^{k}_{d-1}(F) \),

\[
(\nabla p^{k+1}_{T} I^{k}_{T} v, \nabla w)_{T} = - (\pi^{k}_{T} v, \triangle w)_{T} + \sum_{F \in \mathcal{F}_{T}} (\pi^{k}_{F} v, \nabla w \cdot \mathbf{n}_{TF})_{F}
\]

\[
= - (v, \triangle w)_{T} + \sum_{F \in \mathcal{F}_{T}} (v, \nabla w \cdot \mathbf{n}_{TF})_{F} = (\nabla v, \nabla w)_{T}
\]

This shows that \( p^{k+1}_{T} I^{k}_{T} \) is the elliptic projector on \( \mathbb{P}^{k+1}_{d}(T) \):

\[
(\nabla p^{k+1}_{T} I^{k}_{T} v - \nabla v, \nabla w)_{T} = 0 \quad \forall w \in \mathbb{P}^{k+1}_{d}(T)
\]

The approximation properties follow
Stabilization I

- The following naive choice is not stable

\[ a_T(u, v) \approx (\nabla p_{T}^{k+1} u_T, \nabla p_{T}^{k+1} v_T)_T \]

- To remedy, we add a local stabilization term

\[ (\nabla p_{T}^{k+1} u_T, \nabla p_{T}^{k+1} v_T)_T + s_T(u_T, v_T) \]

- Coercivity and boundedness are expressed w.r.t. to

\[ \| v_T \|_{1,T}^2 := \| \nabla v_T \|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} \| v_F - v_T \|_F^2 \]
Define, for $T \in \mathcal{T}_h$, the stabilization bilinear form $s_T$ as

$$s_T(u_T, v_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1}(\pi_F^k(\hat{p}^{k+1}_T u_T - u_F), \pi_F^k(\hat{p}^{k+1}_T v_T - v_F))_F,$$

with $\hat{p}^{k+1}_T$ high-order correction of cell DOFs based on $p^{k+1}_T$

$$\hat{p}^{k+1}_T v_T := v_T + (p^{k+1}_T v_T - \pi_T^k p^{k+1}_T v_T)$$

With this choice, $a_T$ satisfies, for all $v_T \in U^k_T$,

$$\|v_h\|_{1,T}^2 \leq a_T(v_T, v_T) \leq \|v_T\|_{1,T}^2$$
Stabilization III

Lemma (High-order consistency of $s_T$)

$s_T$ preserves the approximation properties of $\nabla p_T^{k+1}$.

- For all $u \in H^{k+2}(T)$, letting $\hat{u}_T := I_T^k u = (\pi_T^k u, (\pi_F^k u)_{F \in T})$, $\pi_F^k (\hat{p}_T^{k+1} \hat{u}_T - \hat{u}_F) \|_F = \pi_F^k (\pi_T^k u + p_T^{k+1} \hat{u}_T - \pi_T^k p_T^{k+1} \hat{u}_T - \pi_F^k u) \|_F \\
  \leq \pi_F^k (p_T^{k+1} \hat{u}_T - u) \|_F + \pi_T^k (u - p_T^{k+1} \hat{u}_T) \|_F \\
  \lesssim h_T^{-1/2} \| p_T^{k+1} \hat{u}_T - u \|_T$

- Recalling the approximation properties of $p_T^{k+1}$, this yields

$$\left\{ \| \nabla (p_T^{k+1} \hat{u}_T - u) \|_T^2 + s_T(\hat{u}_T, \hat{u}_T) \right\}^{1/2} \lesssim h_T^{k+1} \| u \|_{k+2,T}$$
We enforce boundary conditions strongly considering the space

$$\overline{U}_{h,0}^k := \left\{ v_h \in \overline{U}_h^k \mid v_F \equiv 0 \quad \forall F \in \mathcal{F}_h^b \right\}$$

The discrete problem reads: Find \( u_h \in \overline{U}_{h,0}^k \) s.t.

$$a_h(u_h, v_h) := \sum_{T \in T_h} a_T(u_T, v_T) = \sum_{T \in T_h} (f, v_T)_T \quad \forall v_h \in \overline{U}_{h,0}^k$$

Well-posedness follows from the coercivity of \( a_h \)
Theorem (Energy-norm error estimate)

Assume \( u \in H^{k+2}(\mathcal{T}_h) \) and let

\[
\widehat{u}_h := (\pi^k_T u)_{T \in \mathcal{T}_h}, (\pi^k_F u)_{F \in \mathcal{F}_h} \in U^k_{h,0}.
\]

Then, we have the following energy error estimate:

\[
\max \left( \|u_h - \widehat{u}_h\|_{1,h}, \|u_h - \widehat{u}_h\|_{a,h} \right) \leq h^{k+1} \|u\|_{H^{k+2}(\Omega)},
\]

with

\[
\|v_h\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|v_T\|_{1,T}^2.
\]
Theorem ($L^2$-norm error estimate)

Further assuming elliptic regularity and $f \in H^1(\Omega)$ if $k = 0$,

$$\max (\|\tilde{u}_h - u\|, \|\hat{u}_h - u_h\|) \lesssim h^{k+2}N_k,$$

with $N_0 := \|f\|_{H^1(\Omega)}$, $N_k := \|u\|_{H^{k+2}(\mathcal{T}_h)}$ if $k \geq 1$, and, $\forall T \in \mathcal{T}_h$,

$$\tilde{u}_h|_T := p_T^{k+1}u_T, \quad \hat{u}_h|_T := p_T^{k+1}I_T^k u, \quad u_h|_T := u_T.$$
Convergence for a smooth 2d solution

Figure: Energy (left) and $L^2$-norm (right) of the error vs. $h$ for uniformly refined triangular (top) and hexagonal (bottom) mesh families, $u(x) = \sin(\pi x_1) \sin(\pi x_2)$
Figure: Assembly/solution time for triangular (left) and hexagonal (right) mesh families, sequential implementation
Mesh adaptivity: Fichera’s 3d test case I

- Let $\Omega := (-1, 1)^3 \setminus [0, 1]^3$
- We consider the following exact solution:
  \[
  u(\boldsymbol{x}) = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{4}}
  \]
  corresponding to the forcing term
  \[
  f(\boldsymbol{x}) = -\frac{3}{4}(x_1^2 + x_2^2 + x_3^2)^{-\frac{3}{4}}
  \]
- We consider an adaptive procedure driven by guaranteed residual-based a posteriori estimators [DP & Specogna, 2015]
Mesh adaptivity: Fichera’s 3d test case II

Figure: HHO solution on a sequence of adaptively refined meshes
Mesh adaptivity: Fichera’s 3d test case III

Figure: Energy error vs. $\dim(U_h^k)$
Mesh adaptivity: Fichera’s 3d test case IV

**Figure**: Estimated (left) and true (right) error distribution
Outline

1. Poisson

2. Variable diffusion and local conservation

3. Linear elasticity
Let \( \nu : \Omega \rightarrow \mathbb{R}^{d \times d} \) be a SPD tensor-valued field s.t.

\[
\forall T \in \mathcal{T}_h, \quad 0 < \nu_T \leq \lambda(\nu) \leq \bar{\nu}_T
\]

Consider the variable diffusion problem

\[
-\nabla \cdot (\nu \nabla u) = f \quad \text{in } \Omega \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad u = 0 \quad \text{on } \partial \Omega
\]

We confer built-in homogeneization features to \( p_T^{k+1} \)

\[
(\nu \nabla p_T^{k+1} \nu_T, \nabla w)_T = (\nu \nabla \nu_T, \nabla w)_T + \sum_{F \in \mathcal{F}_T} (\nu_F - \nu_T, \nu \nabla w \cdot n_{TF})_F
\]
There is $C$ independent of $h_T$ and $\nu$ s.t., for all $v \in H^{k+2}(T)$, it holds with $\alpha = \frac{1}{2}$ if $\nu$ is piecewise constant and $\alpha = 1$ otherwise:

$$\|v - p_T^{k+1} I_T^k v\|_T + h_T \|\nabla (v - p_T^{k+1} I_T^k v)\|_T \leq C \rho_T^\alpha h_T^{k+2} \|v\|_{k+2,T},$$

with local heterogeneity/anisotropy ratio

$$\rho_T := \frac{\bar{\nu}_T}{\nu_T} \geq 1.$$
Theorem (Energy-error estimate)

Assume that \( u \in H^{k+2}(\mathcal{T}_h) \) and modify the bilinear form as

\[
a_{\nu,T}(u_T, v_T) := (\nu \nabla p_T^{k+1} u_T, \nabla p_T^{k+1} v_T)_T + s_{\nu,T}(u_T, v_T)
\]

where, setting \( \nu_{TF} := \| n_{TF} \cdot \nu |_{T} \cdot n_{TF} \|_{L^\infty(F)} \),

\[
s_{\nu,T}(u_T, v_T) := \sum_{F \in \mathcal{F}_T} \frac{\nu_{TF}}{h_F} (\pi_F^k (\hat{p}_T^{k+1} u_T - u_F), \pi_F^k (\hat{p}_T^{k+1} v_T - v_F))_F.
\]

Then, with \( \hat{u}_h \) and \( \alpha \) as above,

\[
\| \hat{u}_h - u_h \|_{\nu,h} \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \nu_T \rho_T^{1+2\alpha} h_T^{2(k+1)} \| u \|_{k+2,T}^2 \right\}^{1/2}.
\]
A highly prized property in practice is local conservation

At the discrete level, we wish to mimic the local balance

\[(\nu \nabla u, \nabla v)_T - \sum_{F \in \mathcal{F}_T} (\nu_{|T} \nabla u \cdot \mathbf{n}_{TF}, v)_F = (f, v)_T \quad \forall v \in H^1(T)\]

where, for all interface \(F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2},\)

\[\nu_{|T_1} \nabla u \cdot \mathbf{n}_{T_1F} + \nu_{|T_2} \nabla u \cdot \mathbf{n}_{T_2F} = 0\]

This requires to identify numerical fluxes
Define the face residual operator \( R^k_T : \mathbb{P}_d^k(F_T) \rightarrow \mathbb{P}_d^k(F_T) \) s.t.

\[
R^k_T \varphi|_F = \pi^k_F \left( \varphi|_F - p^{k+1}_T(0, \varphi) + \pi_T^k p^{k+1}_T(0, \varphi) \right)
\]

Denote by \( R^{*,k}_T \) its adjoint and let \( \tau_{\partial T} \) and \( u_{\partial T} \) be s.t.

\[
\tau_{\partial T}|_F = \frac{\nu_{TF}}{h_F} \quad \text{and} \quad u_{\partial T}|_F = u_F \quad \forall F \in F_T
\]

The penalty term can be rewritten in conservative form as

\[
s_T(u_T, v_T) = \sum_{F \in F_T} \left( R^{*,k}_T (\tau_{\partial T} R^k_T (u_{\partial T} - u_T)), v_F - v_T \right)_F
\]
Lemma (Flux formulation)

The HHO solution $u_h \in \mathcal{U}_{h,0}^k$ satisfies, for all $T \in \mathcal{T}_h$ and all $v_T \in \mathcal{P}_d^k(T)$

$$(\nu \nabla p_T^{k+1} u_T, \nabla v_T)_T - \sum_{F \in \mathcal{F}_T} (\Phi_{TF}(u_T), v_T)_F = (f, v_T)_T,$$

with numerical flux

$$\Phi_{TF}(u_T) := \nu |_T \nabla p_T^{k+1} u_T \cdot n_T F - R_T^*(\tau_{\partial F} R_T^k(u_{\partial F} - u_T)),$$

s.t., for all interface $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$,

$$\Phi_{T_1 F}(u_{T_1}) + \Phi_{T_2 F}(u_{T_2}) = 0.$$
The flux formulation shows that $\text{HHO} = \text{HDG on steroids}$

Smaller local problems to eliminate flux unknowns:

$$\nabla \mathbb{P}^{k+1}_d(T) \ vs. \ \mathbb{P}^k_d(T)^d$$

Superconvergence of the potential in the $L^2$-norm

$$h^{k+2} \ vs. \ h^{k+1}$$

$\text{HHO}$ can be adapted into existing HDG codes!
Outline

1. Poisson

2. Variable diffusion and local conservation

3. Linear elasticity
Consider the linear elasticity problem: Find \( u : \Omega \to \mathbb{R}^d \) s.t.

\[
-\nabla \cdot \sigma(u) = f \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial \Omega
\]

with real Lamé parameters \( \lambda \geq 0 \) and \( \mu > 0 \) and

\[
\sigma(u) = 2\mu \nabla_s u + \lambda (\nabla \cdot u) I_d
\]

When \( \lambda \to +\infty \) we need to approximate nontrivial incompressible displacement fields.
Rigid body motions

- Applied to vector fields, the operator $\nabla_s$ yields strains
- Its kernel $\text{RM}(\Omega)$ contains rigid-body motions

$$\text{RM}(\Omega) := \{ \mathbf{v} \in H^1(\Omega)^3 \mid \exists \alpha, \omega \in \mathbb{R}^3, \mathbf{v}(\mathbf{x}) = \alpha + \omega \otimes \mathbf{x} \}$$

- We note for further use that

$$\mathbb{P}^0_d(\Omega)^d \subset \text{RM}(\Omega) \subset \mathbb{P}^1_d(\Omega)^d$$
DOFs and reduction map I

\[ U^k_T \] for \( k \in \{1, 2\} \)

- For \( k \geq 1 \) and all \( T \in \mathcal{T}_h \), we define the local space of DOFs

\[
U^k_T := \mathbb{P}^k_d(T)^d \times \left\{ \prod_{F \in \mathcal{F}_T} \mathbb{P}^k_{d-1}(F)^d \right\}
\]

- The global space has single-valued interface DOFs

\[
U^k_h := \left\{ \prod_{T \in \mathcal{T}_h} \mathbb{P}^k_d(T)^d \right\} \times \left\{ \prod_{F \in \mathcal{F}_h} \mathbb{P}^k_{d-1}(F)^d \right\}
\]
Let $T \in \mathcal{T}_h$. The local displacement reconstruction operator $p^{k+1}_T : \underline{U}_T^k \to \underline{\mathbb{P}}_{d}^{k+1}(T)^d$ is s.t., for all $\underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$ and $\underline{w} \in \underline{\mathbb{P}}_{d}^{k+1}(T)^d$,

\[
(\nabla s p^{k+1}_T v_T, \nabla s w)_T = - (v_T, \nabla \cdot \nabla s w)_T + \sum_{F \in \mathcal{F}_T} (v_F, \nabla s w n_{TF})_F
\]

Rigid-body motions are prescribed from $\underline{v}_T$ setting

\[
\int_T p^{k+1}_T v_T = \int_T v_T, \quad \int_T \nabla s p^{k+1}_T v_T = \sum_{F \in \mathcal{F}_T} \int_F \frac{1}{2} (n_{TF} \otimes v_F - v_F \otimes n_{TF})
\]
Lemma (Approximation properties for $p_T^{k+1} I_T^k$)

There exists $C > 0$ independent of $h_T$ s.t., for all $v \in H^{k+2}(T)^d$,

$$
\| v - p_T^{k+1} I_T^k v \|_T + h_T \| \nabla (v - p_T^{k+1} I_T^k v) \|_T \leq C h_T^{k+2} \| v \|_{H^{k+2}(T)^d}.
$$

Proceeding as for Poisson, one can show that

$$
(\nabla_s p_T^{k+1} I_T^k v - \nabla_s v, \nabla_s w)_T = 0 \quad \forall w \in P_d^{k+1}(T)^d,
$$

and the approximation properties follow.
Define, for \( T \in \mathcal{T}_h \), the stabilization bilinear form \( s_T \) as

\[
s_T(u_T, v_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1}(\pi_F^k(\hat{p}_T^{k+1} u_T - u_F), \pi_F^k(\hat{p}_T^{k+1} v_T - v_F))_F,
\]

with displacement reconstruction \( \hat{p}_T^{k+1} : U_T^k \rightarrow P_{d+1}^k(T)^d \) s.t.

\[
\forall v_T \in U_T^k, \quad \hat{p}_T^{k+1} v_T := v_T + (p_T^{k+1} v_T - \pi_T^k p_T^{k+1} v_T).
\]

Stability can be proved in terms of the discrete strain norm

\[
\| v_T \|^{2, T}_{\varepsilon, T} := \| \nabla_s v_T \|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \| v_F \|_F^2
\]
Lemma (Stability and approximation)

Let $T \in T_h$ and assume $k \geq 1$. Then,

$$\|v_T\|_{\varepsilon,T}^2 \lesssim \|\nabla_s p_T^{k+1} v_T\|_T^2 + s_T(v_T, v_T) \lesssim \|v_T\|_{\varepsilon,T}^2.$$  

Moreover, for all $v \in H^{k+2}(T)^d$, we have

$$\left\{ \|\nabla_s (I_T^k v - v)\|_T^2 + s_T(I_T^k v, I_T^k v) \right\}^{1/2} \lesssim h_T^{k+1} \|v\|_{H^{k+2}(T)^d}.$$  

Generalization of a classical result: Crouzeix–Raviart does not meet Korn!
For all $F \in \mathcal{F}_T$ one has, inserting $\pm \pi_F^{k} \hat{p}_T^{k+1} \nu_T$,

$$\|\nu_F - \nu_T\|_F \lesssim \|\pi_F^k (\nu_F - \hat{p}_T^{k+1} \nu_T)\|_F + h_F^{-1/2} \|p_T^{k+1} \nu_T - \pi_T^k \hat{p}_T^{k+1} \nu_T\|_T$$

For any function $w \in H^1(T)^d$ with rigid-body motions $w_{\text{RM}}$,

$$\|w - \pi_T^k w\|_T = \|(w - w_{\text{RM}}) - \pi_T^k (w - w_{\text{RM}})\|_T \lesssim h_T \|\nabla_s w\|_T$$

where $\pi_T^k w_{\text{RM}} = w_{\text{RM}}$ requires $k \geq 1$ to have

$$\text{RM}(T) \subset P_d^k(T)^d$$

Clearly, this reasoning breaks down for $k = 0$
Divergence reconstruction

- We define the **local local discrete divergence operator**

\[ D^k_T : U^k_T \rightarrow \mathbb{P}^k_d(T) \]

s.t., for all \( \mathbf{v}_T = (\mathbf{v}_T, (\mathbf{v}_F)_{F \in \mathcal{F}_T}) \in U^k_T \) and all \( q \in \mathbb{P}^k_d(T) \),

\[(D^k_T \mathbf{v}_T, q)_T := -(\mathbf{v}_T, \nabla q)_T + \sum_{F \in \mathcal{F}_T} (\mathbf{v}_F \cdot \mathbf{n}_{TF}, q)_F\]

- By construction, we have the following commuting diagram:

\[
\begin{array}{ccc}
H^1(T) & \xrightarrow{\nabla} & L^2(T) \\
\downarrow I^k_T & & \downarrow \pi^k_T \\
U^k_T & \xrightarrow{D^k_T} & \mathbb{P}^k_d(T)
\end{array}
\]
We define the local bilinear form $a_T$ on $U_T^k \times U_T^k$ as

$$a_T(u_T, v_T) := 2\mu(\nabla_s p_T^{k+1} u_T, \nabla_s p_T^{k+1} v_T) + \lambda(D_T^k u_T, D_T^k v_T) + (2\mu)s_T(u_T, v_T)$$

The discrete problem reads: Find $u_h \in U_{h,0}^k$ s.t.

$$a_h(u_h, v_h) := \sum_{T \in \mathcal{T}_h} a_T(u_T, v_T) = \sum_{T \in \mathcal{T}_h} (f, v_T)_T \quad \forall v_h \in U_{h,0}^k$$

with $U_{h,0}^k$ incorporating boundary conditions
Theorem (Energy-norm error estimate)

Assume $k \geq 1$ and the additional regularity

$$\mathbf{u} \in H^{k+2}(\mathcal{T}_h)^d \text{ and } \nabla \cdot \mathbf{u} \in H^{k+1}(\mathcal{T}_h).$$

Then, there exists $C > 0$ independent of $h$, $\mu$, and $\lambda$ s.t.

$$(2\mu)^{1/2}\|\mathbf{u}_h - \hat{\mathbf{u}}_h\|_{a,h} \leq Ch^{k+1} B(\mathbf{u}, k),$$

with

$$B(\mathbf{u}, k) := (2\mu)\|\mathbf{u}\|_{H^{k+2}(\mathcal{T}_h)^d} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^{k+1}(\mathcal{T}_h)}.$$
- **Locking-free** if $B(u, k)$ is bounded uniformly in $\lambda$
- For $d = 2$ and $\Omega$ convex, one has using Cattabriga’s regularity

\[ B(u, 0) = \|u\|_{H^2(\Omega)^d} + \lambda \|\nabla \cdot u\|_{H^1(\Omega)} \leq C_\mu \|f\| \]

- More generally, for $k \geq 1$, we need the regularity shift

\[ B(u, k) \leq C_\mu \|f\|_{H^k(\Omega)^d} \]

- **Key point:** commuting property for $D^k_T$
Theorem ($L^2$-error estimate for the displacement)

Let $e_h \in \mathbb{P}^k_d(\mathcal{T}_h)^d$ be s.t.

$$e_h|_T := u_T - \pi^k_T u \quad \forall T \in \mathcal{T}_h.$$  

Then, assuming elliptic regularity for $\Omega$ and provided that

$$u \in H^{k+2}(\mathcal{T}_h)^d \text{ and } \nabla \cdot u \in H^{k+1}(\mathcal{T}_h),$$

it holds with $C>0$ independent of $\lambda$ and $h$,

$$\|e_h\| \leq C h^{k+2} B(u, k).$$
We consider the following exact solution:

\[ u(x) = \left( \sin(\pi x_1) \sin(\pi x_2) + (2\lambda)^{-1} x_1, \cos(\pi x_1) \cos(\pi x_2) + (2\lambda)^{-1} x_2 \right) \]

The solution \( u \) has vanishing divergence in the limit \( \lambda \to +\infty \):

\[ \nabla \cdot u(x) = \frac{1}{\lambda} \]
Figure: Energy error with $\lambda = 1$ (above) and $\lambda = 1000$ (below) vs. $h$ for the triangular (left) and hexagonal (right) mesh families.
Figure: Energy (above) and displacement (below) error vs. $\tau_{\text{tot}}$ (s) for the triangular and hexagonal mesh families
\textit{h}p\text{-version composite discontinuous Galerkin methods for elliptic problems on complicated domains.}

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