Skew-Symmetric Schemes for compressible and incompressible flows

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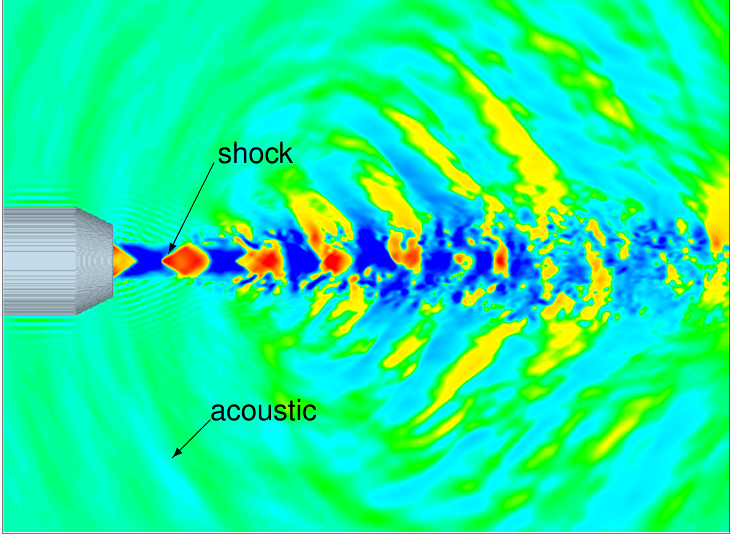
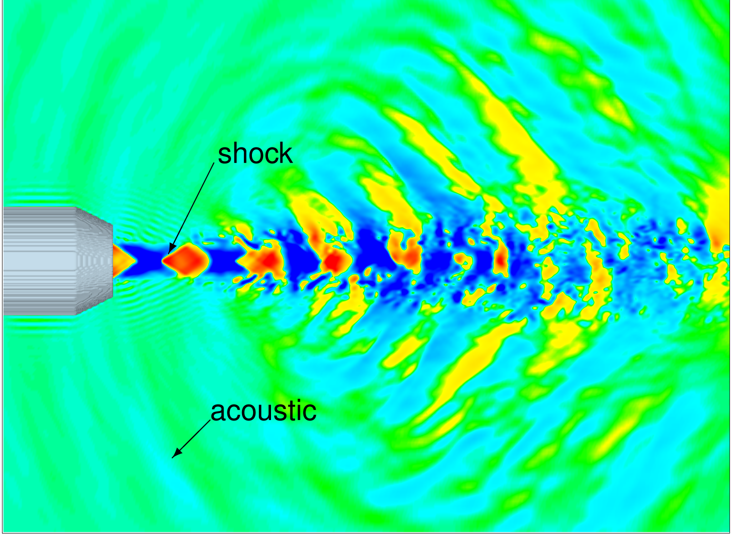


Use Finite Difference or use Finite Volume?



Use Finite Difference or use Finite Volume?
$\rightarrow$ skew symmetric, conservative FD

Overview

The concept

$$\partial_t u + \partial_x f(u) = 0$$

Discretised:

$$\partial_t u + D^u u = 0$$

- $\partial_t \int u \, \mathrm{d}x$

  $\mathbf{1}^T D^u = 0$ **telescoping sum**

- $\partial_t \int u^2 \, \mathrm{d}x$

  $(D^u)^T = -D^u$ **skew symmetry**

The concept

$$\partial_t u + \partial_x f(u) = 0$$

Discretised:

$$\partial_t u + D^u u = 0$$

- $\partial_t \int u \,\mathrm{d}x$

  $\mathbf{1}^T D^u = 0$ **telescoping sum** $\longrightarrow$ Momentum Conservation

- $\partial_t \int u^2 \,\mathrm{d}x$

  $(D^u)^T = -D^u$ **skew symmetry** $\longrightarrow$ Kin. Energy Conservation

Literature: Skew Symmetry

- Feiereisen W.C.Reynolds J.H. Ferziger, *Numerical simulation of a compressible homogeneous, turbulent shear flow*, NASA-CR-164953; SU-TF-13
- E. Tadmor, *Skew-Selfadjoint Form for Systems of Conservation Laws*, J. Math. Ana. Appl. 103, p428, (1984)
- Y. Morinishi and T. S. Lund and O. V. Vasilyev and P. Moin, *Fully conservative higher order finite difference schemes for incompressible flow* JCP 143, p 90, (1998)
- R. W. C. P. Verstappen, A. E. P. Veldman. *Symmetry-preserving discretization of turbulent flow*. JCP 187, p343 , (2003)
- J.C. Kok, *A high-order low-dispersion symmetry-preserving finite-volume method...* JCP 228, p 6811, (2009)
- Y. Morinishi, *Skew-symmetric form of convective terms...* , JCP 229, p276 (2010)
- S. Pirozzoli, *Generalized conservative approximations of split convective derivative operators*, p7180 JCP 229, (2010)

Skew–Symmetric discretisation

Simple example: Burgers' equation

divergence and convection form

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0 \qquad (D)$$

$$\partial_t u + u\partial_x u = 0 \qquad (C)$$

by $(2 \cdot [D] + [C])/3$:

skew–symmetric form

$$\partial_t u + \frac{1}{3}[\partial_x u \cdot + u\partial_x \cdot]u = 0 \qquad (S)$$

Skew–Symmetric discretisation

Burgers' in skew–symmetric form

$$\partial_t u + \frac{1}{3}[\partial_x u \cdot + u \partial_x \cdot] u = 0$$

Discrete

$$\partial_t \mathbf{u} + \underbrace{\frac{1}{3}\left(DU + UD\right)}_{D^u} \mathbf{u} = 0$$

$U = \mathrm{diag}(u)$, derivative D with $D^T = -D$

$$\boxed{\partial_t \mathbf{u} + D^u \mathbf{u} = 0}$$

Skew Symmetric discretisation

Kinetic Energy Conservation

$\frac{1}{2}\mathbf{u}^T\mathbf{u} = \frac{1}{2}\sum_i u_i^2$

$$\begin{aligned} \partial_t \mathbf{u}^T\mathbf{u} &= (\partial_t \mathbf{u})^T\mathbf{u} + \mathbf{u}^T\partial_t\mathbf{u} \\ &= -(D^u\mathbf{u})^T\mathbf{u} - \mathbf{u}^T D^u\mathbf{u} \\ &= -\mathbf{u}^T\left[(D^u)^T + D^u\right]\mathbf{u} \end{aligned}$$

Skew Symmetric discretisation

Kinetic Energy Conservation

$\frac{1}{2}\mathbf{u}^T\mathbf{u} = \frac{1}{2}\sum_i u_i^2$

$$\begin{aligned}\partial_t \mathbf{u}^T\mathbf{u} &= (\partial_t \mathbf{u})^T\mathbf{u} + \mathbf{u}^T\partial_t\mathbf{u} \\ &= -(D^u\mathbf{u})^T\mathbf{u} - \mathbf{u}^T D^u\mathbf{u} \\ &= -\mathbf{u}^T\left[(D^u)^T + D^u\right]\mathbf{u}\end{aligned}$$

Symmetry of transport term D^u

$$(D^u)^T = \frac{1}{3}(DU + UD)^T = \frac{1}{3}(U^T D^T + D^T U^T) = -D^u$$

$$U = \mathrm{diag}(u) = U^T, \quad D^T = -D$$

Skew Symmetric discretisation

Kinetic Energy Conservation

$\frac{1}{2}\mathbf{u}^T\mathbf{u} = \frac{1}{2}\sum_i u_i^2$

$$
\begin{aligned}
\partial_t \mathbf{u}^T\mathbf{u} &= (\partial_t \mathbf{u})^T\mathbf{u} + \mathbf{u}^T\partial_t\mathbf{u} \\
&= -(D^u\mathbf{u})^T\mathbf{u} - \mathbf{u}^T D^u\mathbf{u} \\
&= -\mathbf{u}^T \underbrace{\left[(D^u)^T + D^u\right]}_{=0} \mathbf{u} = 0
\end{aligned}
$$

Symmetry of transport term D^u

$$(D^u)^T = \frac{1}{3}(DU + UD)^T = \frac{1}{3}(U^T D^T + D^T U^T) = -D^u$$

Skew symmetry implies conservation of E_{kin}

Skew Symmetric discretisation

Momentum Conservation

$\mathbf{1}^T\mathbf{u} = \sum_i 1 \cdot u_i$

$$\begin{aligned} \partial_t \mathbf{1}^T\mathbf{u} &= \mathbf{1}^T\partial_t\mathbf{u} \\ &= -\mathbf{1}^T D^u\mathbf{u} \end{aligned}$$

Telescoping

$$\mathbf{1}^T D^u\mathbf{u} = \frac{1}{3}(\underbrace{\mathbf{1}^T D}_{=0} U + \mathbf{1}^T UD)\mathbf{u} = \frac{1}{3}\underbrace{\mathbf{u}^T D\mathbf{u}}_{=0} = 0$$

with $D^T = -D$

Skew Symmetric discretisation

Momentum Conservation

$\mathbf{1}^T\mathbf{u} = \sum_i 1 \cdot u_i$

$$\begin{aligned}\partial_t \mathbf{1}^T\mathbf{u} &= \mathbf{1}^T\partial_t\mathbf{u}\\ &= -\underbrace{\mathbf{1}^T D^u\mathbf{u}}_{=0} = 0\end{aligned}$$

Telescoping

$$\mathbf{1}^T D^u\mathbf{u} = \frac{1}{3}(\underbrace{\mathbf{1}^T D}_{=0} U + \mathbf{1}^T UD)\mathbf{u} = \frac{1}{3}\underbrace{\mathbf{u}^T D\mathbf{u}}_{=0} = 0$$

Telescoping sum property implies conservation of momentum

Time Integration

Implicit midpoint rule[1]

Fully discrete

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3} D^{u^{n+1/2}} u^{n+1/2} = 0$$

with $u^{n+1/2} = \frac{1}{2}(u^n + u^{n+1})$

[1] Verstappen, Veldman, J. Com. Phys 187, p. 343 (2003)

Time Integration

Implicit midpoint rule[1]

Fully discrete

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3} D^{u^{n+1/2}} u^{n+1/2} = 0$$

with $u^{n+1/2} = \frac{1}{2}(u^n + u^{n+1})$

Multiplying by $(u^{n+1/2})^T$

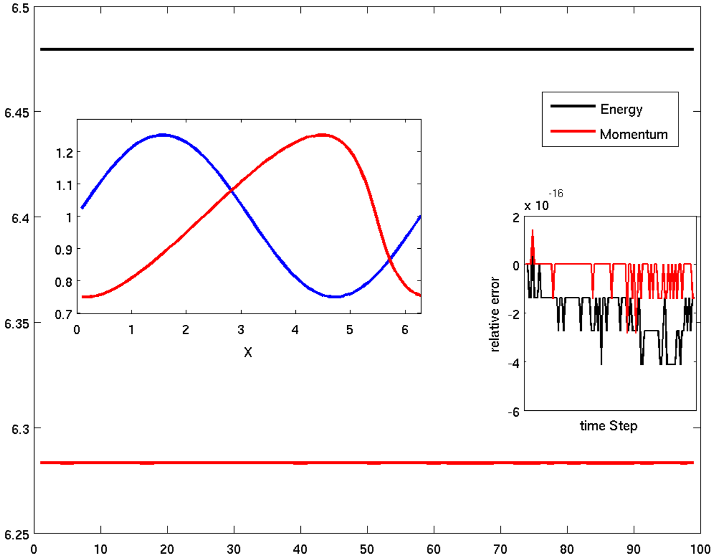
$$(u^{n+1/2})^T (u^{n+1} - u^n) + \frac{1}{3} \underbrace{(u^{n+1/2})^T D^{u^{n+1/2}} u^{n+1/2}}_{=0}$$

$$= \frac{1}{2}(u^n + u^{n+1})^T (u^{n+1} - u^n)$$

$$= \frac{(u^{n+1})^2}{2} - \frac{(u^n)^2}{2} = 0$$

[1] Verstappen, Veldman, J. Com. Phys 187, p. 343 (2003)

Numerical example: Burgers' Equation



Overview

Skew Symmetric discretisation

Euler Momentum Equation

divergence and convection form

$$\begin{aligned} \partial_t(\varrho u) + \partial_x(\varrho u^2) + \partial_x p &= 0 \quad (D) \\ \varrho\partial_t(u) + \varrho u \partial_x(u) + \partial_x p &= 0 \quad (C) \end{aligned}$$

by ([D]+[C])/2

Skew symmetric form

$$\frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u \varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p = 0 \quad (S)$$

Skew Symmetric discretisation

Euler Equations

$$\begin{aligned} \partial_t \varrho + \partial_x(u\varrho) &= 0 \\ \frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u \varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p &= 0 \end{aligned}$$

Skew Symmetric discretisation

Euler Equations

$$
\begin{aligned}
\partial_t \varrho + \partial_x (u \varrho) &= 0 \\
\frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u \varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p &= 0 \\
\partial_t \left(\frac{p}{\gamma - 1} \qquad \right) + \partial_x \left(u \left(\frac{p}{\gamma - 1} + p \qquad \right) \right) &+ \\
\partial_t \left(\qquad \varrho u^2/2 \right) + \partial_x \left(u \left(\qquad \varrho u^2/2 \right) \right) &= 0
\end{aligned}
$$

Skew Symmetric discretisation

Skew symmetric Euler Equations

$$\begin{aligned} \partial_t \varrho + \partial_x (u\varrho) &= 0 \\ \frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u\varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p &= 0 \\ \frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1}\partial_x (up) - u\partial_x p &= 0 \end{aligned}$$

Skew Symmetric discretisation

Skew symmetric Euler Equations

$$\begin{aligned}
\partial_t \varrho + \partial_x (u\varrho) &= 0 \\
\frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u \varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p &= 0 \\
\frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1}\partial_x \left(up\right) - u\partial_x p &= 0
\end{aligned}$$

Discretised

$$\begin{aligned}
\partial_t \varrho + (DU)\varrho &= 0 \\
\frac{1}{2}(\partial_t \varrho + \varrho \partial_t)u + \frac{1}{2}(DUR + RUD)u + Dp &= 0 \\
\frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1}(DU)p - (UD)p &= 0
\end{aligned}$$

with $U = diag(u)$, $R = diag(\varrho)$, Derivative $D = -D^T$

Skew Symmetric discretisation

Skew symmetric Navier–Stokes Equations

$$
\begin{aligned}
\partial_t \varrho + \partial_x(u\varrho) &= 0 \\
\frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u \varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p &= \partial_x \tau \\
\frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1}\partial_x (up) - u\partial_x p &= -u\partial_x \tau + \partial_x \tau
\end{aligned}
$$

Discretised

$$
\begin{aligned}
\partial_t \varrho + (DU)\varrho &= 0 \\
\frac{1}{2}(\partial_t \varrho + \varrho \partial_t)u + \frac{1}{2}(DUR + RUD)u + Dp &= D\tau \\
\frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1}(DU)p - (UD)p &= -UD\tau + D\tau u
\end{aligned}
$$

with $U = diag(u)$, $R = diag(\varrho)$, Derivative $D = -D^T$, $\tau = \mu \partial_x u$

Conservation, time continuous

$$\begin{aligned}\partial_t \varrho + DU\varrho &= 0\\ \frac{1}{2}(\partial_t \varrho + \varrho\partial_t)u + \frac{1}{2}\underbrace{(DUR + RUD)}_{D^{\mathbf{u}\varrho}}u + Dp &= 0\\ \frac{1}{\gamma-1}\partial_t p + \frac{\gamma}{\gamma-1}DUp - UDp &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{1}^T(mass) = 0 \quad &\rightarrow \quad \text{mass Conservation}\\ \mathbf{1}^T(mom) + u^T(mass)/2 = 0 \quad &\rightarrow \quad \text{Momentum Conservation}\\ \mathbf{1}^T(innerE) + u^T(mom) = 0 \quad &\rightarrow \quad \text{Energy Conservation}\end{aligned}$$

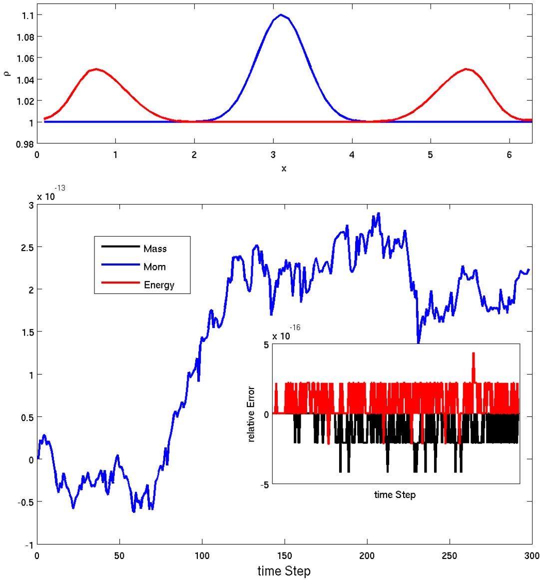
Conservation, time continuous

$$
\begin{aligned}
\partial_t \varrho + DU\varrho &= 0 \\
\frac{1}{2}(\partial_t \varrho + \varrho \partial_t) u + \frac{1}{2} \underbrace{(DUR + RUD)}_{D^{\mathbf{u}\varrho}} u + Dp &= 0 \\
\frac{1}{\gamma - 1} \partial_t p + \frac{\gamma}{\gamma - 1} DUp - UDp &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{1}^T(mass) = 0 &\quad \rightarrow \quad \text{mass Conservation} \\
\mathbf{1}^T(mom) + u^T(mass)/2 = 0 &\quad \rightarrow \quad \text{Momentum Conservation} \\
\mathbf{1}^T(innerE) + u^T(mom) = 0 &\quad \rightarrow \quad \text{Energy Conservation}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\gamma - 1} \partial_t \mathbf{1}^T p + \frac{1}{2} u^T (\partial_t \varrho + \varrho \partial_t) u &= \partial_t \left(\frac{1}{\gamma - 1} \mathbf{1}^T p + u^T (\varrho u)/2 \right) \\
+ \frac{\gamma}{\gamma - 1} \mathbf{1}^T DUp + \frac{1}{2} u^T D^{\mathbf{u}\varrho} u &= 0
\end{aligned}
$$

Numerical example 1D



Time Integration: Leapfrog–like

Time discretization

$$\frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u \varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p = 0$$

One Step methods:

- Morinishi's rewriting $\frac{1}{2}\left(\partial_t \rho \cdot + \rho \partial_t \cdot\right) u \quad = \quad \sqrt{\rho}\partial_t \sqrt{\rho} u$
- use adopted midpoint rule
- $\longrightarrow$ similar to Morinishi , Subbareddy et. al., JCP 2009
- generalises to higher order

Time discretization

$$\frac{1}{2}\left(\partial_t \varrho \cdot + \varrho \partial_t \cdot\right) u + \frac{1}{2}\left(\partial_x u \varrho \cdot + \varrho u \partial_x \cdot\right) u + \partial_x p = 0$$

One Step methods:

- Morinishi's rewriting $\frac{1}{2}\left(\partial_t \rho \cdot + \rho \partial_t \cdot\right) u \quad = \quad \sqrt{\rho}\partial_t \sqrt{\rho} u$
- use adopted midpoint rule
- $\longrightarrow$ similar to Morinishi , Subbareddy et. al., JCP 2009
- generalises to higher order

$$\sqrt{\rho}\partial_t \left(\sqrt{\rho} u\right) + \frac{1}{2} D^{\mathbf{u}\rho} u + D_x p = 0$$

Time discretization

$$\begin{aligned}
\sqrt{\rho}\partial_t\sqrt{\rho} + \frac{1}{2}B^{\mathbf{u}}\rho &= 0 \\
\sqrt{\rho}\partial_t\left(\sqrt{\rho}u\right) + \frac{1}{2}D^{\mathbf{u}\rho}u + D_x p &= 0 \\
\frac{1}{\gamma-1}\partial_t p + \frac{\gamma}{\gamma-1}B^{\mathbf{u}}p - C^{\mathbf{u}}p &= 0
\end{aligned}$$

$$\begin{aligned}
\sqrt{\rho}^{n+1/2}\frac{\left(\sqrt{\rho}^{n+1} - \sqrt{\rho}^{n}\right)}{\Delta t} + \frac{1}{2}B^{\mathbf{u}^{n+a}}\rho^{n+b} &= 0 \\
\sqrt{\rho}^{n+1/2}\frac{\left(\sqrt{\rho}u\right)^{n+1} - \left(\sqrt{\rho}u\right)^{n}}{\Delta t} + \frac{1}{2}D^{\mathbf{u}^{n+a}\rho^{n+b}}u^{n+1/2} + D_x p^{n+c} &= 0
\end{aligned}$$

$$\sqrt{\rho}^{n+1/2} = \tfrac{1}{2}\left(\sqrt{\rho}^{n} + \sqrt{\rho}^{n+1}\right)$$

Time discretization

$$
\begin{aligned}
\sqrt{\rho}\partial_t\sqrt{\rho} + \frac{1}{2}B^{\mathbf{u}}\rho &= 0\\
\sqrt{\rho}\partial_t\left(\sqrt{\rho}u\right) + \frac{1}{2}D^{\mathbf{u}\rho}u + D_x p &= 0\\
\frac{1}{\gamma-1}\partial_t p + \frac{\gamma}{\gamma-1}B^{\mathbf{u}}p - C^{\mathbf{u}}p &= 0
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\rho}^{n+1/2}\frac{\left(\sqrt{\rho}^{n+1}-\sqrt{\rho}^{n}\right)}{\Delta t} + \frac{1}{2}B^{\mathbf{u}^{n+a}}\rho^{n+b} &= 0\\
\sqrt{\rho}^{n+1/2}\frac{\left(\sqrt{\rho}u\right)^{n+1}-\left(\sqrt{\rho}u\right)^{n}}{\Delta t} + \frac{1}{2}D^{\mathbf{u}^{n+a}\rho^{n+b}}u^{n+1/2} + D_x p^{n+c} &= 0
\end{aligned}
$$

$$\sqrt{\rho}^{n+1/2} = \tfrac{1}{2}\left(\sqrt{\rho}^{n} + \sqrt{\rho}^{n+1}\right) \qquad u^{n+1/2} = \frac{1}{2}\frac{(\sqrt{\rho}u)^{n+1} + (\sqrt{\rho}u)^{n}}{\sqrt{\rho}^{n+1/2}}$$

Time discretization

Higher Order

$$
\begin{aligned}
\sqrt{\rho}\partial_t\sqrt{\rho} + \frac{1}{2}B^{\mathbf{u}}\rho &= 0 \\
\sqrt{\rho}\partial_t\left(\sqrt{\rho}u\right) + \frac{1}{2}D^{\mathbf{u}\rho}u + D_x p &= 0 \\
\frac{1}{\gamma-1}\partial_t p + \frac{\gamma}{\gamma-1}B^{\mathbf{u}}p - C^{\mathbf{u}}p &= 0
\end{aligned}
$$

- Implicit Midpoint rule is Gauss-collocation method
- *All* Gauss-collocation methods preserve skew-symme. & cons.†

Midpoint (2^{nd} order)

$$
\begin{array}{c|c}
\frac{1}{2} & \frac{1}{2} \\
\hline
 & 1
\end{array}
$$

Midpoint (4^{th} order)

$$
\begin{array}{c|cc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

†Brouwer, Reiss, in prep.

Overview

Euler Equations, more dimensions

Skew–symmetric momentum equation

$$\sqrt{\rho}\partial_t\left(\sqrt{\rho}u_\alpha\right) + \frac{1}{2}\left[\partial_{x_\beta}\rho u_\beta \cdot + \rho u_\beta \partial_{x_\beta}\cdot\right] u_\alpha + \partial_{x_\alpha} p = 0.$$

Main problem: Keep skew symmetric structure on distorted grids.[a]

Local base $\mathbf{e}_\alpha = \partial_{\xi^\alpha}\mathbf{r}$, $\mathbf{r} = \mathbf{r}(\xi, \eta, \zeta) = (x, y, z)^T$

[a] Veldman, Rinzema, Playing with nonuniform grids, J. Engin. and Math.26, p 119, (1992), VV2003

Euler Equations, more dimensions

Skew–symmetric momentum equation

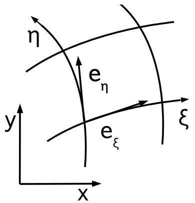
$$\sqrt{\rho}\partial_t\left(\sqrt{\rho}u_\alpha\right) + \frac{1}{2}\left[\partial_{x_\beta}\rho u_\beta \cdot + \rho u_\beta \partial_{x_\beta}\cdot\right] u_\alpha + \partial_{x_\alpha} p = 0.$$

Main problem: Keep skew symmetric structure on distorted grids.

Local base $\mathbf{e}_\alpha = \partial_{\xi^\alpha}\mathbf{r}$, $\mathbf{r} = \mathbf{r}(\xi,\eta,\zeta) = (x,y,z)^T$

Use divergence as

$$\begin{aligned}\frac{\partial u_\beta}{\partial x_\beta} &= \frac{1}{J}\sum_{\alpha,cy}\partial_{\xi^\alpha}(\mathbf{e}_\beta \times \mathbf{e}_\gamma)\mathbf{u} \\ &= \frac{1}{J}\sum_{\alpha,cy}(\mathbf{e}_\beta \times \mathbf{e}_\gamma)\partial_{\xi^\alpha}\mathbf{u}.\end{aligned}$$



Euler Equations, more dimensions

Skew–symmetric momentum equation

$$\sqrt{\rho}\partial_t(\sqrt{\rho}u_\alpha) + \frac{1}{2}\left[\partial_{x_\beta}\rho u_\beta \cdot + \rho u_\beta \partial_{x_\beta}\cdot\right] u_\alpha + \partial_{x_\alpha} p = 0.$$

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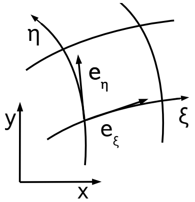
$$\frac{\partial u_\beta}{\partial x_\beta} = \frac{1}{J}\sum_{\alpha, cy} \partial_{\xi^\alpha}(\mathbf{e}_\beta \times \mathbf{e}_\gamma)\mathbf{u}$$

$$= \frac{1}{J}\sum_{\alpha, cy} (\mathbf{e}_\beta \times \mathbf{e}_\gamma)\partial_{\xi^\alpha}\mathbf{u}.$$



$$[\partial_{x_\beta}\rho u_\beta \cdot \quad + \quad \rho u_\beta \partial_{x_\beta}\cdot] u_\alpha$$

$$\sim \sum_{\alpha, cy} [\partial_{\xi^\alpha}(\mathbf{e}_\beta \times \mathbf{e}_\gamma)\mathbf{u}\varrho \cdot \quad + \quad \mathbf{u}\varrho(\mathbf{e}_\beta \times \mathbf{e}_\gamma)\partial_{\xi^\alpha}\cdot] u_\alpha$$

Euler Equations in 2D

Semidiscrete

$$
\begin{aligned}
J\partial_t\rho + B^u\rho &= 0\\
J\sqrt{\rho}\partial_t(\sqrt{\rho}u) + \frac{1}{2}D^{\mathbf{u}\rho}u + D_x p &= 0\\
J\sqrt{\rho}\partial_t(\sqrt{\rho}v) + \frac{1}{2}D^{\mathbf{u}\rho}v + D_y p &= 0\\
J\frac{1}{\gamma-1}\partial_t p + \frac{\gamma}{\gamma-1}B^u p - C^u p &= 0
\end{aligned}
$$

with

$$
\begin{aligned}
B^{\mathbf{u}} &= D_\xi\tilde{U} + D_\eta\tilde{V}\\
D^{\mathbf{u}\rho} &= (D_\xi\tilde{U}R + R\tilde{U}D_\xi) + (D_\eta\tilde{V}R + R\tilde{V}D_\eta)\\
C^{\mathbf{u}} &= UD_x + VD_y
\end{aligned}
$$

where $\tilde{U} = (UY_\eta - VX_\eta)$ and $\tilde{V} = (VX_\xi - UY_\xi)$
and $D_x = D_\xi Y_\eta - D_\eta Y_\xi$, $D_y = \ldots$

Overview

Boundary conditions

Flux vs. Value, 1D mass equation

$$\partial_t \rho + D_x u\rho = 0 \qquad \longrightarrow \qquad \partial_t \mathbf{1}^T \rho + \underbrace{\mathbf{1}^T D_x}_{b^T} u\rho = 0$$

$$b^T = \mathbf{1}^T \begin{pmatrix} -1 & 1 & & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & & \\ & -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & & -1 & 1 \end{pmatrix} = (-\frac{3}{2}, \frac{1}{2}, 0, \cdots, 0, -\frac{1}{2}, \frac{3}{2})$$

mass flux over boundaries:

$$b^T u\rho = -\left((\rho u)_1 \frac{3}{2} - (\rho u)_2 \frac{1}{2}\right) + \left((\rho u)_N \frac{3}{2} - (\rho u)_{N-1} \frac{1}{2}\right) = -f_0 + f_N$$

Flux no-zero even for $u_1 = 0$

Boundary conditions

Would like $u_1 = 0 \Leftrightarrow f_0 = 0$

$$Wu' = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & -\frac{1}{2} & 0 & \frac{1}{2} & \\ & & \ddots & \ddots & \ddots \\ & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} u.$$

$\longrightarrow b^T = (-1, 0, \ldots, 0, 1)$.

weigh- matrix W (norm)

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$\longrightarrow b^T = (-1, 0, \dots, 0, 1)$.

weigh- matrix W (norm)

$$W_{ij} = diag(\frac{1}{2}, 1, 1, \dots, 1, \frac{1}{2})$$

Summation By Parts (SBP)-Property

Boundary conditions: SBP

For SBP derivatives there is a norm, such that[2]

- Telescoping sum is broken at boundary
- Skew Symmetry only broken at $D_{1,1}$ and $D_{N,N}$

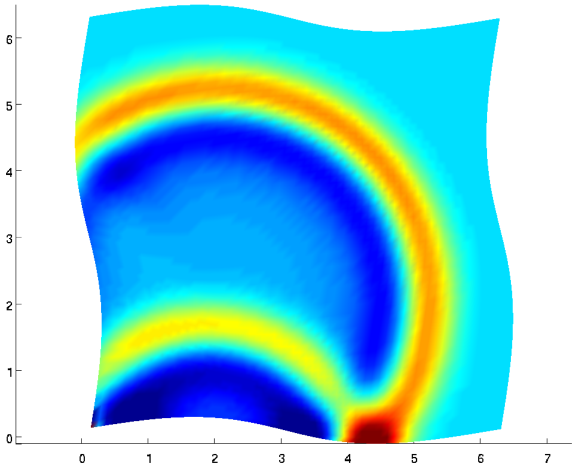
Boundary flux of our scheme depends entirely

- on boundary values,
- addition flux to enforce BC

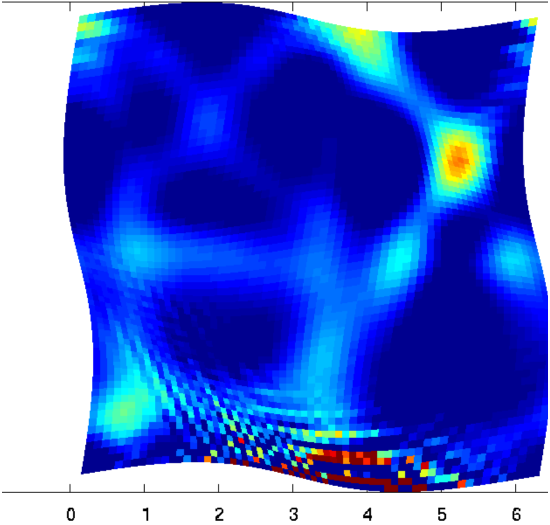
Enforcing BC e.g. in Carpenter[3] give global energy estimates.
(Work in progress. . .)

[2] Strand, JCP 110, p47, 1994
[3] Carpenter et al, JCP 108, p272, 1993

Skew Symmetric discretisation

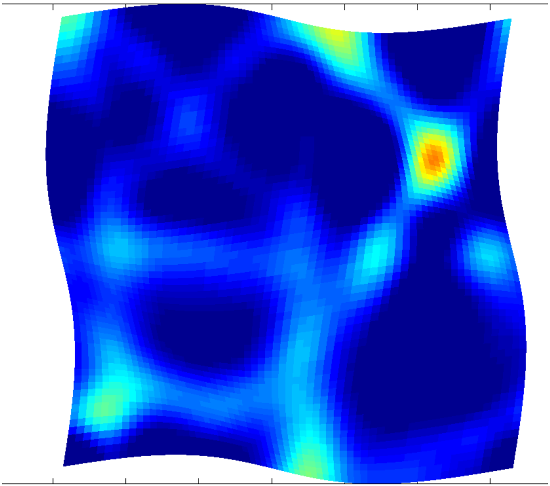


Skew Symmetric discretisation



Without SBP

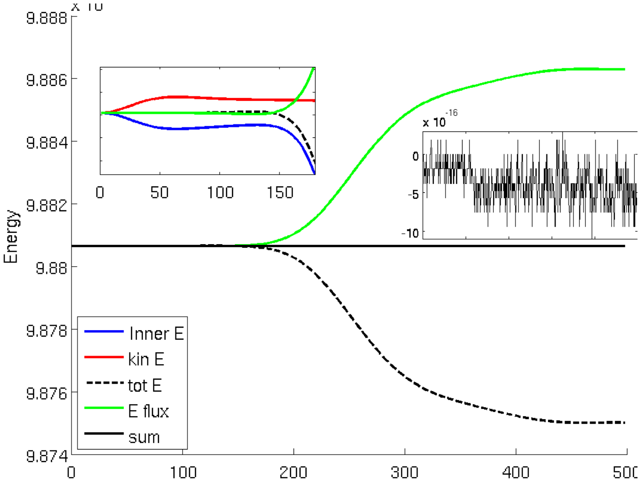
Skew Symmetric discretisation



With SBP

Skew Symmetric discretisation

Conservation with boundary



Skew Symmetric discretisation

First Conclusions

- Skew symmetry leads to kinetic energy conservation
- ... with telescoping sum property full conservation
- strict skew symmetry & perfect conservation on *transformed* grids
- Procedure easy to implement & numerically efficient
- Derivatives of any order in space
- Time stepping of any order

Skew Symmetric discretisation

First Conclusions

- Skew symmetry leads to kinetic energy conservation
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- Procedure easy to implement & numerically efficient
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Outlook

- Implementation for shock acoustic simulations in progress
- Big DNS of turbulent flows

Bonus Part

Schemes for the **incompressible** Navier-Stokes equation

Bonus Part

Schemes for the **incompressible** Navier-Stokes equation

How to avoid odd-even decoupling?

Idea:

- energy and momentum conserving, and **collocated**, by
  - use of skew symmetry
    &
  - the combination of non-symmetric derivatives
- Grid transformations preserving these properties
  - general in 2D
  - restricted in 3D

Incompressible Navier-Stokes

$$
\begin{aligned}
\partial_t \rho + \partial_{x_\alpha} \rho u_\alpha &= 0 \\
\sqrt{\rho}\partial_t \left(\sqrt{\rho} u_\alpha\right) + \frac{1}{2}\left(\partial_{x_\beta} \rho u_\beta \cdot + \rho u_\beta \partial_{x_\beta} \cdot\right) u_\alpha + \partial_{x_\alpha} p &= \partial_\beta \tau_{\alpha,\beta} \\
\frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1}\partial_{x_\beta}(u_\beta p) - u_\beta \partial_{x_\beta} p &= 0
\end{aligned}
$$

$$
\begin{aligned}
\partial_t u_\alpha + \frac{1}{2}\left(\partial_{x_\beta} u_\beta \cdot + u_\beta \partial_{x_\beta} \cdot\right) u_\alpha + \partial_{x_\alpha} p &= \nu \Delta u_\alpha \\
\partial_{x_\alpha} u_\alpha &= 0 \qquad \alpha, \beta = 1, 2, 3
\end{aligned}
$$

Pressure Poission Equation:

$$
\partial_{x_\alpha} \partial_{x_\alpha} p = \partial_{x_\alpha} \left(-D^{\mathbf{u}} u_\alpha + \nu \Delta u_\alpha\right)
$$

Incompressible Navier-Stokes

$$\begin{aligned} \partial_t u_\alpha + \frac{1}{2}\left(\partial_{x_\beta} u_\beta \cdot + u_\beta \partial_{x_\beta} \cdot\right) u_\alpha + \partial_{x_\alpha} p &= \nu \Delta u_\alpha \\ \partial_{x_\alpha} u_\alpha &= 0 \qquad \alpha, \beta = 1, 2, 3 \end{aligned}$$

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Pressure Poission Equation:

$$\partial_{x_\alpha} \partial_{x_\alpha} p = \partial_{x_\alpha} \left(-D^{\mathbf{u}} u_\alpha + \nu \Delta u_\alpha\right)$$

Discrete

$$\partial_{x_\alpha} \partial_{x_\alpha} p \sim D_\alpha G_\alpha p$$

$$\underbrace{(Gp)_i \sim (p_{i+1} - p_{i-1}) \quad \& \quad (Du)_i \sim (u_{i+1} - u_{i-1})}_{DGp \sim (p_{i+2} - 2p_i + p_{i-2})}$$

Incompressible Navier-Stokes

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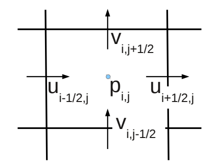
$$DGp \sim (p_{i+2} - 2p_i + p_{i-2})$$

Odd-even decoupling, nasty to solve!

Avoiding decoupling of Δp

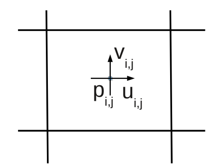
Staggered Grid



- (+) Simple
- (+) No extra damping

- (-) Boundary non-simple
- (-) Different geometry factors for p, u_α

Collocated Grid



- (+) Nice Boundary
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- (-) Upwind: Large damping
- (-) Rhie-Chow: Smaller Damping

Avoiding decoupling of Δp

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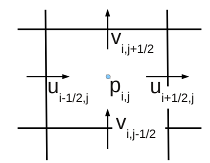

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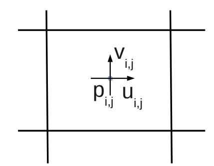

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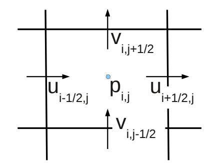
Skew Symmetric Schemes

Avoiding decoupling of Δp

Staggered Grid



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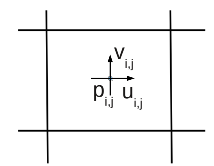

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Skew Symmetric Schemes

— ??? —

Discretisation

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\begin{aligned}
\partial_{x_\alpha} u_\alpha &= 0 \\
\partial_t u_\alpha + \frac{1}{2}\left(\partial_{x_\beta} u_\beta \cdot + u_\beta \partial_{x_\beta} \cdot\right) u_\alpha + \partial_{x_\alpha} p &= \nu \Delta u_\alpha
\end{aligned}
$$

$$
\begin{aligned}
D_\alpha u_\alpha &= 0 \\
\partial_t u_\alpha + \underbrace{\frac{1}{2}\left(D_\beta U_\beta + U_\beta G_\beta\right)}_{D^{\mathbf{u}}} u_\alpha + G_\alpha p &= \nu L u_\alpha
\end{aligned}
$$

Transport term is skew symmetric

$$
\boxed{{D^{\mathbf{u}}}^T = \tfrac{1}{2}(U_\beta D_\beta^T + G_\beta^T U_\beta) \equiv -D^{\mathbf{u}}}
$$

provided $\boxed{D_\beta = -G_\beta^T}$ **not necessary skew sym.!**

Discrete Poission Equation

$$\begin{aligned} D_\alpha u_\alpha &= 0 \\ \partial_t u_\alpha + D^{\mathbf{u}} u_\alpha + G_\alpha p &= \nu L u_\alpha \end{aligned}$$

$$D_\alpha G_\alpha p = D_\alpha \left(-D^{\mathbf{u}} u_\alpha + \nu L u_\alpha\right)$$

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$$

$$
\underbrace{D_\beta = -G_\beta^T}_{\rightarrow \text{ Energy cons.!}} = \frac{1}{\Delta h} \begin{pmatrix} \ddots & & \\ -1 & 1 & 0 \\ & & \ddots \end{pmatrix}
$$

Not upwind!

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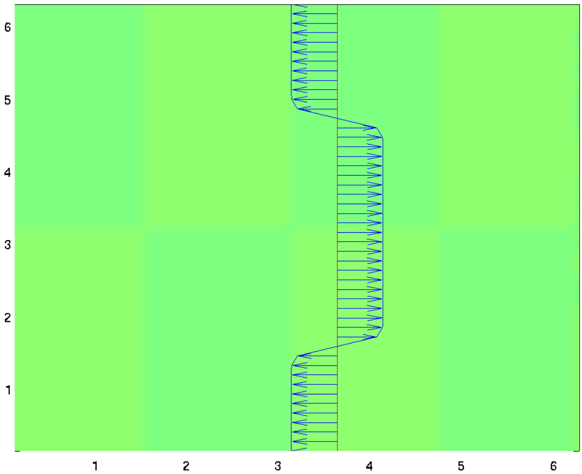
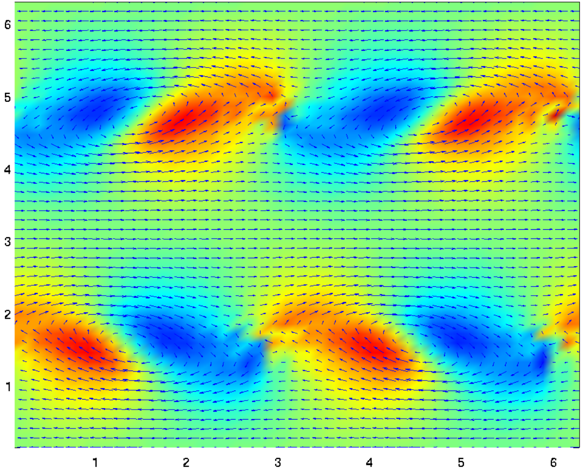
Not upwind!

$$D_\beta = \begin{pmatrix} & \ddots & & & \\ 1/6 & -1 & 1/2 & 1/3 & 0 \end{pmatrix}$$

Numerical example



Numerical example



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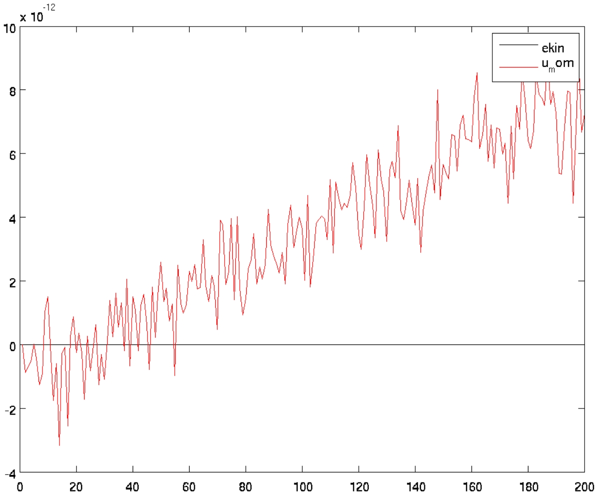

Transformed Grids

$$\begin{aligned} D_\alpha u_\alpha &= 0 \\ \partial_t u_\alpha + \frac{1}{2}\left(D_\beta U_\beta + U_\beta G_\beta\right) u_\alpha + G_\alpha p &= \nu L u_\alpha \end{aligned}$$

Transformed Grids

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\end{aligned}
$$

$$
\begin{aligned}
\bar{D}_\beta M_{\alpha,\beta} u_\alpha &= 0 \\
J\partial_t u_\alpha + \frac{1}{2}\left(\bar{D}_\gamma M_{\beta,\gamma} U_\beta + U_\beta M_{\beta,\gamma} \bar{G}_\gamma\right) u_\alpha + M_{\alpha,\gamma} \bar{G}_\gamma p &= \nu L u_\alpha
\end{aligned}
$$

$$
\boxed{\bar{D}_\beta = -\bar{G}_\beta^T}
$$

Transformed Grids

$$\begin{aligned} D_\alpha u_\alpha &= 0 \\ \partial_t u_\alpha + \frac{1}{2}\left(D_\beta U_\beta + U_\beta G_\beta\right) u_\alpha + G_\alpha p &= \nu L u_\alpha \end{aligned}$$

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$$\boxed{\bar{D}_\beta = -\bar{G}_\beta^T}$$

$$\Delta\, p \sim \bar{D}_\beta \frac{M_{\alpha,\beta} M_{\alpha,\gamma}}{J} \bar{G}_\gamma\, p = \ldots$$

Transformed Grids: Conservation?

$$\begin{aligned} \bar{D}_\beta M_{\alpha,\beta} u_\alpha &= 0 \\ J\partial_t u_\alpha + \frac{1}{2}\left(\bar{D}_\gamma M_{\beta,\gamma} U_\beta + U_\beta M_{\beta,\gamma} \bar{G}_\gamma\right) u_\alpha + M_{\alpha,\gamma} \bar{G}_\gamma p &= \nu L u_\alpha \end{aligned}$$

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$$\boxed{\bar{D}_\beta = -\bar{G}_\beta^T}$$

Energy conservation

$$\begin{aligned} & u_\alpha^T M_{\alpha,\gamma} \bar{G}_\gamma p \\ = \; & p^T \bar{G}_\gamma^T M_{\alpha,\gamma} u_\alpha \\ = \; & -p^T \bar{D}_\gamma M_{\alpha,\gamma} u_\alpha \\ = \; & 0 \end{aligned}$$

Transformed Grids: Conservation?

$$\bar{D}_\beta M_{\alpha,\beta} u_\alpha = 0$$

$$J\partial_t u_\alpha + \frac{1}{2}\left(\bar{D}_\gamma M_{\beta,\gamma} U_\beta + U_\beta M_{\beta,\gamma} \bar{G}_\gamma\right) u_\alpha + M_{\alpha,\gamma} \bar{G}_\gamma p = \nu L u_\alpha$$

$$\boxed{\bar{D}_\beta = -\bar{G}_\beta^T}$$

Energy conservation

$$\begin{aligned} & u_\alpha^T M_{\alpha,\gamma} \bar{G}_\gamma p \\ = \; & p^T \bar{G}_\gamma^T M_{\alpha,\gamma} u_\alpha \\ = \; & -p^T \bar{D}_\gamma M_{\alpha,\gamma} u_\alpha \\ = \; & 0 \end{aligned}$$

Momentum conservation

$$\begin{aligned} & \mathbf{1}^T M_{\alpha,\gamma} \bar{G}_\gamma p \\ = \; & p^T \bar{G}_\gamma^T m_{\alpha,\gamma} \\ = \; & -p^T \bar{D}_\gamma m_{\alpha,\gamma} \\ = \; & ? \end{aligned}$$

Transformed Grids: Momentum Conservation

Transformed Grids: Momentum Conservation

2D

$$\bar{D}_\gamma m_{\alpha,\gamma} = \begin{pmatrix} \bar{D}_\xi y_\eta - \bar{D}_\eta y_\xi \\ \dots \end{pmatrix} = 0$$

Provided $y_\xi = \bar{D}_\xi y \qquad y_\eta = \bar{D}_\eta y$

Transformed Grids: Momentum Conservation

2D

$$\bar{D}_\gamma m_{\alpha,\gamma} = \begin{pmatrix} \bar{D}_\xi y_\eta - \bar{D}_\eta y_\xi \\ \dots \end{pmatrix} = 0$$

Provided $\quad y_\xi = \bar{D}_\xi y \qquad y_\eta = \bar{D}_\eta y$

3D

$$\bar{D}_\gamma^T m_{\alpha,\gamma} =$$
$$\begin{pmatrix} \bar{D}_\xi(y_\eta z_\zeta - y_\zeta z_\eta) + \bar{D}_\eta(y_\zeta z_\xi - y_\xi z_\zeta) + \bar{D}_\zeta(y_\xi z_\eta - y_\eta z_\xi) \\ \dots \end{pmatrix}$$

Only zero for restricted transformations:

$$x(\xi,\eta), y(\xi,\eta), z(\zeta)$$

Second Conclusions

We found
Schemes for the incompressible Navier-Stokes equation, wich are

- collocated, energy and momentum conserving, by
  - use of skew symmetry
    &
  - the combination of non-symmetric derivatives
- Grid transformations preserving these properties
  - general in 2D
  - restricted in 3D
- Very simple to implement

Conclusions

Skew symmetric Schemes are

- Skew symmetric schemes respect kinetic energy
- Avoid numerical damping
- Stable
- Simple to implement
- ... worth a try!

Thank you for your attention!

Question?

Euler Equations, more dimensions

$$\sqrt{\rho}\partial_t\left(\sqrt{\rho}u_\alpha\right) + \frac{1}{2}\left[\partial_{x_\beta}\rho u_\beta \cdot + \rho u_\beta \partial_{x_\beta}\cdot\right] u_\alpha + \partial_{x_\alpha} p \quad = \quad 0.$$

Momentum equation in skew symmetric form, 2D, $u_1 \equiv u$ component :

$$\begin{aligned}&\frac{J}{\sqrt{\rho}}\partial_t\left(\sqrt{\rho}u\right)\\ &\quad+\frac{1}{2}\Big[(\partial_\xi \varrho(y_\eta u - x_\eta v)\cdot + \varrho(y_\eta u - x_\eta v)\partial_\xi \cdot)\\ &\qquad + (\partial_\eta \varrho(-y_\xi u + x_\xi v)\cdot + \varrho(-y_\xi u + x_\xi v)\partial_\eta \cdot)\Big] u\\ &\qquad + (\partial_\xi y_\eta - \partial_\eta y_\xi) p = 0\end{aligned}$$

Shock & Artifical Damping

$$\frac{1}{2}(\partial_t R + R\partial_t)u + \frac{1}{2}(DUR + RUD)u + Dp \quad = \quad D\tau$$

Friction Term $\tau = \mu D u$

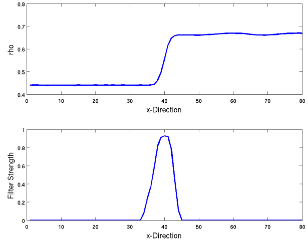
$$D\mu D u \quad \rightarrow \quad -F\sigma F^T u$$

with adaptive sigama:

$$\sigma = \frac{1}{2}\left(1 - \frac{r_{th}}{r_i} + \left|1 - \frac{r_{th}}{r_i}\right|\right)$$

with with shock detector building on dilatation[1]

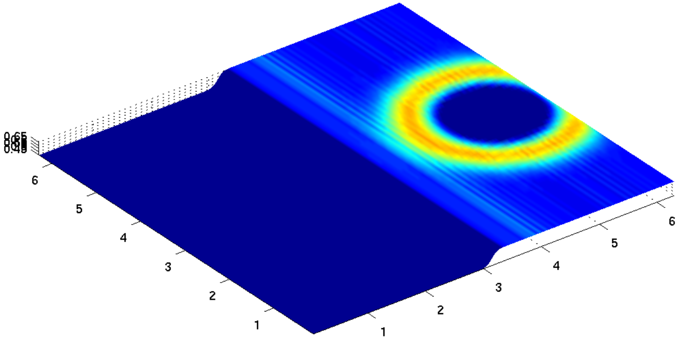
$$r_i = r_i(\nabla u)$$



[4]Bogey et al, JCP 228, 1447, 2009

Where are we now?

Shock & Acousic Pulse



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Skew Symmetric discretisation

Euler Equations, time

$$
\begin{aligned}
\partial_t \varrho + B^u \varrho &= 0 \\
\frac{1}{2}[\partial_t R + R\partial_t]u + \frac{1}{2} D^{u\varrho} u + D_x p &= 0 \\
\frac{1}{\gamma - 1}\partial_t p + \frac{\gamma}{\gamma - 1} B^u p - C^u p &= 0
\end{aligned}
$$

Leap–Frog like scheme

$$
\begin{aligned}
\tfrac{1}{2\Delta t}\left(\varrho^{n+1} - \varrho^{n-1}\right) \quad & + B^{u^n} \varrho^n = 0 \\
\tfrac{1}{4\Delta t}\left((R^{n+1} + R^n)u^{n+1} - (R^{n-1} + R^n)u^{n-1}\right) \quad & + \frac{1}{2} D^{u^n \varrho^n} u^n + D_x p^n = 0 \\
\tfrac{1}{\gamma-1}\tfrac{1}{2\Delta t}\left(p^{n+1} - p^{n-1}\right) \quad & + \gamma \frac{B^{u^n} p^n}{\gamma - 1} - C^{u^n} p^n = 0
\end{aligned}
$$

Skew Symmetric discretisation

Euler Equations, implicit time

mass

$$\frac{1}{2\Delta t}\left(\varrho^{n+1}-\varrho^{n-1}\right)+\frac{1}{8}D\left((u\varrho)^{n-1}+6(u\varrho)^{n}+(u\varrho)^{n+1}\right)=0$$

velocity

$$\begin{aligned}&\frac{1}{4\Delta t}\left((R^{n+1}+R^{n})u^{n+1}-(R^{n-1}+R^{n})u^{n-1}\right)+\\&\frac{1}{2}\frac{1}{8}\left(D^{(u\varrho)^{n-1}}u^{n-1}+D^{(u\varrho)^{n}}(u^{n-1}+4u^{n}+u^{n+1})+D^{(u\varrho)^{n+1}}u^{n+1}\right)\\&+\frac{1}{4}D_x(p^{n-1}+2p^{n}+p^{n+1})=0\end{aligned}$$

pressure

$$\begin{aligned}0=&\frac{1}{\gamma-1}\frac{1}{2\Delta t}\left(p^{n+1}-p^{n-1}\right)+\\&\frac{\gamma}{4\gamma-1}D\left((up)^{n-1}+2(up)^{n}+(up)^{n+1}\right)-\frac{1}{4}u^{n}D(p^{n-1}+2p^{n}+p^{n+1})\end{aligned}$$

Time Integration

Energy conservation

Time integration by midpoint rule produces energy conservation

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3} D^{u^{n+1/2}} u^{n+1/2} = 0$$

Multiplying by $(u^{n+1/2})^T = \frac{1}{2}(u^n + u^{n+1})^T$

$$\frac{1}{2}(u^n + u^{n+1})^T \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3} \underbrace{(u^{n+1/2})^T D^{u^{n+1/2}} u^{n+1/2}}_{=0} = 0$$

Time Integration
Energy conservation

Time integration by midpoint rule produces energy conservation

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3} D^{u^{n+1/2}} u^{n+1/2} = 0$$

Multiplying by $(u^{n+1/2})^T = \frac{1}{2}(u^n + u^{n+1})^T$

$$\frac{1}{2}(u^n + u^{n+1})^T \frac{u^{n+1} - u^n}{\Delta t} = 0$$

gives

$$\frac{1}{2}\left((u^{n+1})^T u^{n+1} - (u^n)^T u^n\right) = 0$$

Time Integration

Momentum conservation

Time integration by midpoint rule produces momentum conservation
Multiplying by $(\vec{1})^T = (1, 1, 1, 1, 1, \dots)$

$$(\vec{1})^T \frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{3}(\vec{1})^T D^{u^{n+1/2}} u^{n+1/2} = 0$$

with

$$\begin{aligned} & (\vec{1})^T D^{u^{n+1/2}} u^{n+1/2} \\ = \; & (\vec{1})^T D U^{n+1/2} + (u^{n+1/2})^T D u^{n+1/2} = 0 \end{aligned}$$

gives

$$\frac{1}{2}\left(\sum_i u_i^{n+1} - \sum_i u_i^n\right) = 0$$

Euler Equations, more dimensions

$$\frac{1}{2}\left(\partial_t \varrho + \varrho \partial_t\right) u_i + \frac{1}{2}\left[\nabla \varrho \vec{u} + \varrho \vec{u} \cdot \nabla\right] u_i + \nabla p \quad = \quad 0.$$

Use divergence as

$$\nabla \mathbf{u} = \frac{1}{J} \sum_{i,cy} \partial_{\xi^i} (\mathbf{e}_j \times \mathbf{e}_k) \mathbf{u} \quad = \quad \frac{1}{J} \sum_{i,cy} (\mathbf{e}_j \times \mathbf{e}_k) \partial_{\xi^i} \mathbf{u}.$$

Momentum equation in skew symmetric form

$$\frac{1}{2}\left(\partial_t \varrho + \varrho \partial_t\right) u_\alpha + \frac{1}{2}\frac{1}{J} \sum_{i,cy} \left(\partial_{\xi^i} (\mathbf{e}_j \times \mathbf{e}_k) \varrho \mathbf{u} + (\mathbf{e}_j \times \mathbf{e}_k) \varrho \mathbf{u} \partial_{\xi^i}\right) u_\alpha$$

$$+ \left(\frac{1}{J} \sum_{i,cy} \partial_{\xi^i} (\mathbf{e}_j \times \mathbf{e}_k) p\right)_\alpha = 0$$

. . . is indeed skew symmetric!

Time discretization

$$\begin{aligned} \partial_t \varrho + \quad & RHS_\varrho &=& \; 0 \\ \frac{1}{2}[\partial_t R + R\partial_t]u + \quad & RHS_u &=& \; 0 \\ \frac{1}{\gamma - 1}\partial_t p + \quad & RHS_p &=& \; 0 \end{aligned}$$

Central scheme 3–step

$$\begin{aligned} \tfrac{1}{2\Delta t}\left(\varrho^{n+1} - \varrho^{n-1}\right) \quad & +RHS_\varrho = 0 \\ \tfrac{1}{4\Delta t}\left((R^{n+1} + R^n)u^{n+1} - (R^{n-1} + R^n)u^{n-1}\right) \quad & +RHS_u = 0 \\ \tfrac{1}{\gamma-1}\tfrac{1}{2\Delta t}\left(p^{n+1} - p^{n-1}\right) \quad & +RHS_p = 0 \end{aligned}$$

Skew Symmetric discretisation

Euler Equations, time

$$mass^{n+\frac{1}{2}} = \mathbf{1}^T \frac{(\varrho^n + \varrho^{n+1})}{2}$$

$$mom^{n+\frac{1}{2}} = \frac{(\varrho^n + \varrho^{n+1})^T}{2} \frac{(u^n + u^{n+1})}{2}$$

$$e_{tot}^{n+1/2} = \frac{\mathbf{1}^T}{\gamma - 1}(p^n + p^{n+1})/2 + \frac{(u^n)^T(\varrho^n + \varrho^{n+1})u^{n+1}}{4}$$

E.g. momentum conservation

$$mom^{1/2} - mom^{N-1/2} = \sum_{n=1}^{N-1} f_{1/2} - \sum_{n=1}^{N-1} f_{N-1/2}$$

$\rightarrow$ equivalent to FV for $\Delta t \rightarrow 0$

End