Ring Star Problems Solver

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Joint work with Fabian Castaño, André Rossi and Sonia Toubaline

JuliaOptDays 2024, 29th of October

Some Objectives

- Hear some French
- What is Operations Research?
- What is the Ring Star Problem?

  https://en.wikipedia.org/wiki/Ring_star_problem
- Survivable and Resilient Ring Star Problems
- A Solver for RSP and its variants

  https://github.com/jkhamphousone/RingStarProblems.jl
- 2-S-RSP

Introduction and Ring Star Network

- Network Design
- Applications in fiber-optic networks, transportation networks
- Backbone and tributary architecture

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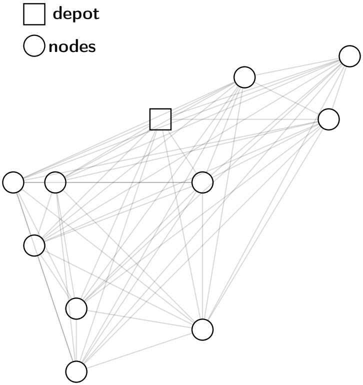

Complete graph

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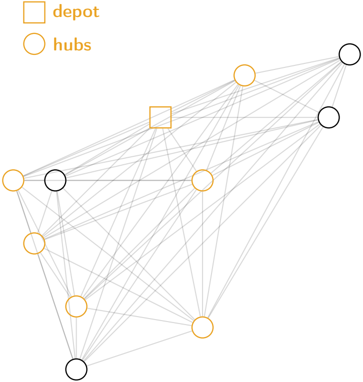

Hubs selection

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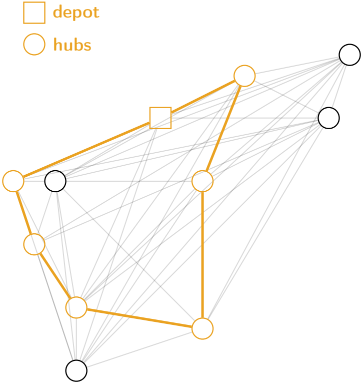
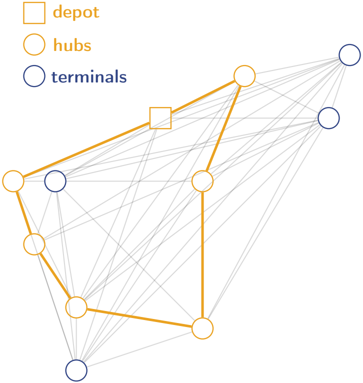


Ring design

- Network Design
- Applications in fiber-optic networks, transportation networks
- Backbone and tributary architecture



Terminals selection

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- Applications in fiber-optic networks, transportation networks
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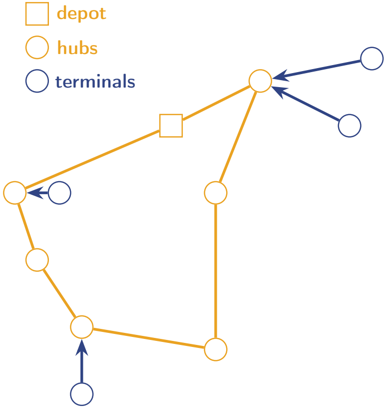


Ring Star network

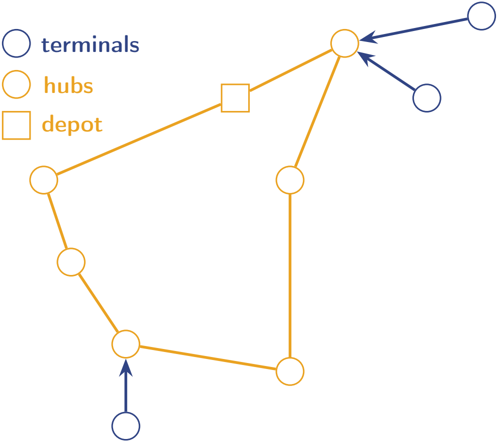
- Exact formulations

  1999 Xu *et al.*
  2004 Labbé *et al.*
  2010 Kedad-Sidhoum et Nguyen
  2011 Simonetti *et al.*

- Heuristics

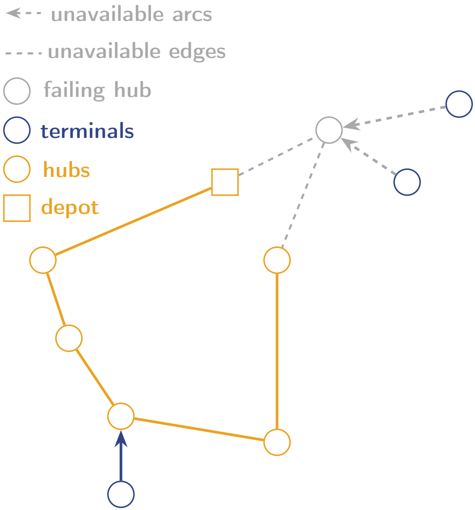
  2006 Dias *et al.*
  2013 Calvete *et al.*
  2020 Zang *et al.*

Ring Star Problem (RSP) - Failing Hub



What if top right hub node fails?

Ring Star Problem (RSP) - Failing Hub



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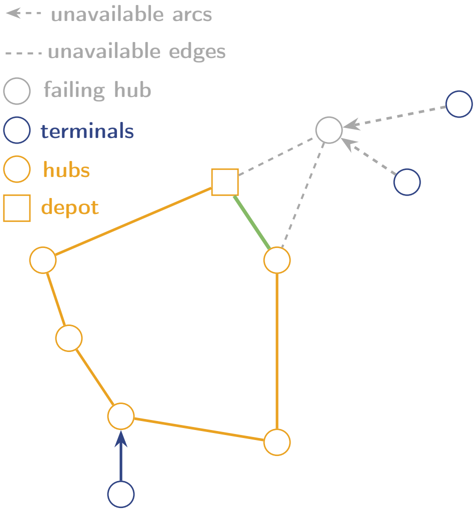

Ring repair

Star re-allocation

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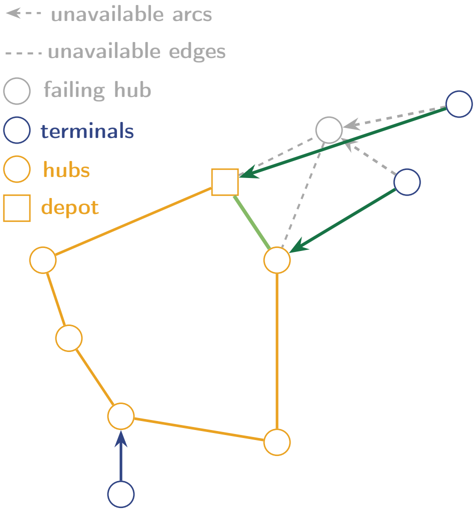

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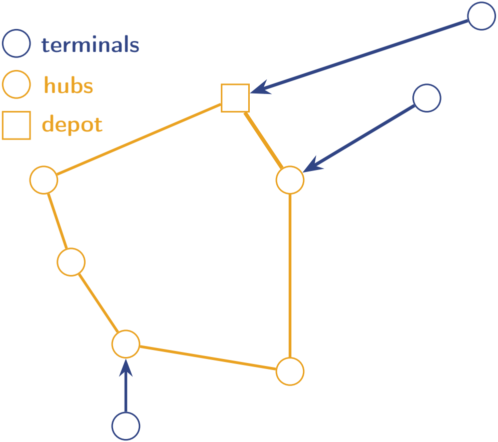
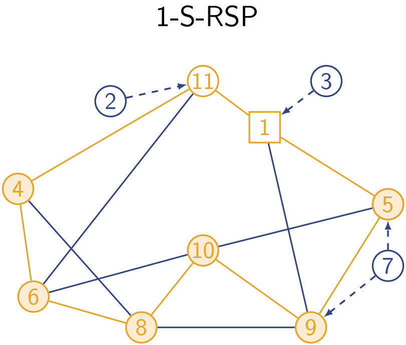


Repaired RSP solution

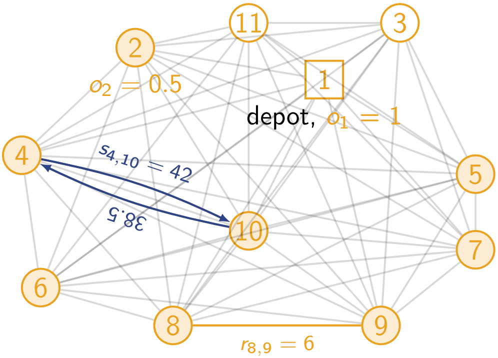
Survivable and Resilient Ring Star Problem



Survivable Ring Star Problem

Inputs

- A weighted complete mixed graph $G = (V, E \cup A)$
- ☐ A depot
- $o_i, \quad \forall i \in V$
- $r_{ij}, \quad \forall (i,j) \in E$
- $s_{ij}, \quad \forall ij \in A$
- $\widetilde{V} \subseteq V$: uncertain nodes



Outputs

A minimum cost Survivable ring star network

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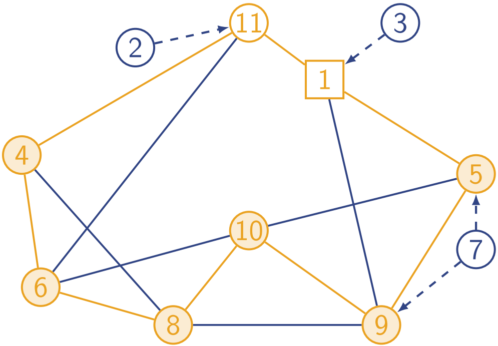

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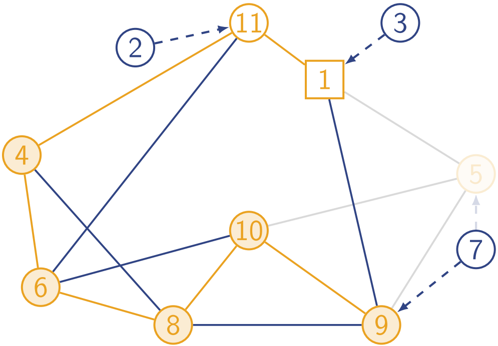

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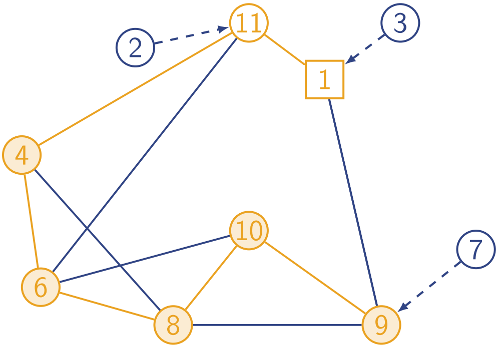


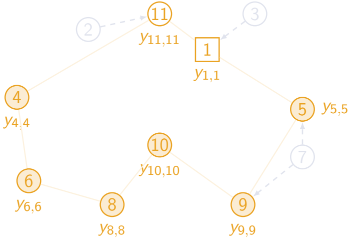
Survivable RSP — ILP Formulation

Ring

$y_{ii} = 1$ if i is selected as a hub

$x_{ij} = 1$ if $ij \in E$ is selected as an edge for connecting hubs i and j

$x'_{ij} = 1$ if $ij \in E$ is selected as a backup edge



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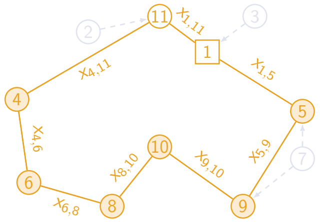

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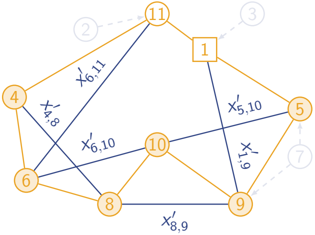


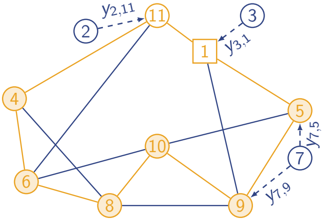
Survivable RSP — ILP Formulation

Ring

y_{ii} x_{ij} x'_{ij}

Star

$y_{ij} = 1$ if $(i, j) \in A$ is selected as an arc for connecting terminal i and hub j



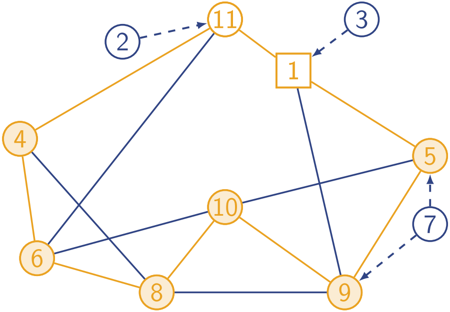
Survivable RSP — ILP Formulation

Ring

y_{ii} x_{ij} x'_{ij}

Star

y_{ij}



Objective

$$\text{Min} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E} r_{ij}(x_{ij} + x'_{ij}) + \sum_{(i,j) \in A} s_{ij} y_{ij}$$

$$\text{Min} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E} r_{ij}(x_{ij} + x'_{ij}) + \sum_{(i,j) \in A} s_{ij} y_{ij}$$

$$\sum_{\substack{j \in V \\ i<j}} x_{ij} + \sum_{\substack{j \in V \\ i>j}} x_{ji} = 2 y_{ii} \qquad \forall i \in V \quad \text{Connectivity constraints} \qquad (1)$$

$$x(\delta(S)) \geq 2 \sum_{j \in S} y_{ij} \qquad \forall S \subset V, 1 \notin S, i \in S \quad \text{Subtour elimination} \qquad (2)$$

$$\sum_{i \in V} \sum_{j \in V, i<j} x_{ij} \geq 5 \qquad \text{Ring size} \qquad (3)$$

$$y_{11} = 1 \qquad \text{Depot is a hub} \qquad (4)$$

$$\sum_{\substack{j \in V \setminus \widetilde{V} \\ i \neq j}} 2 y_{ij} + \sum_{\substack{j \in \widetilde{V} \\ i \neq j}} y_{ij} = 2(1 - y_{ii}) \qquad \forall i \in V \quad \text{Nodes are hubs or terminals} \qquad (5)$$

$$y_{ij} \leq y_{jj} \qquad \forall (i,j) \in A \quad \text{Terminals are connected to hubs} \qquad (6)$$

$$x_{ij} + x_{jk} \leq 1 + x'_{ik} \qquad \forall (i,j,k) \in \widetilde{J} \quad \text{Backup edges} \qquad (7)$$

$$\widetilde{J} = \{(i,j,k) \in V^3 : j \in \widetilde{V}, i \neq j, j \neq k, i < k\}$$

$$x_{ij} \in \mathbb{B} \qquad \forall ij \in E \qquad (8)$$

$$y_{ij} \in \mathbb{B} \qquad \forall (i,j) \in V^2 \qquad (9)$$

$$x'_{ij} \in \mathbb{B} \qquad \forall ij \in E \qquad (10)$$

Variables separation for the Branch-and-Benders-cut

$$\text{Min} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E} r_{ij}(x_{ij} + x'_{ij}) + \sum_{(i,j) \in A} s_{ij} y_{ij}$$

s.t. (1) — (10)

Complicating variables

- y_{ii} $\forall i \in V$
- x_{ij} $\forall (i,j) \in V^2 : i < j$

} Master: builds the ring

Non-complicating variables

- x'_{ij} $\forall (i,j) \in V^2 : i < j$
- y_{ij} $\forall (i,j) \in V^2 : i \neq j$

} Subproblem: builds backup edges and the star

Benders Decomposition — Master Problem, MP

$$\text{Min} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E} r_{ij} x_{ij} + \lambda$$

$$\sum_{\substack{j \in V \\ i<j}} x_{ij} + \sum_{\substack{j \in V \\ j>i}} x_{ij} = 2y_{ii} \quad \forall i \in V \quad \text{Connectivity constraints}$$

$$\sum_{i \in V} \sum_{j \in V, i<j} x_{ij} \geq 5 \quad \text{Ring size}$$

$$y_{11} = 1 \quad \text{Depot is a hub}$$

$$x_{ij} \in \mathbb{B} \quad \forall ij \in E$$

$$y_{ii} \in \mathbb{B} \quad \forall i \in V$$

$$\lambda \in \mathbb{R}_+$$

Benders Decomposition — Subproblem, SP$(\hat{x}, \hat{y})$

For given fixed $\hat{y} = \{y_{ii},\ i \in V\}$ and $\hat{x} = \{x_{ij},\ (i,j) \in V^{\neq}\}$

$$\text{Min } \lambda = \sum_{ij \in E} r_{ij} x'_{ij} + \sum_{(i,j) \in A} s_{ij} y_{ij}$$

$$\hat{x}_{ij} + \hat{x}_{jk} \leq 1 + x'_{ik}, \qquad \forall (i,j,k) \in \widetilde{J} \qquad \text{Backup ring}$$

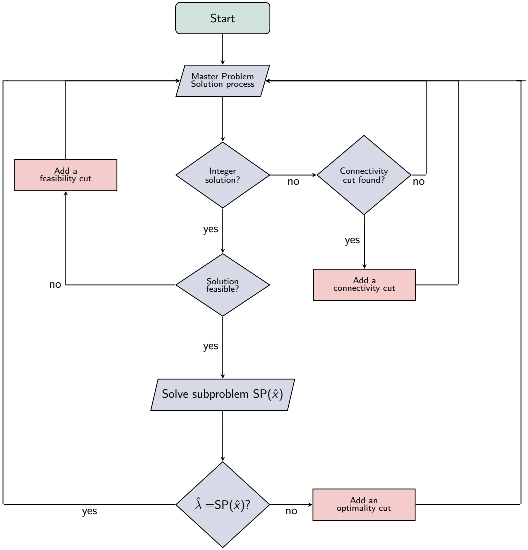
$$\sum_{\substack{j \in V \setminus \widetilde{V} \\ i \neq j}} 2y_{ij} + \sum_{\substack{j \in \widetilde{V} \\ i \neq j}} y_{ij} = 2(1 - \hat{y}_{ii}), \qquad \forall i \in V \qquad \text{Terminals connections}$$

$$y_{ij} \leq \hat{y}_{jj}, \qquad \forall (i,j) \in V^{\neq} \qquad \text{Terminals to hubs}$$

$$y_{ij} \in \mathbb{B}, \qquad \forall (i,j) \in V^2$$

$$x'_{ij} \in \mathbb{B}, \qquad \forall (i,j) \in V^2, i < j$$

Branch-and-Benders-cut scheme



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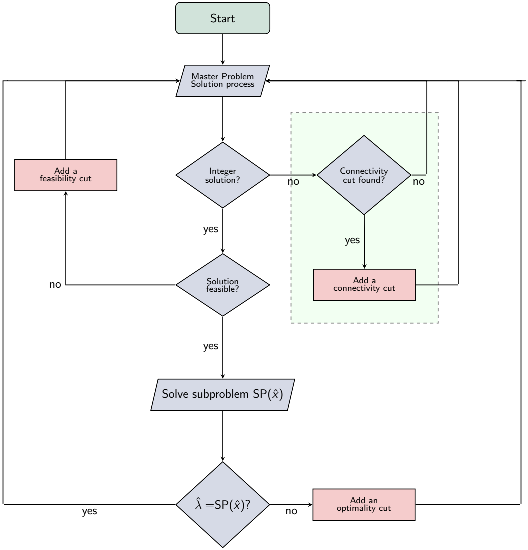

Subproblem's ILP formulation can be stated as LP

Valid inequalities

$$y_{ik} \leq \sum_{j \in \widetilde{V} \setminus \{k\} : \hat{y}_{jj}=1} y_{ij} \quad \text{For a given terminal } i \in V, \forall k \in \Omega_i$$

$$\Omega_i = \left\{ j \in \widetilde{V} \setminus \{i\} : \hat{y}_{jj} = 1, s_{ij} = \min_{j \in \widetilde{V} : y_{jj}=1} s_{ij} \right\}$$

Interpretation

Terminal and connected to an uncertain hub $\implies$ connected to another different uncertain hub.

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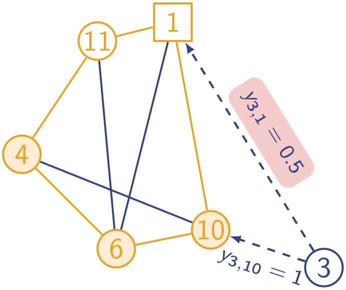
$$1 = y_{3,10} \not\leq y_{3,6} + y_{3,4} = 0$$



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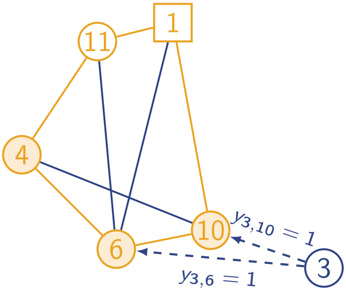

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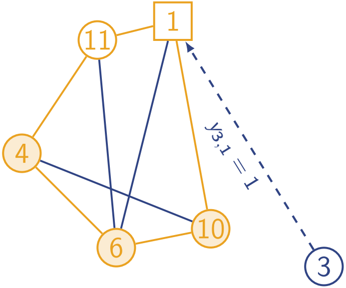
$$0 = y_{3,10} \leq y_{3,6} + y_{3,4} = 0$$



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Solving the dual of this LP

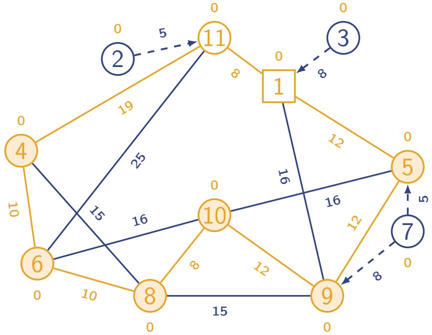
With a custom quadratic time algorithm

Improvements for 1-S-RSP

1. Connectivity cuts on fractional nodes
2. Instance transformation
3. Heuristic improvement of the ring

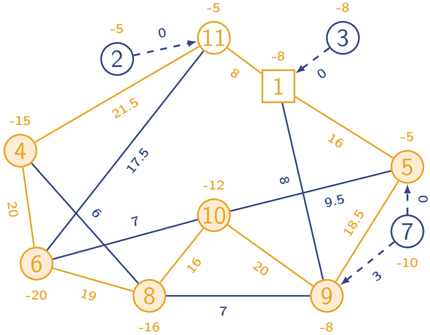
2. Instance transformation for 1-S-RSP

Original instance



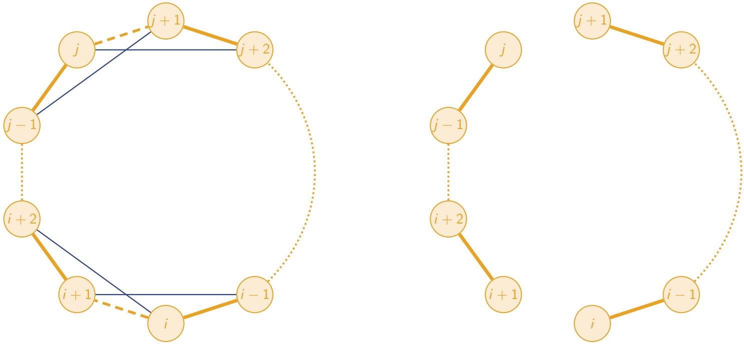
Master Problem cost = 0 + 91
Subproblem cost = 129
Total cost = 220

Transformed instance



Master Problem cost = 112 + 50
Subproblem cost = 58
Total cost = 220

3. Heuristic improvement of the ring for 1-S-RSP



ring edges removed by 2-opt backup

backup edges removed by 2-opt backup

unchanged ring edges

ring edges added by 2-opt backup

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unchanged ring edges

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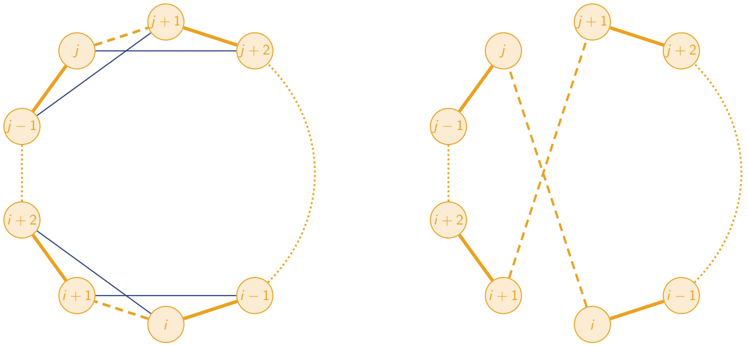

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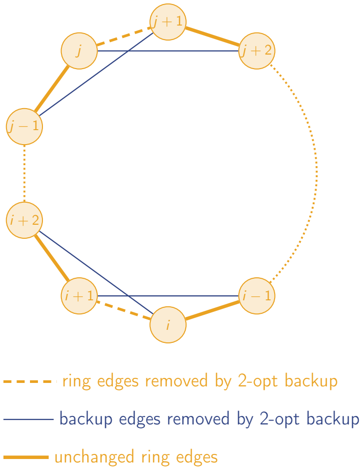
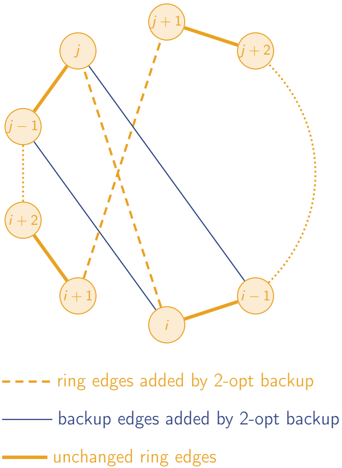
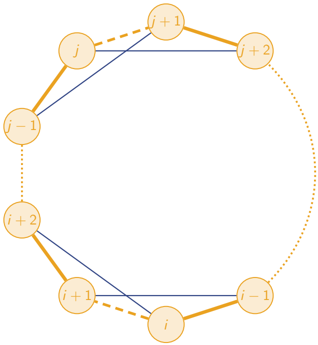




3. Heuristic improvement of the ring for 1-S-RSP



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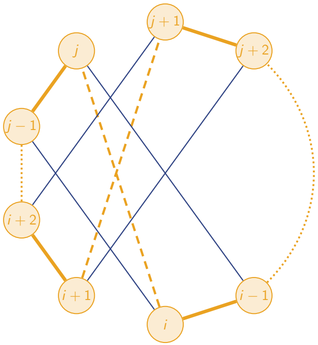


ring edges added by 2-opt backup

backup edges added by 2-opt backup

unchanged ring edges

Survivable RSP - Experimental Results

Ressources

julia v1.8.5, JuMP v.1.10, GUROBI OPTIMIZATION v10.0.1

CPU Intel(R) Core i7-7700 CPU @ 3.6GHz

RAM 16GBytes

Instances

TSPLIB 96 instances from 51 to 200 nodes.

$o_i = 0$	Hub selection costs
$\alpha \in \{3, 5, 7\}$	Ring and Star costs ratio
$r_{ij} = \lceil \alpha l_{ij} \rceil$	Ring costs
$s_{ij} = \lceil (10 - \alpha) l_{ij} \rceil$	Star costs

Random 40 new artificially generated instances.

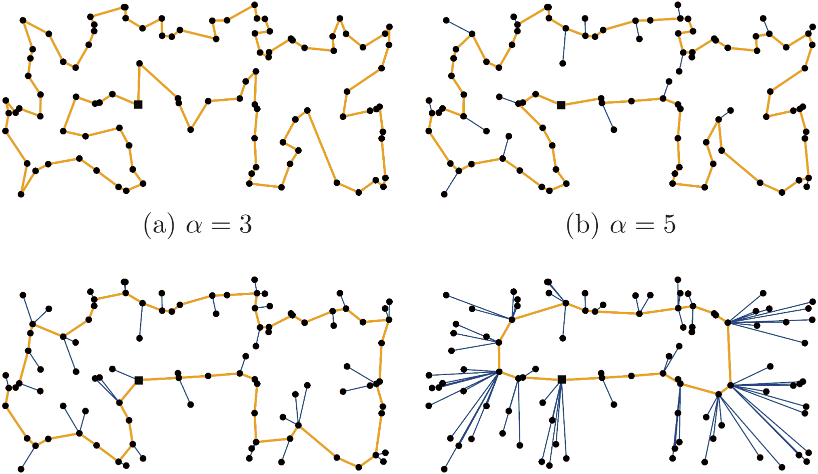
$o_i = \mathrm{RAND}(\{0, \ldots, 1000\})$

$r_{ij} = s_{ij} = \lceil l_{ij} \rceil$

Time Limit: 1hour

1-S-RSP — Numerical Experiments — Parameter α

4 optimal RSP solutions with the same instance except the ring and star costs [Calvete et al., 2013]



(a) $\alpha = 3$ (b) $\alpha = 5$

(c) $\alpha = 7$ (d) $\alpha = 9$

Survivable RSP — Numerical Experiments (1/3)

TSPLIB instances Class I

Instance n–α		ILP							Branch-and-Benders-cut								
$n = \|V\|$	CPU BF	CPU	Gap	LB	UB	$\|H\|$	n_subtour	Nodes	CPU	CPU SP	Gap	LB	UB	$\|H\|$	n_subtour	n_cut	Nodes
eil 51-3	0.31	TL	37.0%	2,257	3,588	50	121	78,603	TL	0.56	24.0%	2,457	3,236	50	5,176	2,423	468,554
eil 51-5	0.32	TL	32.4%	3,656	5,415	42	107	54,287	TL	16.42	28.9%	3,607	5,078	27	13,233	5,278	708,228
eil 51-7	0.37	TL	15.3%	3,532	4,171	4	16	32,403	TL	22.44	27.7%	3,013	4,171	4	10,705	5,552	945,219
berlin 52-3	0.34	TL	36.8%	39,849	63,065	52	167	63,312	TL	0.78	28.5%	41,225	57,728	52	6,364	2,709	423,356
berlin 52-5	0.41	TL	37.0%	62,860	99,903	4	287	67,975	TL	11.32	33.7%	59,510	89,875	25	11,946	3,989	618,813
berlin 52-7	0.32	TL	2.4%	63,689	65,307	4	20	41,342	TL	18.19	19.8%	52,348	65,307	4	11,064	4,686	853,647
brazil 58-3	0.50	TL	41.5%	133,661	228,852	57	68	37,606	TL	0.62	38.3%	125,103	202,818	58	3,969	2,510	431,328
brazil 58-5	0.53	TL	45.0%	204,560	372,410	52	102	27,887	TL	9.59	46.7%	171,404	321,835	36	14,922	2,547	606,453
brazil 58-7	0.56	TL	31.8%	199,565	292,931	4	29	36,218	TL	14.39	57.4%	124,501	292,931	4	18,845	2,951	951,187
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
d 198-3	217.97	OUT OF MEMORY							TL	0.80	40.0%	74,440	124,172	198	4,355	139	205,655
d 198-5	183.70	OUT OF MEMORY							TL	13.26	66.0%	94,576	278,430	90	15,103	92	506,090
d 198-7	206.90	OUT OF MEMORY							OUT OF MEMORY								
kroB 200-3	225.57	OUT OF MEMORY							TL	0.72	35.8%	149,463	232,886	200	3,378	100	282,332
kroB 200-5	227.45	OUT OF MEMORY							TL	36.56	57.0%	203,088	472,837	108	10,584	274	406,513
kroB 200-7	248.91	OUT OF MEMORY							TL	28.74	69.4%	143,055	468,114	60	9,713	186	333,348

Numerical Experiments 1-S-RSP (2/3)

Variation of the results when 2-opt backup is not used

Instance n–α	ILP				Branch-and-Benders-cut			
$n = \|V\|$	CPU	Gap	LB	UB	CPU	Gap	LB	UB
eil 51-3	TL	+14.05%	−1.02%	+7.80%	TL	+187.92%	−0.60%	+144.78%
eil 51-5	TL	+15.43%	−2.43%	+5.36%	TL	+42.91%	−0.36%	+20.76%
eil 51-7	TL	0.00%	0.00%	0.00%	TL	0.00%	0.00%	0.00%
pr 124-3	TL	0.00%	0.00%	0.00%	TL	+113.92%	+2.41%	+647.11%
pr 124-5	TL	0.00%	0.00%	0.00%	TL	+51.13%	−1.59%	+213.46%
pr 124-7	TL	0.00%	0.00%	0.00%	TL	+17.09%	−1.53%	+72.19%
ClassII - 20.1	−1.54%	0.00%	0.00%	0.00%	+20.86%	0.00%	0.00%	0.00%
ClassII - 30.1	TL	+38.00%	+0.97%	+3.05%	TL	+17.75%	+1.46%	+7.16%
ClassII - 40.1	TL	+35.16%	−1.32%	+4.07%	TL	+42.00%	+0.45%	+16.81%
ClassII - 60.1	TL	−1.52%	+0.27%	−0.09%	TL	+41.43%	−1.37%	+22.54%
ClassII - 100.1	TL	+22.16%	+0.26%	+5.00%	TL	+40.89%	+1.09%	+24.28%
ClassII - 125.1	OUT OF MEMORY				TL	+22.49%	+0.16%	+13.33%

Numerical Experiments 1-S-RSP (3/3)

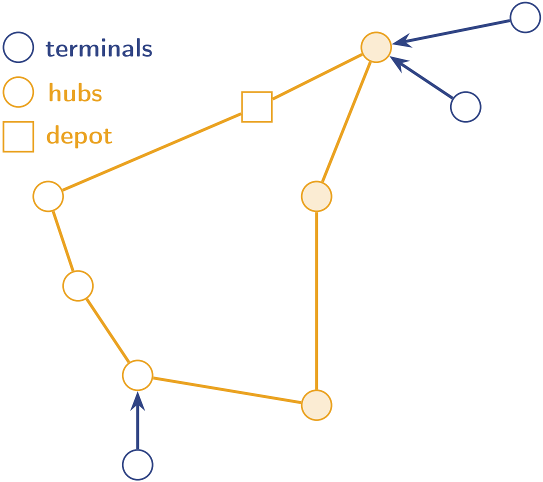
Impact of $\widetilde{V}$'s cardinality

Instance n.ID		ILP				
$n = \|V\|$	$\widetilde{V}$	CPU	Gap	LB	UB	n_constraints
	$\emptyset$ (Ring Star Problem)	5.59	0%	6,912	6,912	871
	$\{2, \ldots, 8\}$	5.44	0%	7,545	7,545	3,713
ClassII - 30.1	$\{2, \ldots, 15\}$	5.86	0%	8,335	8,335	6,555
	$\{2, \ldots, 23\}$	5.10	0%	8,800	8,800	9,803
	$\{2, \ldots, 30\} = V \setminus \{1\}$	TL	5.0%	13,535	14,247	12,645
	$\emptyset$ (Ring Star Problem)	5.92	0%	9,906	9,906	3,541
	$\{2, \ldots, 15\}$	6.24	0%	9,920	9,920	27,495
ClassII - 60.1	$\{2, \ldots, 30\}$	7.63	0%	10,862	10,862	53,160
	$\{2, \ldots, 45\}$	22.76	0%	13,980	13,980	78,825
	$\{2, \ldots, 60\} = V \setminus \{1\}$	TL	19.8%	18,626	23,229	104,490

Resilient Ring Star Problem Inputs and Outputs

Inputs

- $G = (V, E \cup A)$, c, d, $\widetilde{V}$
- $c' \in \mathbb{R}_+^E$ **in euros per unit of time**
- $d' \in \mathbb{R}_+^A$ **in euros per unit of time**
- F, total duration of breakdowns during the considered horizon



Resilient Ring Star Problem Inputs and Outputs

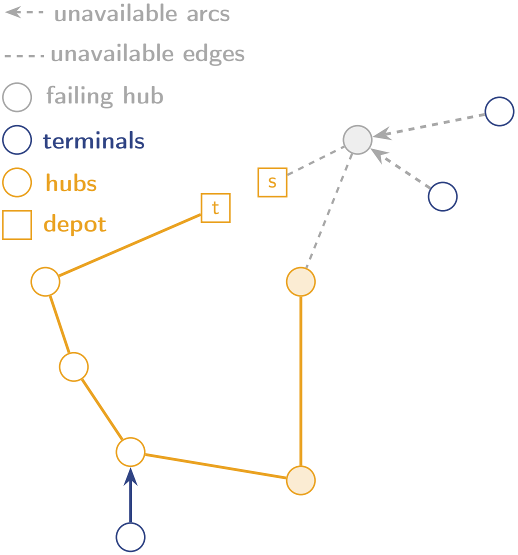
Inputs

- $G = (V, E \cup A)$, c, d, $\widetilde{V}$
- $c' \in \mathbb{R}_+^E$ **in euros per unit of time**
- $d' \in \mathbb{R}_+^A$ **in euros per unit of time**
- F, total duration of breakdowns during the considered horizon

Outputs A ring star network that minimizes the cost which is the sum of

Costs of classical RSP

+ **Corrective costs in the worst case**



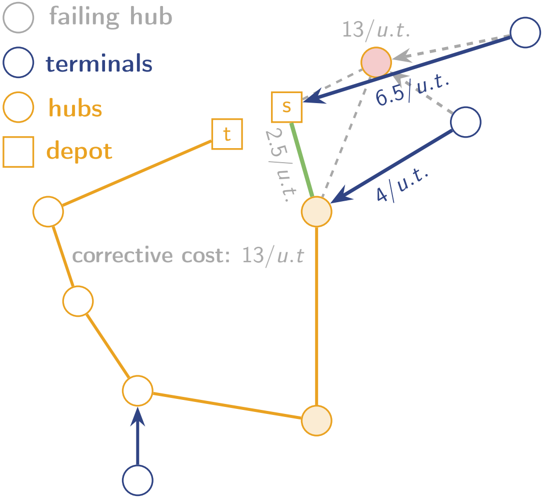
Resilient Ring Star Problem Inputs and Outputs

Outputs A ring star network that minimizes the cost which is the sum of

Costs of classical RSP

+ **Corrective costs in the worst case**

Largest cost $\Longrightarrow$ 13€/unit of time?



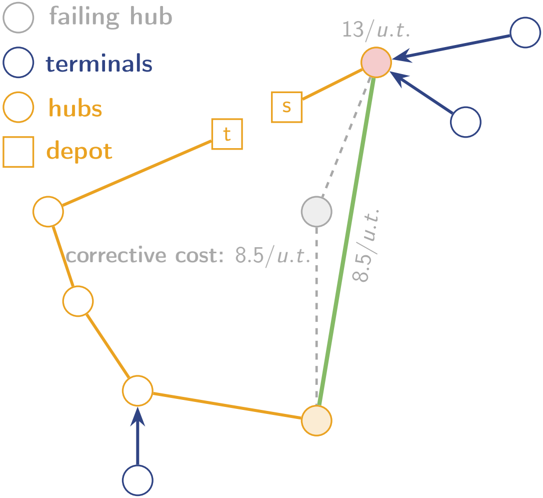
Resilient Ring Star Problem Inputs and Outputs

Outputs A ring star network that minimizes the cost which is the sum of

Costs of classical RSP

+ **Corrective costs in the worst case**

Largest cost $\Longrightarrow$ 13€/unit of time > 8.5?



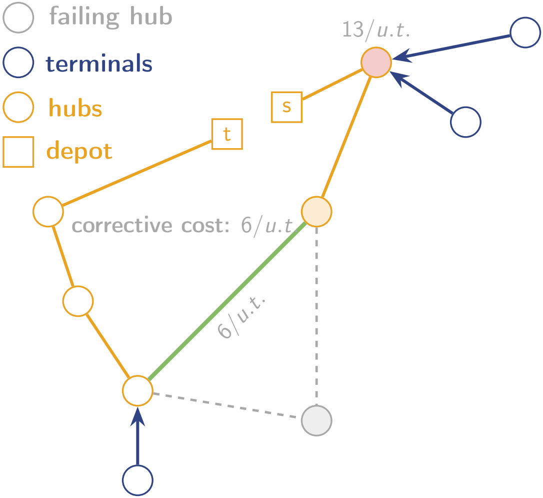
Resilient Ring Star Problem Inputs and Outputs

Outputs A ring star network that minimizes the cost which is the sum of

Costs of classical RSP

\+ **Corrective costs in the worst case**

Largest cost $\Longrightarrow 13/u.t.$



Resilient Ring Star Problem Objective

Outputs A ring star network that minimizes the cost which is the sum of

Costs of classical RSP

\+ **Corrective costs in the worst case**

Largest cost $\Longrightarrow B$

Objective

$$\text{Minimize} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E'} c_{ij} x_{ij} + \sum_{(i,j) \in A^{\neq}} d_{ij} y_{ij} + FB$$

B is the maximum total cost incurred by the corrections per unit of time in the worst case

F is a parameter, the total duration of breakdowns during the considered horizon

$$\text{Min} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E'} c_{ij} x_{ij} + \sum_{(i,j) \in A^{\neq}} c_{ij} y_{ij} + FB$$

$$\sum_{i \in V} y_{ii} \geq 4 \quad (11)$$

$$\sum_{ij \in \delta(i)} x_{ij} = 2 y_{ii} \qquad \forall i \in V' \setminus \{s, t\} \quad (12)$$

$$x(\delta(S)) \geq 2 \sum_{j \in S} y_{ij} \qquad \forall S \subset V' \setminus \{s, t\}, \forall i \in S \quad (13)$$

$$\sum_{i \in V \setminus \{s\}} x_{si} = 1, \quad \sum_{i \in V \setminus \{s\}} x_{it} = 1 \quad (14)$$

$$y_{si} = 0, \quad y_{tt} = 1, \quad y_{ss} = 1 \qquad \forall i \in V' \setminus \{s, t\} \quad (15)$$

$$\sum_{j \in V} y_{ij} = 1 \qquad \forall i \in V \quad (16)$$

$$x'_{ik} \geq x_{ij} + x_{jk} - 1 \qquad \forall (i, j, k) \in \widetilde{J} \quad (17)$$

$$x'_{ik} \geq x_{si} + x_{kt} - 1 \qquad \forall (i, k) \in (V \setminus \{s\})^2, i < k \quad (18)$$

$$y_{jj} \geq x_{ij} + y_{ij} + y'_{ij} + x'_{ij} \qquad \forall (i, j) \in A^{\neq} \quad (19)$$

$$1 - y_{ii} - \sum_{j \in V \setminus \{\widetilde{V} \cup \{i\}\}} y_{ij} = \sum_{(i,j) \in A^{\neq}} y'_{ij} \qquad \forall i \in V \quad (20)$$

$$\sum_{k \in V \setminus \{i,j\}} d'_{ik} (y'_{ik} + y_{ij} - 1) \leq \theta_{ij} \qquad \forall (i, j) \in V \times \widetilde{V} \setminus \{i\} \quad (21)$$

$$c'_{ik} (x'_{ik} + x_{ij} + x_{jk} - 2) + \sum_{t \in V \setminus \{j\}} \theta_{tj} \leq B \qquad \forall (i, j, k) \in \widetilde{J} \quad (22)$$

And domain constraints

Variables separation for the Branch-and-Benders-cut

$$\text{Min} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E'} c_{ij} x_{ij} + \sum_{(i,j) \in A^{\neq}} c_{ij} y_{ij} + F B$$

s.t. (11) — (22)

Complicating variables

- $y_{ii} \quad \forall i \in V$
- $x_{ij} \quad \forall ij \in E'$
- $y_{ij} \quad \forall (i,j) \in A^{\neq}$

} Master: builds a RSP solution

Non-complicating variables

- B
- $x'_{ij} \quad \forall (i,j) \in V \times V', i < j$
- $y'_{ij} \quad \forall (i,j) \in A^{\neq}$
- $\theta_{ij} \quad \forall i \in V, j \in \widetilde{V} \setminus \{i\}$

} Subproblem: builds the corrective operations

Benders Decomposition — Master Problem, MP

$$\text{Min} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E'} c_{ij} x_{ij} + \sum_{(i,j) \in A^{\neq}} d_{ij} y_{ij} + \lambda$$

$$\sum_{i \in V} y_{ii} \geq 4$$

$$\sum_{ij \in \delta(i)} x_{ij} = 2 y_{ii} \qquad \forall i \in V' \setminus \{s, t\}$$

$$x(\delta(S)) \geq 2 \sum_{j \in S} y_{ij} \qquad \forall S \subset V' \setminus \{s, t\}, \forall i \in S$$

$$\sum_{i \in V \setminus \{s\}} x_{si} = 1, \qquad \sum_{i \in V \setminus \{s\}} x_{it} = 1$$

$$y_{si} = 0, \quad y_{tt} = 1, \quad y_{ss} = 1 \qquad \forall i \in V' \setminus \{s, t\}$$

$$\sum_{j \in V} y_{ij} = 1 \qquad \forall i \in V$$

$$y_{jj} - x_{ij} \geq y_{ij} \qquad \forall (i, j) \in A^{\neq}$$

$$x_{ij} \in \mathbb{B}, \quad y_{ij} \in \mathbb{B} \qquad \lambda \in \mathbb{R}_+$$

Benders Decomposition — Subproblem, $SP(\hat{x}, \hat{y})$

For given fixed $\hat{y} = \{y_{ij}, \forall (i,j) \in A\}$, $\hat{x} = \{x_{ij}, \forall (i,j) \in V^{\neq}\}$
Min $\lambda = FB$

$$x'_{ik} \geq \widehat{x}_{ij} + \widehat{x}_{jk} - 1 \qquad \forall (i,j,k) \in \widetilde{J}$$

$$\sum_{(i,j)\in A^{\neq}} y'_{ij} = 1 - \widehat{y}_{ii} - \sum_{j\in V\setminus\{\widetilde{V}\cup\{i\}\}} \widehat{y}_{ij} \qquad \forall i \in V$$

$$\theta_{ij} - \sum_{k\in V\setminus\{i,j\}} d'_{ik} y'_{ik} \geq \sum_{k\in V\setminus\{i,j\}} d'_{ik}(\widehat{y}_{ij} - 1) \qquad \forall i \in V, \forall j \in \widetilde{V} \setminus \{i\}$$

$$B - c'_{ik} x'_{ik} - \sum_{t\in V\setminus\{j\}} \theta_{tj} \geq c'_{ik}(\widehat{x}_{ij} + \widehat{x}_{jk} - 2) \qquad \forall (i,j,k) \in \widetilde{J}$$

$$x'_{ij} + y'_{ij} \leq \widehat{y}_{jj} - \widehat{x}_{ij} - \widehat{y}_{ij} \qquad \forall (i,j) \in A^{\neq}$$

$$x'_{ij} \in \mathbb{B} \qquad (i,j) \in V \times V', i < j$$

$$y'_{ij} \in \mathbb{B} \qquad \forall (i,j) \in A^{\neq}$$

$$\theta_{ij} \in \mathbb{R}_+ \qquad \forall i \in V, \forall j \in \widetilde{V} \setminus \{i\}$$

$$B \in \mathbb{R}_+$$

1-R-RSP - Experimental Results

Ressources

julia v1.8.0, JuMP v.1.2.1, GUROBI OPTIMIZATION v9.0.3

CPU Intel(R) Core(TM) i7-10610U CPU @ 1.8GHz

RAM 16GBytes

Instances

TSPLIB 96 instances from 51 to 200 nodes.

$\alpha \in \{3, 5, 7\}$	Ring and Star costs ratio
$o_i = 0$	Hub selection costs
$c_{ij} = \lceil \alpha l_{ij} \rceil$	Ring costs
$c'_{ij} = 0.01 c_{ij}$	Ring corrective costs/u.t.
$d_{ij} = \lceil (10 - \alpha) l_{ij} \rceil$	Star costs
$d'_{ij} = 0.01 d_{ij}$	Star corrective costs/u.t.

$F \in \{0, 7, 31, 183\}$ days

Time horizon: 365 days

1-R-RSP — Numerical Experiments

Instance n–α		ILP							Branch-and-Benders-cut								
$n = \|V\|$	F	CPU	Gap	LB	UB	$\|H\|$	n_subtour	Nodes	CPU	CPU SP	Gap	LB	UB	$\|H\|$	n_subtour	n_cut	Nodes
eil51-3	0 (no failures)	40.41	0%	1311.00	1311.00	51	13	1	7.79	0.00	0%	1311.00	1311.00	51	16	0	1
eil51-3	7 (one week)	42.67	0%	1315.76	1315.76	51	16	1	9.32	0.16	0%	1315.76	1315.76	51	29	5	1
eil51-3	31 (one month)	41.31	0%	1332.08	1332.08	51	11	1	9.59	0.27	0%	1332.08	1332.08	51	32	14	31
eil51-3	183 (six months)	102.10	0%	1435.44	1435.44	51	56	397	50.90	2.29	0%	1435.44	1435.44	51	1457	232	8246
eil51-5	0 (no failures)	17.50	0%	2042.00	2042.00	37	1	1	10.42	0.00	0%	2042.00	2042.00	37	335	0	390
eil51-5	7 (one week)	22.42	0%	2052.08	2052.08	37	3	1	11.61	0.17	0%	2052.08	2052.08	37	483	7	567
eil51-5	31 (one month)	45.88	0%	2086.64	2086.64	37	17	1	19.76	0.32	0%	2086.64	2086.64	37	1311	22	1194
eil51-5	183 (six months)	996.89	0%	2290.41	2290.41	40	791	5455	TL	5.00	5.8%	2157.00	2290.41	40	14263	479	225870
eil51-7	0 (no failures)	284.10	0%	2147.00	2147.00	16	173	259	14.88	0.00	0%	2147.00	2147.00	17	234	0	280
eil51-7	7 (one week)	301.48	0%	2168.51	2168.51	16	211	492	34.95	0.53	0%	2168.51	2168.51	17	1184	37	1813
eil51-7	31 (one month)	2017.40	0%	2232.86	2232.86	20	1483	6945	912.89	5.29	0%	2232.86	2232.86	19	2006	520	274318
eil51-7	183 (six months)	TL	11.0%	2246.73	2524.89	23	2000	4134	TL	50.10	12.9%	2200.52	2526.72	24	2815	5665	375421
pr76-3	0 (no failures)	OUT OF MEMORY							34.08	0.00	0%	324510.00	324510.00	76	525	0	2281
pr76-3	7 (one week)	493.67	0%	325972.72	325972.72	76	176	1716	33.76	0.39	0%	325972.72	325972.72	76	616	6	2134
pr76-3	31 (one month)	435.28	0%	330987.76	330987.76	76	132	1266	157.83	1.44	0%	330987.76	330987.76	76	2000	29	18954
pr76-3	183 (six months)	OUT OF MEMORY							TL	48.56	1.1%	358901.03	362749.66	76	2998	1327	1038213
pr76-5	0 (no failures)	2029.15	0%	500431.00	500431.00	61	172	7	3240.21	0.00	0%	500428.50	500431.00	61	12118	0	486372
pr76-5	7 (one week)	OUT OF MEMORY							TL	0.64	21.4%	486662.33	619456.84	66	12282	13	107716
pr76-5	31 (one month)	960.01	0%	510667.05	510667.05	61	360	717	TL	1.04	23.7%	486734.83	637978.88	61	12300	25	102296
pr76-5	183 (six months)	OUT OF MEMORY							TL	0.37	NO SOLUTION FOUND	486431.64			10407	6	0
pr76-7	0 (no failures)	OUT OF MEMORY							158.88	0.00	0%	555888.00	555888.00	39	1754	0	2401
pr76-7	7 (one week)	2555.39	0%	560503.45	560503.45	39	1162	4291	243.31	1.69	0%	560503.45	560503.45	39	2000	32	6184
pr76-7	31 (one month)	TL	4.0%	556861.08	579846.96	44	612	1056	TL	8.02	1.2%	565851.01	572902.66	42	2065	212	511976
pr76-7	183 (six months)	OUT OF MEMORY							TL	47.38	12.3%	559427.44	637935.65	45	4587	1386	156188

1. Preserving the ring-star structure under a hub failure
2. Survivable Ring Star Problem — 1-S-RSP
3. Branch-and-Benders-cut for 1-S-RSP
4. Improvements for 1-S-RSP
5. Resilient Ring Star Problem — 1-R-RSP
6. Resilient or Survivable Ring Star Problems?
7. Conclusions

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

Survivable or Resilient RSP ?

Depends on input parameter $F \implies$ Solve 1-R-RSP for all F

Objective

$$\text{Minimize} \sum_{i \in V} o_i y_{ii} + \sum_{ij \in E'} c_{ij} x_{ij} + \sum_{(i,j) \in A^{\neq}} d_{ij} y_{ij} + F B$$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

Survivable or Resilient RSP ?

Depends on input parameter $F \implies$ Solve 1-R-RSP for all F

Objective

For a given RSP solution $\mathcal{S} \in \mathbf{S}$:

$$\underbrace{\sum_{i \in V} o_i y_{ii} + \sum_{ij \in E'} c_{ij} x_{ij} + \sum_{(i,j) \in A^{\neq}} d_{ij} y_{ij}}_{K^{\mathcal{S}}} + F B^{\mathcal{S}}$$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

Survivable or Resilient RSP ?

Depends on input parameter $F \Longrightarrow$ Solve 1-R-RSP for all F

Objective

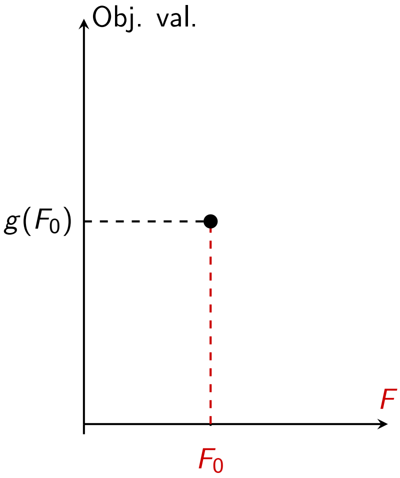
For a given RSP solution $\mathcal{S} \in \mathbf{S}$:

$$\underbrace{\sum_{i \in V} o_i y_{ii} + \sum_{ij \in E'} c_{ij} x_{ij} + \sum_{(i,j) \in A^{\neq}} d_{ij} y_{ij}}_{K^{\mathcal{S}}} + F B^{\mathcal{S}}$$

$$\min_{\mathcal{S} \in \mathbf{S}} \quad g(F) = K^{\mathcal{S}} + F B^{\mathcal{S}}$$

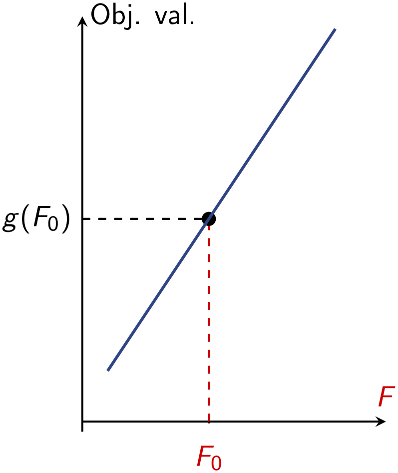
Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

$$\min_{\mathcal{S} \in \mathbf{S}} \quad g(F) = K^{\mathcal{S}} + F B^{\mathcal{S}}$$



Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

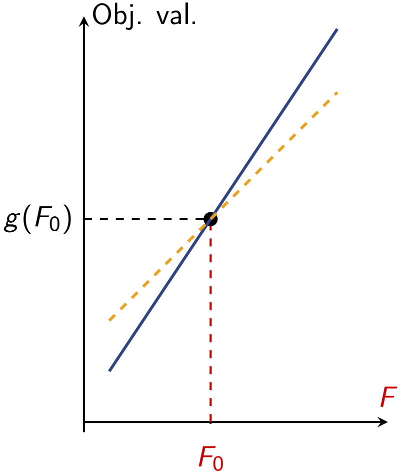
$$\min_{\mathcal{S} \in \mathbf{S}} \quad g(F) = K^{\mathcal{S}} + F B^{\mathcal{S}}$$



- $\mathcal{S}_1 : K^{\mathcal{S}_1} + F_0 B^{\mathcal{S}_1}$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

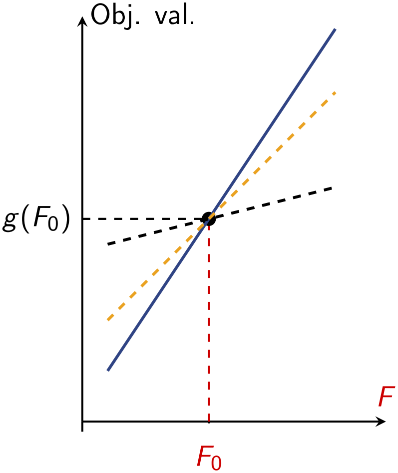
$$\min_{\mathcal{S} \in \mathbf{S}} \quad g(F) = K^{\mathcal{S}} + F B^{\mathcal{S}}$$



- $\mathcal{S}_1 : K^{\mathcal{S}_1} + F_0 B^{\mathcal{S}_1}$
- $\mathcal{S}_2 : K^{\mathcal{S}_2} + F_0 B^{\mathcal{S}_2}$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

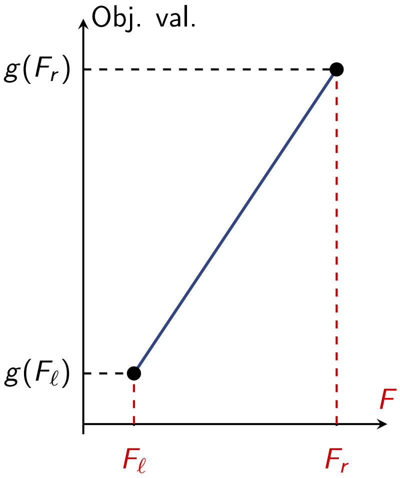
$$\min_{\mathcal{S} \in \mathbf{S}} \quad g(F) = K^{\mathcal{S}} + F B^{\mathcal{S}}$$



- $\mathcal{S}_1 : K^{\mathcal{S}_1} + F_0 B^{\mathcal{S}_1}$
- $\mathcal{S}_2 : K^{\mathcal{S}_2} + F_0 B^{\mathcal{S}_2}$
- $\mathcal{S}_3 : K^{\mathcal{S}_3} + F_0 B^{\mathcal{S}_3}$
- $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ all optimal for F_0. Do they remain optimal for $F \geq F_0$?

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

$$\min_{\mathcal{S} \in \mathbf{S}} \quad g(F) = K^{\mathcal{S}} + F B^{\mathcal{S}}$$

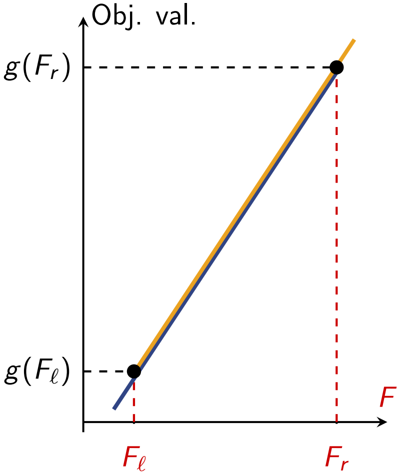


$\mathcal{S}_\ell = \mathcal{S}_r$

Theorem If solving 1-R-RSP on F_ℓ and F_r yields the same optimal solution then it is guaranteed to be optimal on $[F_\ell, F_r]$

Solve 1-R-RSP for $F \in [F_\ell, F_r]$

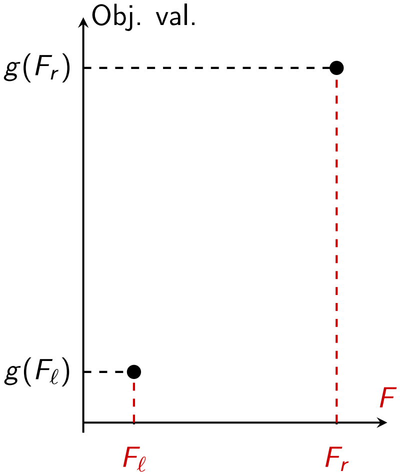
$$\min_{\mathcal{S}} \quad g(F) = K^{\mathcal{S}} + F B^{\mathcal{S}}$$



- **Case 0,** $B^{\mathcal{S}_\ell} = B^{\mathcal{S}_r}$
  - $\mathcal{S}_\ell$ is optimal for $F \in [F_\ell, F_r]$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

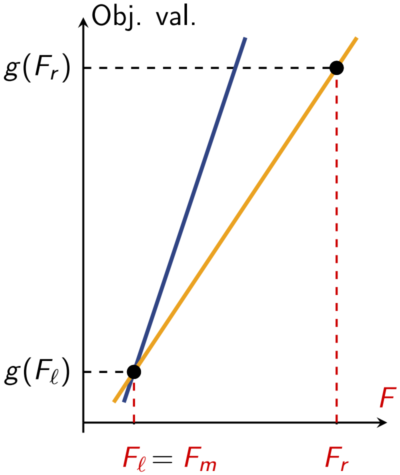
Minimize $K^{\mathcal{S}} + F B^{\mathcal{S}} = g(F)$



- **Case 0,** $B^{S_\ell} = B^{S_r}$
  - $\mathcal{S}_\ell$ is optimal for $F \in [F_\ell, F_r]$
- Compute $F_m \leftarrow \frac{K^{S_r} - K^{S_\ell}}{B^{S_\ell} - B^{S_r}}$
  - F_m : F value corresponding to the intersection of the segment lines of $\mathcal{S}_\ell$ and $\mathcal{S}_r$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

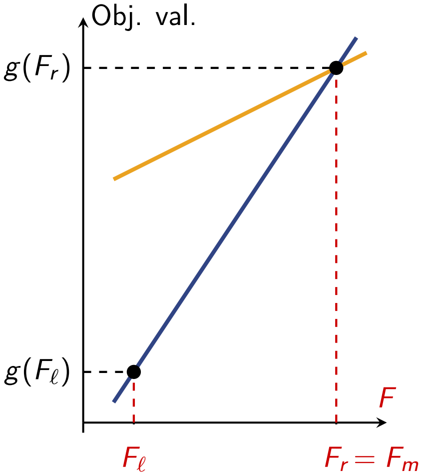
Minimize $K^{\mathcal{S}} + F B^{\mathcal{S}} = g(F)$



- **Case 0,** $B^{S_\ell} = B^{S_r}$
  - $\mathcal{S}_\ell$ is optimal for $F \in [F_\ell, F_r]$
- $F_m \leftarrow \frac{K^{S_r} - K^{S_\ell}}{B^{S_\ell} - B^{S_r}}$
- **Case 1,** $F_\ell = F_m$
  - $\mathcal{S}_r$ is optimal for $F \in [F_\ell, F_r]$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

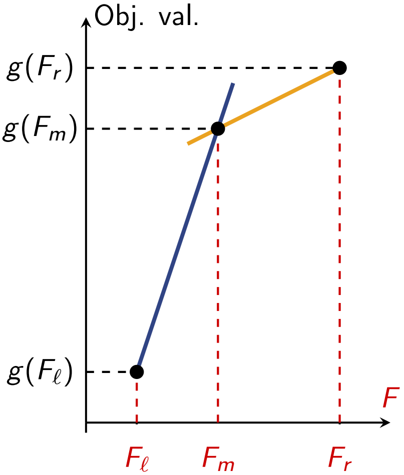
Minimize $K^{\mathcal{S}} + F B^{\mathcal{S}} = g(F)$



- **Case 0,** $B^{S_\ell} = B^{S_r}$
- $F_m \leftarrow \frac{K^{S_r} - K^{S_\ell}}{B^{S_\ell} - B^{S_r}}$
- **Case 1,** $F_\ell = F_m$
- **Case 2,** $F_r = F_m$
  - $\mathcal{S}_\ell$ is optimal for $F \in [F_\ell, F_r]$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

Minimize $K^{\mathcal{S}} + F B^{\mathcal{S}} = g(F)$



Case 3, $F_m \in (F_\ell, F_r)$

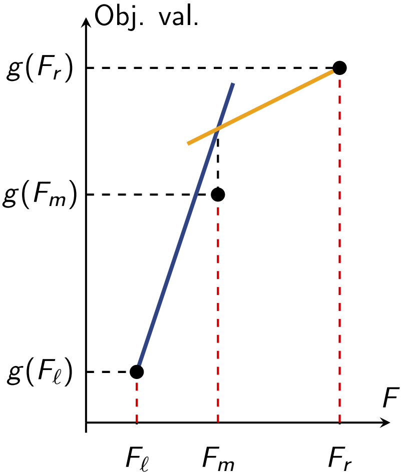
- Subcase 3. a

$$g(F_m) = K^{\mathcal{S}_\ell} + F_m B^{\mathcal{S}_\ell}$$
$$= K^{\mathcal{S}_m} + F_m B^{\mathcal{S}_m}$$

- $\mathcal{S}_\ell$ is optimal for $F \in [F_\ell, F_m]$
- $\mathcal{S}_r$ is optimal for $F \in [F_m, F_r]$

Solve 1-R-RSP for all $F \in [F_\ell, F_r]$

Minimize $K^{\mathcal{S}} + F B^{\mathcal{S}} = g(F)$



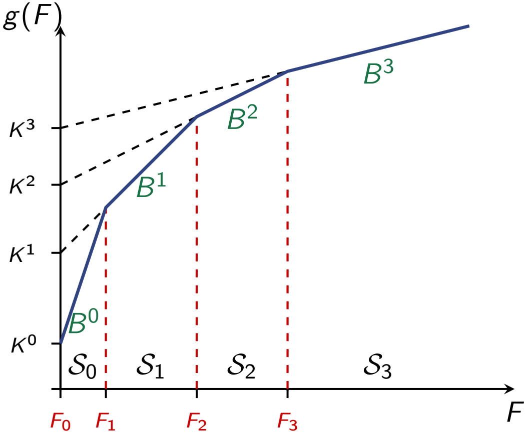
Case 3, $F_m \in (F_\ell, F_r)$

- Subcase 3. b

$$g(F_m) = K^{\mathcal{S}_m} + F_m B^{\mathcal{S}_m} < K^{\mathcal{S}_\ell} + F_m B^{\mathcal{S}_\ell}$$

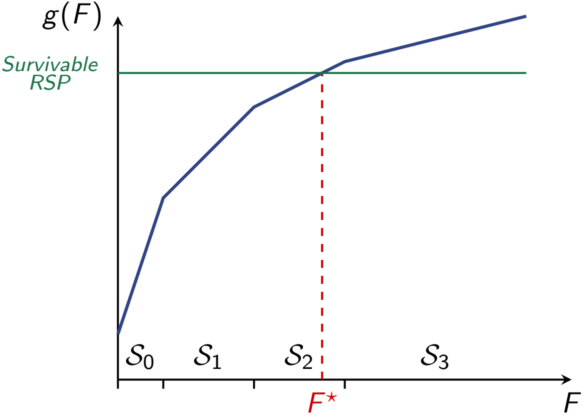
- Recursively explore $[F_\ell, F_m]$
  and
  Recursively explore $[F_m, F_r]$

1-R-RSP for all F



Shape of 1-R-RSP's objective for $F \geq 0$

Resilient or Survivable RSP?



Choose Resilient RSP below $F^\star$ and Survivable RSP above

Resilient or Survivable RSP

Nb. nodes	$\lvert\mathcal{S}_i\rvert$: Nb. opt. Solutions	F values	1-R-RSP(F) CPU Time	1-S-RSP CPU Time	$F^\star$
10	5	0.0, 779.4, 933.3 1025.0, 1544.9	30.87s	3.28s	569.76
12	6	0.0, 0.96, 164.71 728.78, 1727.27, 2866.66	187.66s	286.96s	574.87

Numerical experiments for comparing 1-R-RSP and 1-S-RSP

Overview

- Study of 1-S-RSP and 1-R-RSP
- ILP and Branch-and-Benders-cut formulations
- Generalized 2-opt heuristic for 1-S-RSP
- Introduce an instance transformation to enhance the master problem for 1-S-RSP
- Extensive numerical results are carried for the two models on set of instances up to 200 nodes
- Decision aiding model to choose between 1-S-RSP and 1-R-RSP

Limitations

- Extensive numerical experiments should be carried out for comparing 1-R-RSP and 1-S-RSP
- For 1-S-RSP and the B&BC, we generate a lot of cuts

Future works

- Use the st-chains formulation of 1-R-RSP to take advantage of the blossom inequalities
- Consider the situation where more than one hub can fail in both variants
- Any ILP formulation may be improved by solving the SP of 1-S-RSP
- Other heuristic approaches to improve the ring quality
- Fast way to generate optimality cuts
- Heuristics for large instances

Publications

National Conferences

ROADEF 2019 — M. Diamantini, A. Faye and J. Khamphousone, Calcul des dates d'atterrissage d'une séquence d'avions pour des fonctions de coût convexes et affines par morceaux

ROADEF 2021 — J. Khamphousone, F. Castaño, A. Rossi and S. Toubaline, Introducting the Resilient Ring Star Problem

ROADEF 2022 — J. Khamphousone, F. Castaño, A. Rossi and S. Toubaline, A robust variant of the Ring Star Problem

ROADEF 2023 — J. Khamphousone, F. Castaño, A. Rossi and S. Toubaline, Resilient and Survivable Ring Star Problems

International Conferences

INOC 2022 — J. Khamphousone, F. Castaño, A. Rossi and S. Toubaline, A robust variant of the Ring Star Problem

International Journals

Networks, Wiley, accepted on October 2023 — J. Khamphousone, F. Castaño, A. Rossi and S. Toubaline, A Survivable variant of the Ring Star Problem, https://doi.org/10.1002/net.22193

Survivable RSP — Numerical Experiments

Randomly generated instances Class II

Instance n.ID		ILP							Branch-and-Benders-cut								
$n = \|V\|$	CPU BF	CPU	Gap	LB	UB	$\|H\|$	n_subtour	Nodes	CPU	CPU SP	Gap	LB	UB	$\|H\|$	n_subtour	n_cut	Nodes
ClassII - 20.1	0.00	9.11	0.0%	10,200	10,200	4	3	3,578	156.11	0.77	0.0%	10,200	10,200	4	367	2028	318,762
ClassII - 20.2	0.00	6.57	0.0%	9,748	9,748	4	4	2,456	19.17	0.27	0.0%	9,748	9,748	4	51	613	39,770
ClassII - 20.3	0.00	5.53	0.0%	10,626	10,626	4	0	362	12.05	0.16	0.0%	10,626	10,626	4	32	303	17,521
ClassII - 20.4	0.00	5.49	0.0%	8,973	8,973	4	0	59	8.60	0.06	0.0%	8,973	8,973	4	14	201	6,538
ClassII - 20.5	0.00	6.70	0.0%	10,636	10,636	4	1	2,913	26.49	0.34	0.0%	10,636	10,636	4	45	801	52,752
ClassII - 30.1	0.03	TL	5.0%	13,535	14,247	12	122	336,852	TL	11.31	23.1%	10,818	14,071	9	6,644	11,149	690,368
ClassII - 30.2	0.03	TL	7.3%	15,216	16,422	11	211	432,828	TL	9.85	20.1%	12,579	15,760	8	7,268	11,419	725,196
ClassII - 30.3	0.03	14.09	0.0%	13,562	13,562	4	0	4,318	59.39	1.21	0.0%	13,562	13,562	4	71	1,065	92,113
ClassII - 30.4	0.01	707.83	0.0%	14,226	14,226	4	21	59,512	TL	13.47	14.3%	12,178	14,226	4	3,841	14,313	723,707
ClassII - 30.5	0.03	TL	1.1%	16,665	16,866	4	29	348,546	TL	14.37	19.8%	13,520	16,866	4	6,355	13,213	834,502
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ClassII - 150.1	39.01	OUT OF MEMORY							TL	79.30	34.3%	26,027	39,664	35	8,443	1,036	441,501
ClassII - 150.2	39.19	OUT OF MEMORY							TL	95.37	34.4%	26,594	40,563	27	9,625	1,240	472,963
ClassII - 150.3	39.67	OUT OF MEMORY							TL	94.80	35.7%	25,283	39,330	33	9,983	1,246	514,406
ClassII - 150.4	40.03	OUT OF MEMORY							TL	98.45	38.3%	25,687	41,682	32	10,111	1,325	550,316
ClassII - 150.5	40.03	OUT OF MEMORY							TL	95.33	35.6%	27,252	42,339	32	9,563	1,315	485,729
ClassII - 200.1	173.61	OUT OF MEMORY							TL	110.18	38.7%	30,212	49,353	43	8,407	714	428,736
ClassII - 200.2	174.60	OUT OF MEMORY							TL	108.87	37.1%	29,995	47,761	43	8,276	642	395,399
ClassII - 200.3	173.61	OUT OF MEMORY							TL	113.77	40.4%	29,816	50,077	36	8,263	729	444,891
ClassII - 200.4	180.11	OUT OF MEMORY							TL	107.59	40.0%	29,572	49,344	44	8,822	696	446,267
ClassII - 200.5	173.63	OUT OF MEMORY							TL	111.37	40.2%	29,482	49,379	43	7,345	668	389,473

Effect of instance transformation and 2-opt backup

Instance n–α	Branch-and-Benders-cut						
$n = \|V\|$	CPU	Gap	LB	UB	n_subtour	n_cuts	Nodes
without 2-opt and without instance transformation							
`ClassII - 20.4`	38.47	0.00%	8,973	8,973	87	1347	80,524
with 2-opt and without instance transformation							
`ClassII - 20.4`	46.01	0.00%	8,973	8,973	87	1347	80,524
without 2-opt and with instance transformation							
`ClassII - 20.4`	11.6	0.00%	8,973	8,973	16	183	7,088
with 2-opt and instance transformation							
`ClassII - 20.4`	8.6	0.00%	8,973	8,973	14	201	6,538
without 2-opt and without instance transformation							
`eil 51-3`	TL	81.0%	897	4,714	11,254	4,019	629,186
with 2-opt and without instance transformation							
`eil 51-3`	TL	74.8%	909	3,607	10,390	3,462	665,549
without 2-opt and with instance transformation							
`eil 51-3`	TL	27.7%	2,472	3,418	5,889	3,247	714,858
with 2-opt and instance transformation							
`eil 51-3`	TL	24.0%	2,457	3,236	5,176	2,423	468,554