Vertex Centred discretization of two phase Darcy flows with discontinuous capillary pressures

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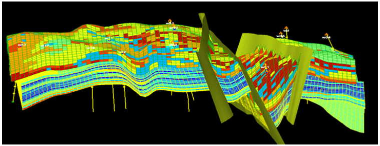
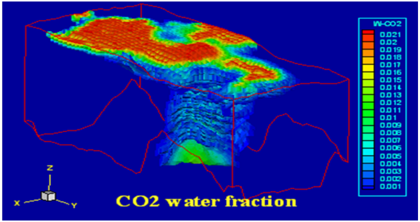
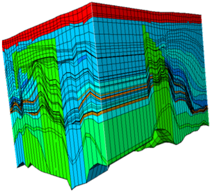
Applications of multiphase Darcy flows in geosciences



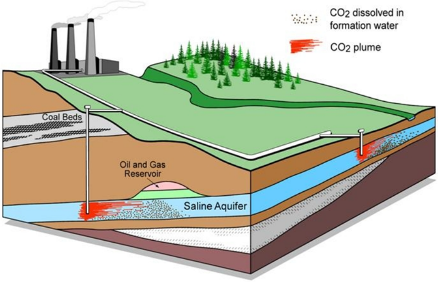




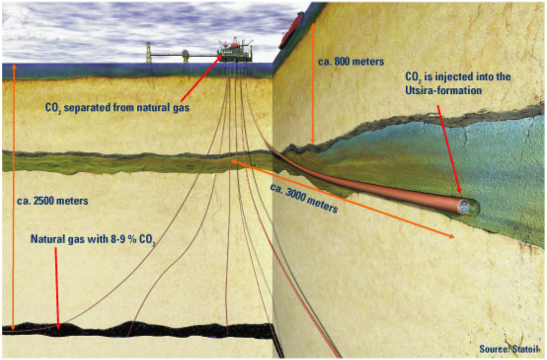
- Reservoir simulations
- CO2 geological storages
- Nuclear waste storages
- Hydrogeology

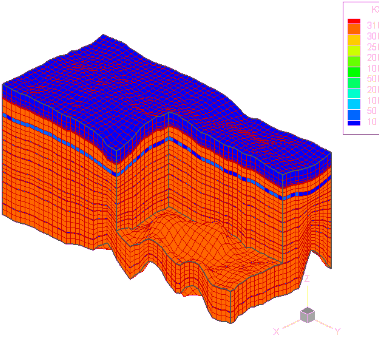
CO2 geological storage in saline aquifers

- Objectives
  - Optimize the CO2 injection
  - Risk assessment (leakage)
- Model
  - Compositional multiphase Darcy flow

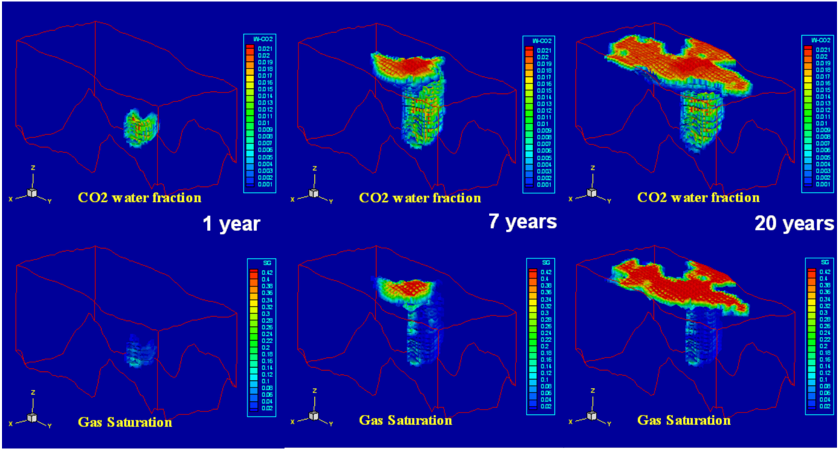


Example of injection of CO2 in the Sleipner field, aquifer Utsira

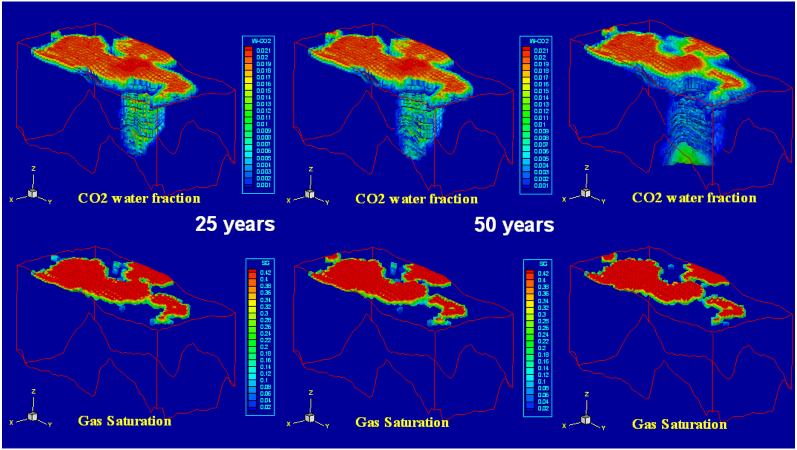




Example of injection of CO2 in the Sleipner field, aquifer Utsira: injection during 20 years



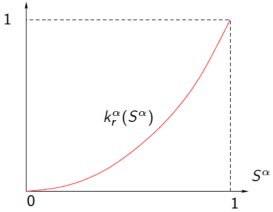
Example of injection of CO2 in the Sleipner field, aquifer Utsira: storage during 1000 years

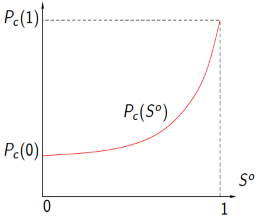


Two Phase (water - oil) Darcy flow model

S^w and S^o: volume fractions
P^w and P^o: phases pressures

$$\begin{cases} \mathbf{U}^w = -\dfrac{k_r^w(S^w)}{\mu^w} K(x) \left(\nabla P^w - \rho^w \mathbf{g}\right), \\[2ex] \mathbf{U}^o = -\dfrac{k_r^o(S^o)}{\mu^o} K(x) \left(\nabla P^o - \rho^o \mathbf{g}\right), \\[2ex] P^o = P^w + P_c(S^o), \\[2ex] S^w + S^o = 1. \end{cases}$$





Immiscible two phase (water-oil) Darcy flow

- Conservation equations

$$\left\{\begin{array}{l} \partial_t\Big(\phi(x)\ \rho^o\ S^o\Big) + \mathrm{div}\,(\rho^o\ \mathbf{U}^o) = 0 \\[2ex] \partial_t\Big(\phi(x)\ \rho^w\ S^w\Big) + \mathrm{div}\,(\rho^w\ \mathbf{U}^w) = 0 \end{array}\right.$$

- Darcy velocities $\left\{\begin{array}{l} \mathbf{U}^o = -\dfrac{k_r^g(S^o)}{\mu^o}\ K(x)\ \Big(\nabla P^o - \rho^o\mathbf{g}\Big) \\ \mathbf{U}^w = -\dfrac{k_r^w(S^w)}{\mu^w}\ K(x)\ \Big(\nabla P^w - \rho^w\mathbf{g}\Big) \end{array}\right.$

- Closure laws $\left\{\begin{array}{l} S^w + S^o = 1, \\ P^o = P^w + P_c(S^o). \end{array}\right.$

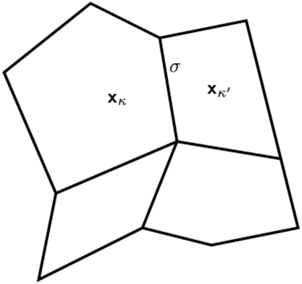
Cell centred finite volume discretization of Multiphase Darcy flows

- Discretization
- Cell centred unknowns: P_κ^α, S_κ^α, $\alpha = w, o$
- Discrete conservation laws on each cell κ:

$$\int_\kappa \Big((\phi\ \rho^o\ S^o)^n - (\phi\ \rho^o\ S^o)^{n-1} \Big) dx + \sum_{\kappa' \in \mathcal{M}_\kappa} \int_{\sigma=\kappa\kappa'} \int_{t^{n-1}}^{t^n} \rho^\alpha\ \mathbf{U}^\alpha \cdot \mathbf{n}_{\kappa\kappa'} d\sigma =$$

- Implicit Euler time integration
- Conservative approximation of the fluxes

$$\int_{\sigma=\kappa\kappa'} \rho^\alpha\ \mathbf{U}^\alpha \cdot \mathbf{n}_{\kappa\kappa'} d\sigma$$



Cell centred finite volume discretization of two phase Darcy flows

$$\phi_\kappa \frac{\left(\rho_\kappa^{\alpha,n}\ S_\kappa^{\alpha,n} - \rho_\kappa^{\alpha,n-1} S_\kappa^{\alpha,n-1}\right)}{\Delta t} + \sum_{\kappa' \in \mathcal{M}_\kappa} \left(\frac{\rho^\alpha\ k_{r_\alpha}(S^\alpha)}{\mu^\alpha}\right)^n_{up^\alpha_{\kappa\kappa'}} \ V^{\alpha,n}_{\kappa\kappa'} = 0$$

for all $\kappa \in \mathcal{M}$, $\alpha = w, o$, with $\phi_\kappa = \int_\kappa \phi(x) dx$ and

- the conservative discretization of the Darcy fluxes

$$V^\alpha_{\kappa\kappa'} = -V^\alpha_{\kappa'\kappa} \sim \int_{\kappa\kappa'} -K(x)\Big(\nabla P^\alpha + \rho^\alpha_{\kappa\kappa'}\ g\ \nabla Z\Big) \cdot \mathbf{n}_{\kappa\kappa'} d\sigma$$

- the upwinding $\quad up^\alpha_{\kappa\kappa'} = \begin{cases} \kappa \ \text{ if } \ V^\alpha_{\kappa\kappa'} \geq 0, \\ \kappa' \ \text{ if } \ V^\alpha_{\kappa\kappa'} < 0. \end{cases}$

Cell centered finite volume discretization of two phase darcy flows

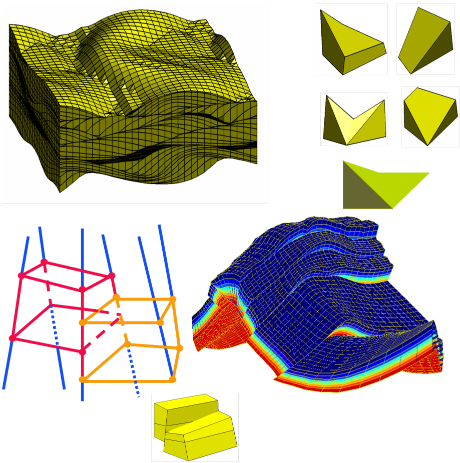
The local closure laws are coupled implicitely to the transport equations

$$\left\{ \begin{array}{l} S_\kappa^{w,n} + S_\kappa^{o,n} = 1, \\ \\ P^{o,n} = P^{w,n} + P_c(S^{o,n}). \end{array} \right.$$

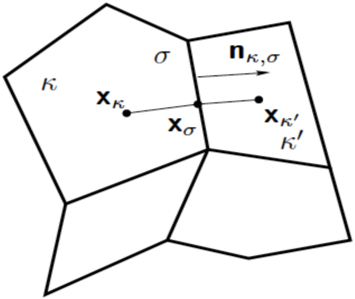
- The system is solved for the unknowns P^w,P^o, S^w, S^o using a Newton type algorithm at each time step
- The local closure laws are locally eliminated from the linear system which reduces to two primary unknown (say P^w, S^w) and two conservation equations for each cell κ.

Darcy fluxes $F_{\kappa\kappa'} \sim \int_{\kappa\kappa'} -K(x)\,\nabla u \cdot n_{\kappa\kappa'}\, ds$ discretization

- Meshes
  - Polyhedral cells
  - Non planar faces
  - Faults
- Heterogeneous anisotropic media



Cell centred Two Point Flux Approximation (TPFA) for isotropic heterogeneous media on admissible meshes



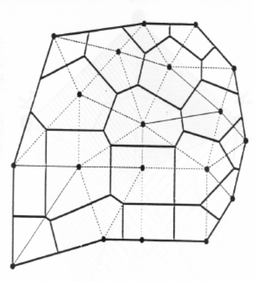
$$F_{\kappa,\kappa'}(u) = \frac{K_\kappa \; m_\sigma}{|x_\kappa x_\sigma|}(u_\kappa - u_\sigma) = \frac{K_{\kappa'} \; m_\sigma}{|x_{\kappa'} x_\sigma|}(u_\sigma - u_{\kappa'}),$$

$$F_{\kappa,\kappa'}(u) = T_{\kappa\kappa'}\Big(u_\kappa - u_{\kappa'}\Big) \text{ with } \frac{|x_\kappa x_{\kappa'}|}{T_{\kappa\kappa'}} = \frac{|x_\kappa x_\sigma|}{K_\kappa \; m_\sigma} + \frac{|x_{\kappa'} x_\sigma|}{K_{\kappa'} \; m_\sigma}.$$

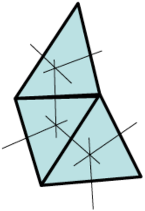
TPFA: admissible meshes

Exemples of admissible meshes

Voronoi



Triangles: angles $\leq \pi / 2$



Cartesian:

Finite volume approximation of Darcy fluxes on general meshes and permeability tensors

- Centred but not symmetric schemes: conditionally coercive and convergent:

$$F_{\kappa\kappa'}(u) = \sum_{L \in \mathcal{S}_{\kappa\kappa'}} T^L_{\kappa\kappa'} u_L$$

  - MPFA O Aavatsmark, Edwards

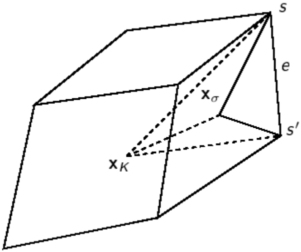
- Symmetric schemes but with with additional unknowns at faces:
  - VFH Eymard, Gallouet, Herbin
  - MFD Brezzi, Lipnikov, Shashkov

- Symmetric schemes but with with additional unknowns at vertices:
  - DDFV [Hermeline, Omnes, Boyer, Hubert, Coudière, ...]
  - VAG

Vertex Approximate Gradient (VAG) scheme [Eymard et al 2010]

- Tetrahedral submesh $\mathcal{T}$
- Interpolation at the face centres x_σ using the face nodal values
- $\mathbb{P}_1$ finite element discretization on $\mathcal{T}$ with interpolation at the face centres
- Nodal basis: $\eta_\kappa, \eta_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{V}_\kappa, \kappa \in \mathcal{M}$

$$\mathbf{x}_\sigma = \sum_{s \in \mathcal{V}_\sigma} \frac{1}{\mathrm{Card}\mathcal{V}_\sigma} \mathbf{x}_{\mathbf{s}}, \quad u_\sigma = \sum_{\mathbf{s} \in \mathcal{V}_\sigma} \frac{1}{\mathrm{Card}\mathcal{V}_\sigma} u_{\mathbf{s}}$$



Variational formulation and fluxes

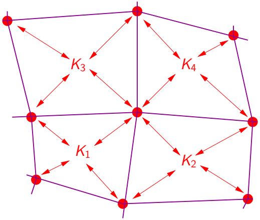
$$a(u_{\mathcal{T}}, v_{\mathcal{T}}) = \int_\Omega K(x)\nabla u_{\mathcal{T}}(x) \cdot \nabla v_{\mathcal{T}}(x)\ dx = \int_\Omega f(x)\ v_{\mathcal{T}}(x)\ dx$$

$$\begin{aligned} a(u_{\mathcal{T}}, v_{\mathcal{T}}) &= \sum_{\kappa\in\mathcal{M}} \sum_{s\in\mathcal{V}_\kappa} \Big(\int_\kappa -K(x)\nabla u_{\mathcal{T}}(x) \cdot \nabla \eta_{\mathbf{s}}(x) dx\Big)\Big(v_\kappa - v_{\mathbf{s}}\Big), \\ &= \sum_{\kappa\in\mathcal{M}} \sum_{s\in\mathcal{V}_\kappa} F_{\kappa,\mathbf{s}}(u_{\mathcal{T}})\ \Big(v_\kappa - v_{\mathbf{s}}\Big) \end{aligned}$$

with the fluxes $F_{\kappa,\mathbf{s}}(u_{\mathcal{T}}) = -F_{\mathbf{s},\kappa}(u_{\mathcal{T}}) = \displaystyle\int_\kappa -K(x)\nabla u_{\mathcal{T}} \cdot \nabla \eta_{\mathbf{s}}(x) dx.$

Equivalent discrete conservation laws

$$\begin{cases} \displaystyle\sum_{s \in \mathcal{V}_\kappa} F_{\kappa,\mathbf{s}}(u_\mathcal{T}) = \int_\kappa f(x)\eta_\kappa(x)\, dx \text{ for all } \kappa \in \mathcal{M}, \\ \displaystyle\sum_{\kappa \in \mathcal{M}_\mathbf{s}} F_{\mathbf{s},\kappa}(u_\mathcal{T}) = \int_\Omega f(x)\eta_\mathbf{s}(x)\, dx \text{ for all } \mathbf{s} \in \mathcal{V} \setminus \partial\Omega \end{cases}$$



Mass lumping: $$\begin{cases} \displaystyle\sum_{s \in \mathcal{V}_\kappa} F_{\kappa,\mathbf{s}}(u_\mathcal{T}) = m_\kappa f(x_\kappa) \text{ for all } \kappa \in \mathcal{M}, \\ \displaystyle\sum_{\kappa \in \mathcal{M}_\mathbf{s}} F_{\mathbf{s},\kappa}(u_\mathcal{T}) = m_\mathbf{s} f(x_\mathbf{s}) \text{ for all } \mathbf{s} \in \mathcal{V} \setminus \partial\Omega \end{cases}$$

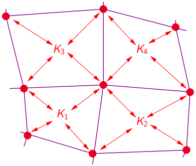
Application to two phase Darcy flows ($\alpha = o, w$).

The Darcy fluxes between κ and $\mathbf{s}$ are discretized by:

$$V^{\alpha}_{\kappa,\mathbf{s}} = F_{\kappa,\mathbf{s}}(P^{\alpha,n}_{\mathcal{T}}) + \rho^{\alpha}_{\kappa,\mathbf{s}}\; g\; F_{\kappa,\mathbf{s}}(Z_{\mathcal{T}}).$$

with the conservativity property $V^{\alpha}_{\mathbf{s},\kappa} = -V^{\alpha}_{\kappa,\mathbf{s}}$.

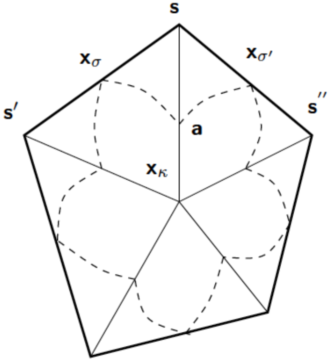
$$\left\{\begin{array}{l}
m_{\kappa}\phi_{\kappa}\dfrac{S^{\alpha,n}_{\kappa} - S^{\alpha,n-1}_{\kappa}}{\Delta t} + \displaystyle\sum_{\mathbf{s}\in\mathcal{V}_{\kappa}} \dfrac{k^{\alpha}_{r}(S^{\alpha,n}_{up^{\alpha}})}{\mu^{\alpha}} V^{\alpha}_{\kappa,\mathbf{s}} = 0, \kappa \in \mathcal{M}, \\[2ex]
m_{\mathbf{s}}\phi_{\mathbf{s}}\dfrac{S^{\alpha,n}_{\mathbf{s}} - S^{\alpha,n-1}_{\mathbf{s}}}{\Delta t} - \displaystyle\sum_{\kappa\in\mathcal{M}_{\mathbf{s}}} \dfrac{k^{\alpha}_{r}(S^{\alpha,n}_{up^{\alpha}})}{\mu^{\alpha}} V^{\alpha}_{\kappa,\mathbf{s}} = 0, \mathbf{s} \in \mathcal{V}\setminus\mathcal{V}_{D}, \\[2ex]
up^{\alpha} = \left\{\begin{array}{l} \kappa \text{ if } V^{\alpha}_{\kappa,\mathbf{s}} \geq 0, \\ \mathbf{s} \text{ if } V^{\alpha}_{\kappa,\mathbf{s}} < 0. \end{array}\right.
\end{array}\right.$$



Control Volume Finite Element (CVFE) interpretation of the VAG Fluxes in 2D

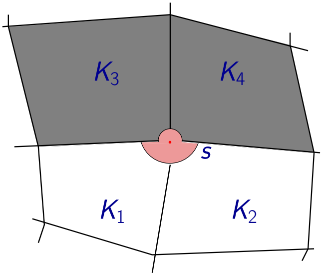
$$F_{\kappa,\mathbf{s}}(u_{\mathcal{T}}) = \int_{\kappa} -K_{\kappa}\nabla u_{\mathcal{T}}(x) \cdot \nabla \eta_{\mathbf{s}}(x)\ dx,$$

$$= \int_{\widehat{x_{\sigma}\mathbf{a}} \cup \widehat{x_{\sigma'}\mathbf{a}}} -K_{\kappa}\nabla u_{\mathcal{T}}(x) \cdot \mathbf{n}_{\kappa} d\sigma.$$



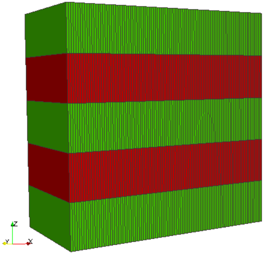
Definition of a porous volume and a rocktype to each vertex

- The porous volume is taken from the surrounding cells
  - proportionaly to the permeability of the cells



Highly heterogeneous test case on coarse grids

- Domain : $[0, 100] \times [0, 50] \times [0, 100]\ m^3$
- Isotropic heterogeneous media initially saturated with water
- Injection of gas at the left end $x = 0$
- Ratio of the drain and barrier permeabilities : 10^4
- Cartesian mesh: $100 \times 1 \times 5$

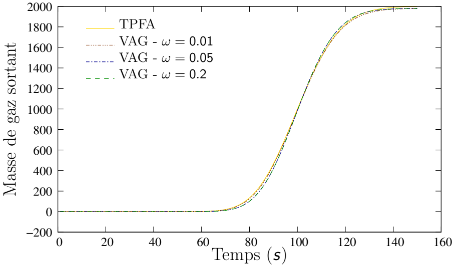


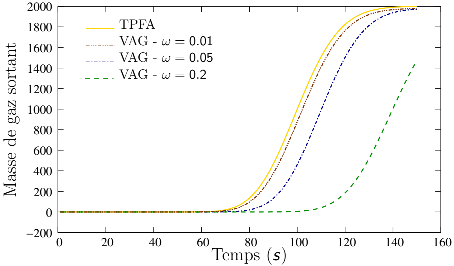
drains
barrières

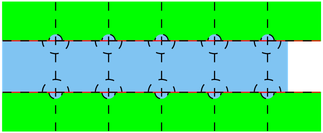
Highly heterogeneous test case

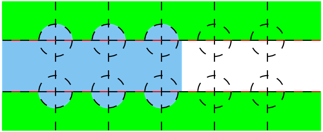
$$\alpha_{\kappa,\mathbf{s}} = \omega \frac{\sum_{\mathbf{s}'} T_\kappa^{\mathbf{ss}'}}{\sum_{\kappa'} \sum_{\mathbf{s}'} T_{\kappa'}^{\mathbf{ss}'}}$$

$$\alpha_{\kappa,\mathbf{s}} = \omega \frac{1}{\mathrm{Card}\mathcal{M}_\mathbf{s}}$$



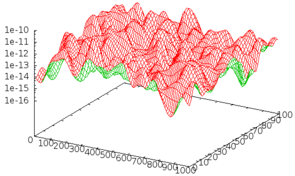




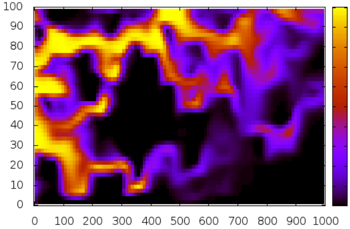


Gas injection in an heterogeneous media saturated with water (log normal permeabiliy field)

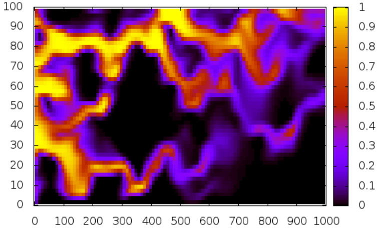
Permeability field



TPFA

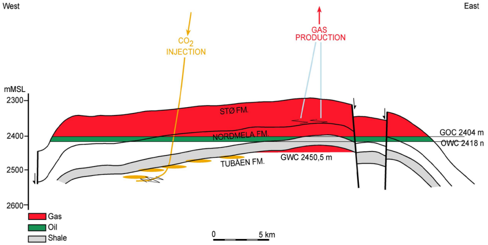


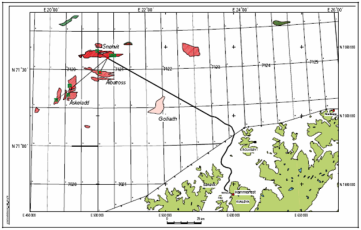
VAG

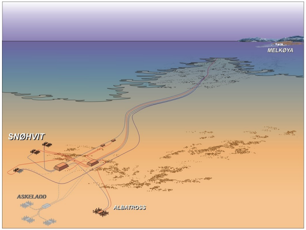


Near well drying and alteration: CO2 injection in the Snohvit field

- Snohvit Gaz field contains 5 to 8 % of CO2
- Reinjection of CO2 in the saline aquifer Tubaen
- 700000 reinjected since 2008
- Unexplained periodic loss of injectivity







Model with three phases *water, gaz and mineral*, and three components *H2O, CO2 and Salt*

$$\boxed{\mathcal{C}^w = \{H2O, CO2, sel\}, \quad \mathcal{C}^g = \{H2O, CO2\}, \quad \mathcal{C}^m = \{sel\}}$$

$$\left\{\begin{array}{l}
\partial_t \phi \left(\rho^w \ S^w \ C^w_{H2O} + \rho^g \ S^g \ C^g_{H20}\right) + \mathrm{div}\Big(C^w_{H2O} \ \rho^w \ \mathbf{U}^w + C^g_{H2O} \ \rho^g \ \mathbf{U}^g\Big) = 0, \\[2ex]
\partial_t \phi \left(\rho^w \ S^w \ C^w_{Salt} + \rho^m \ S^m\right) + \mathrm{div}\Big(C^w_{Salt} \ \rho^w \ \mathbf{U}^w\Big) = 0, \\[2ex]
\partial_t \phi \left(\rho^g \ S^g \ C^g_{CO2} + \rho^w \ S^w \ C^w_{CO2}\right) + \mathrm{div}\Big(C^g_{CO2} \ \rho^g \ \mathbf{U}^g + C^w_{CO2} \ \rho^w \ \mathbf{U}^w\Big) = 0, \\[2ex]
S^w + S^g + S^m = 1, \\
C^w_{H2O} + C^w_{CO2} + C^w_{Salt} = 1 \ \text{ if w present}, \\
C^g_{H20} + C^g_{CO2} = 1 \ \text{ if g present} \\[2ex]
\mathbf{U}^g = -\frac{k_{r_g}(S)}{\mu^g} K\Big(\nabla P^g - \rho^g \mathbf{g}\Big), \\
\mathbf{U}^w = -\frac{k_{r_w}(S)}{\mu^w} K\Big(\nabla \Big[P^g + P_{c,w}(S)\Big] - \rho^w \mathbf{g}\Big).
\end{array}\right.$$

Thermodynamical equilibrium

$$\left\{\begin{array}{ll} C^{w}_{CO2} = K_{CO2}\ C^{g}_{CO2} & \text{if w,g present} \\ C^{g}_{H2O} = K_{H20}\ C^{w}_{H2O} & \text{if w,g present,} \\ C^{w}_{Salt} = K_{Salt} & \text{if w,m present.} \end{array}\right.$$

The model is not closed since we don't know which phases are present:
7 possible set of present phases: w-g-m, w-g, w-m, g-m, w,g,m:

Three phase Flash: phase diagram in compositional space Z at fixed pressure P (and T)

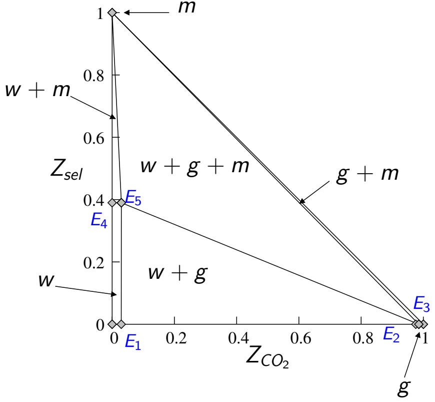
Z_i: total mass fractions of the components $i = CO_2, H_2O, Salt$

$$\left(Z_{CO2}, Z_{salt}\right):$$

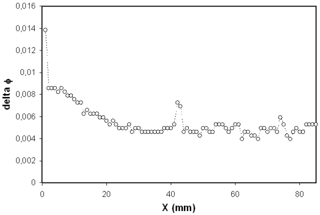
$$Z_{CO2} \geq 0,$$

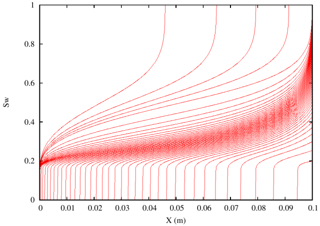
$$Z_{salt} \geq 0,$$

$$Z_{H2O} = 1 - Z_{CO2} - Z_{salt} \geq 0.$$

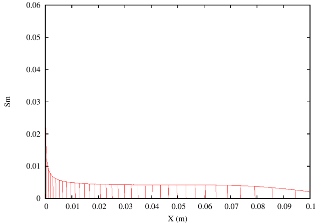


Nearwell drying and alteration: laboratory experiment versus simulation



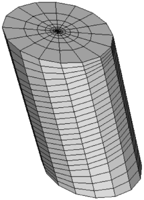


Water saturation

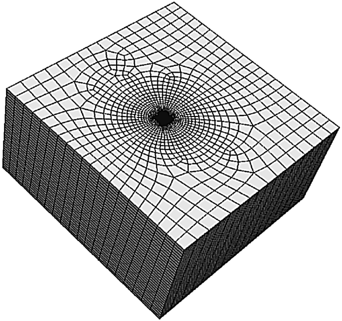


Mineral saturation

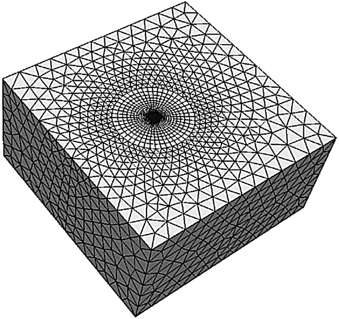
3D nearwell meshes for a deviated well



Radial mesh

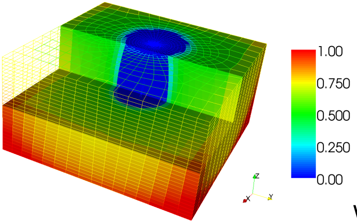


Hexahedral mesh

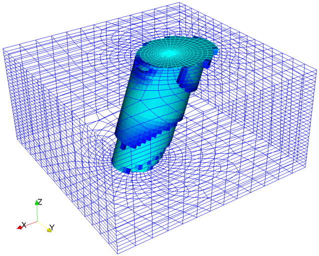


Hybrid mesh

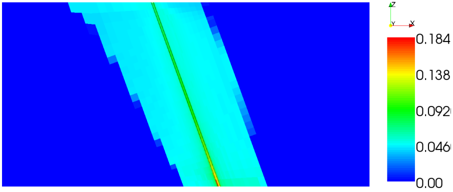
Water and mineral saturations at final time



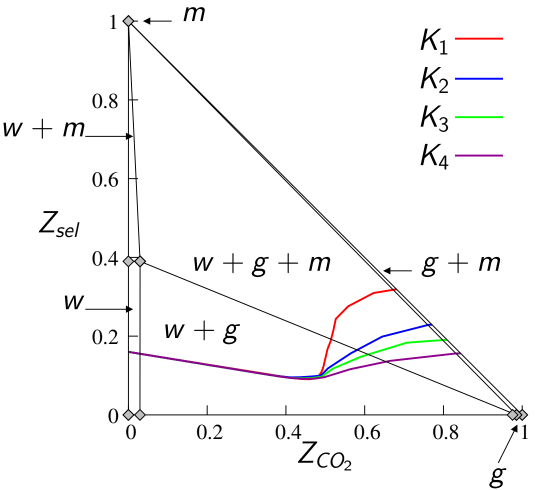
Water saturation



Mineral saturation $|S^m > 0.1\%$

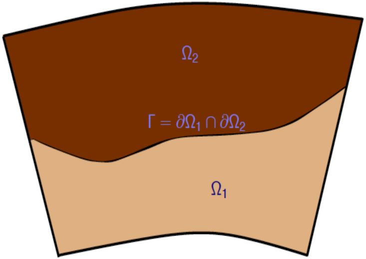


2D xz – mineral saturation S^m



Trajectory of four cells in the compositional space (Z_{CO_2}, Z_{salt})

Discontinuous Capillary pressures



- The relative permeabilities and the capillary pressure depend on the rock type
- $k_r^\alpha(S^\alpha, x) = k_{r,i}^\alpha(S^\alpha)$ on Ω_i
- $P_c(S^o, x) = P_{c,i}(S^o)$ on Ω_i

Multivalued pressures [Cances et al 2011]

- If $S^o \in (0,1)$ both pressures P^o and P^w are defined and must satisfy

$$P^o - P^w = P_c(S^o),$$

- If $S^o = 0$, the oil pressure is not uniquely defined. It satisfies

$$P^o - P^w \leq P_c(0), \quad \text{we can write } p^o = \Big[-\infty, P^w + P_c(0)\Big]$$

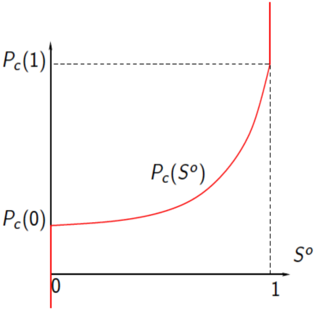
- If $S^o = 1$, the water pressure is not uniquely defined. It satisfies

$$P^o - P^w \geq P_c(1), \quad \text{we can write } p^w = \Big[-\infty, P^o - P_c(1)\Big].$$

Monotone graph of the capillary pressure

We define the monotone graph of the capillary pressure [Cances et al 2011]:

$$\widetilde{P}_c(S^o) = \begin{cases} \left[-\infty, P_c(0)\right] & \text{if} \quad S^o = 0, \\ P_c(S^o) & \text{if} \quad S^o \in (0,1), \\ \left[P_c(1), +\infty\right] & \text{if} \quad S^o = 1. \end{cases}$$

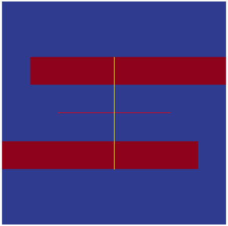


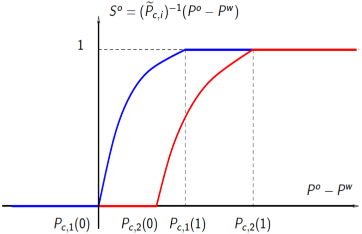
$$P^o - P^w \in \widetilde{P}_c(S^o), \qquad S^o = (\widetilde{P}_c)^{-1}(P^o - P^w).$$

Immiscible incompressible two phase Darcy flows with discontinuous capillary pressures

$$\left\{\begin{array}{rl} \phi\partial_t S^\alpha + \mathrm{div}\mathbf{U}^\alpha &= 0,\ \alpha = w, o,\\ \mathbf{U}^\alpha &= -\frac{k_r^\alpha(S^\alpha,x)}{\mu^\alpha} K(x)(\nabla P^\alpha - \rho^\alpha \mathbf{g}),\ \alpha = w, o,\\ S^w + S^o &= 1,\\ P^o - P^w &= P_c(S^o, x). \end{array}\right.$$

$$\text{Two Rocktypes } i = 1,2 : \left\{\begin{array}{l} P_c(S^o,x) = P_{c,i}(S^o),\\ \\ k_r^\alpha(S^\alpha,x) = k_{r,i}^\alpha(S^\alpha), \end{array}\right. \quad \text{for } x \in \Omega_i,\ i = 1,2.$$



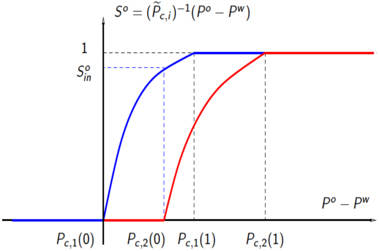


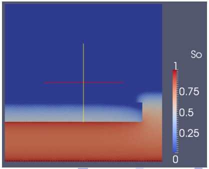
Immiscible incompressible two phase Darcy flows with discontinuous capillary pressures

Matching conditions at the interface $\Gamma = \Omega_1 \cap \Omega_2$ between the two rocktypes [Enchery et al 2008], [Cances et al 2011], [Brenner et al 2011]:

$$\begin{cases} P^o = P_1^o = P_2^o, \\ P^w = P_1^w = P_2^w, \\ S_1^o = (\widetilde{P}_{c,1})^{-1}(P^o - P^w), \\ S_2^o = (\widetilde{P}_{c,2})^{-1}(P^o - P^w) \end{cases}$$

$$\begin{cases} \mathbf{U}_1^o \cdot \mathbf{n}_1 + \mathbf{U}_2^o \cdot \mathbf{n}_1 = 0, \\ \mathbf{U}_1^w \cdot \mathbf{n}_1 + \mathbf{U}_2^w \cdot \mathbf{n}_1 = 0. \end{cases}$$





Extension of the scheme [Brenner et al 2011], [Brenner 2011] to the VAG discretization on general meshes

- Allow for discontinuous saturations at the interfaces between two different rocktypes:

$$S_{\kappa,\mathbf{s}},\ \ \kappa \in \mathcal{M}_{\mathbf{s}},$$

- Fluxes continuity at a given interface $\mathbf{s}$: given by the conservation equations at $\mathbf{s}$

- Phase pressures continuity:

$$\left\{\begin{array}{l} S^o_{\kappa,\mathbf{s}} = P^{-1}_{c,\kappa}(p^o_{\mathbf{s}} - p^w_{\mathbf{s}}),\ \ \kappa \in \mathcal{M}_{\mathbf{s}}, \\ \\ S^o_{\kappa} = P^{-1}_{c,\kappa}(p^o_{\kappa} - p^w_{\kappa}). \end{array}\right.$$

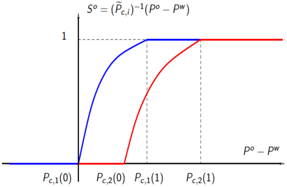
VAG discretization

$$
\left\{
\begin{array}{l}
m_\kappa \phi_\kappa \dfrac{S_\kappa^{\alpha,n} - S_\kappa^{\alpha,n-1}}{\Delta t} + \displaystyle\sum_{\mathbf{s}\in\mathcal{V}_\kappa} \dfrac{k_{r,\kappa}^\alpha(S_\kappa^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^+ + \dfrac{k_{r,\kappa}^\alpha(S_{\kappa,\mathbf{s}}^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^- = 0, \\
\qquad k \in \mathcal{M},\ \alpha = w, o, \\[2ex]
\displaystyle\sum_{\kappa\in\mathcal{M}_\mathbf{s}} m_{\kappa,\mathbf{s}} \phi_\kappa \dfrac{S_{\kappa,\mathbf{s}}^{\alpha,n} - S_{\kappa,s}^{\alpha,n-1}}{\Delta t} - \displaystyle\sum_{\kappa\in\mathcal{M}_\mathbf{s}} \dfrac{k_{r,\kappa}^\alpha(S_\kappa^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^+ + \dfrac{k_{r,\kappa}^\alpha(S_{\kappa,\mathbf{s}}^{\alpha,n})}{\mu^\alpha}(V_{\kappa,\mathbf{s}}^\alpha)^- = 0, \\
\qquad \mathbf{s} \in \mathcal{V} \setminus \mathcal{V}_D,\ \alpha = w, o.
\end{array}
\right.
$$

$$
\left\{
\begin{array}{l}
S_{\kappa,\mathbf{s}}^{o,n} = P_{c,\kappa}^{-1}(p_\mathbf{s}^{o,n} - p_\mathbf{s}^{w,n}),\ \ \kappa \in \mathcal{M}_\mathbf{s}, \mathbf{s} \in \mathcal{V} \setminus \mathcal{V}_D, \\[2ex]
S_\kappa^{o,n} = P_{c,\kappa}^{-1}(p_\kappa^{o,n} - p_\kappa^{w,n}),\ \ \kappa \in \mathcal{M}.
\end{array}
\right.
$$

Problem of non uniqueness of the solution P^w, P^o

Example: initial state: P^w, $S^o = 0$:
$P^o \in \left[-\infty, P^w + P_{c,i}(0)\right]$ not unique !



To avoid this singularity when solving the discrete nonlinear system:

Projections of $P^o_\kappa - P^w_\kappa$ on the interval:

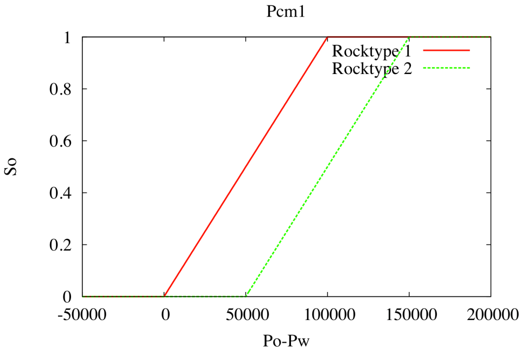
$$\left[\min_{\{p \,|\, (P^{-1}_{c,\kappa})'(p)>0\}} P^{-1}_{c,\kappa}(p), \max_{\{p \,|\, (P^{-1}_{c,\kappa})'(p)>0\}} P^{-1}_{c,\kappa}(p)\right]$$

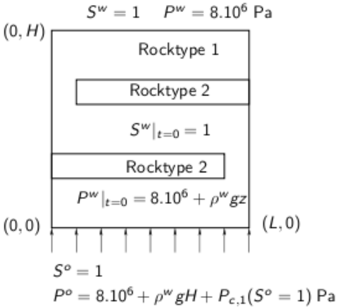
and of $P^o_{\mathbf{s}} - P^w_{\mathbf{s}}$ on

$$\left[\min_{\kappa\in\mathcal{M}_{\mathbf{s}}} \min_{\{p \,|\, (P^{-1}_{c,\kappa})'(p)>0\}} P^{-1}_{c,\kappa}(p), \max_{\kappa\in\mathcal{M}_{\mathbf{s}}} \max_{\{p \,|\, (P^{-1}_{c,\kappa})'(p)>0\}} P^{-1}_{c,\kappa}(p)\right].$$

Test Case: two barriers

Porous media with two rocktypes: $K_1 = K_2 = 1.10^{-12}\, m^2$, $\phi_1 = \phi_2 = 0.1$, $k^{\alpha}_{r,1} = k^{\alpha}_{r,2}$, $\alpha = w, o$, and the following $P^{-1}_{c,1}$, $P^{-1}_{c,2}$:



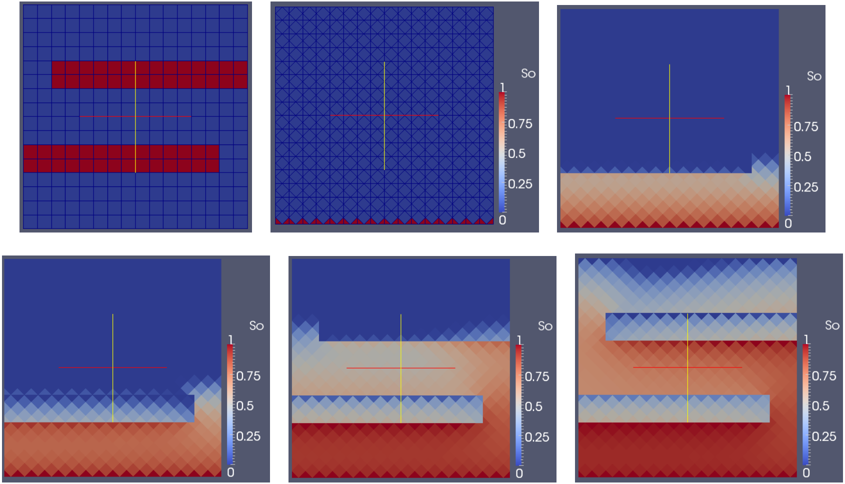


Density driven flow: $\rho^o = 800$, $\rho^w = 1000\ kg/m^3$,

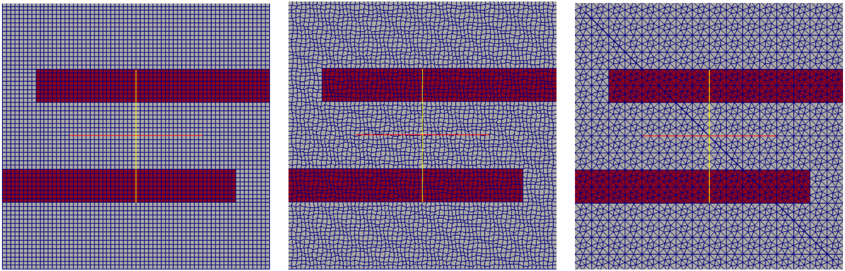
$$k^o_r(S^o) = (S^o)^2,\ \mu^o = 5.10^{-3},$$

$$k^w_r(S^w) = (S^w)^2,\ \mu^w = 1.10^{-3}.$$

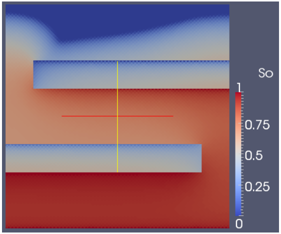
Barriers test case: numerical result on a Cartesian grid 16×16

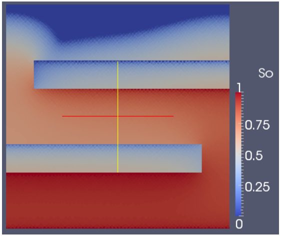


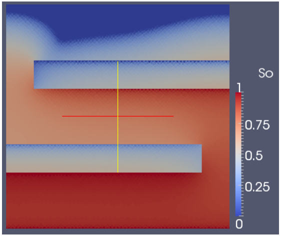
Comparison of the solution at final time on cartesian, random quadrangular and triangular meshes



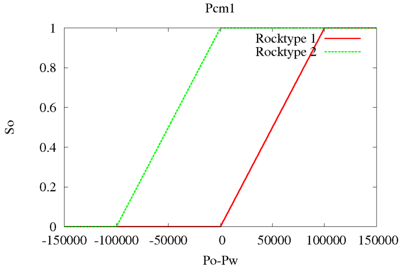
Comparison of the solution on cartesian 64×64, random quadrangular 64×64, and triangular (1900 nodes) meshes

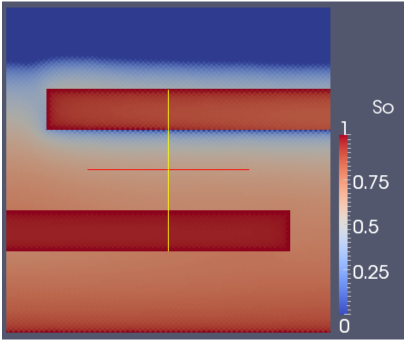






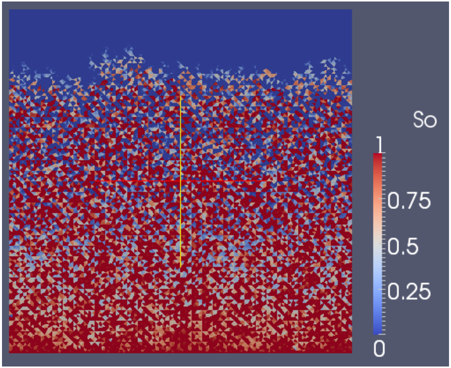
Test case with change of wettability: imbibition in the barriers



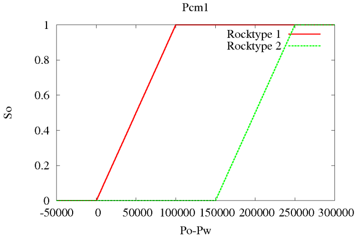


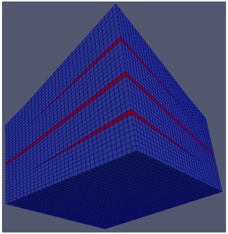
Oil migration in a porous media with random P_c

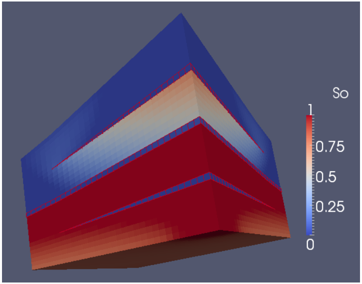
$$(P_c)(S^o) = (S^o + a)\ 10^5 \text{ with } a \in (-1, 1).$$

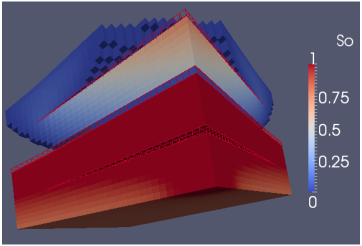


Oil migration in a 3D basin with barriers









Conclusion and perspectives

- Vertex centred discretization of Darcy flows:
    - adapted to general meshes,
    - very efficient on simpletic meshes compared with cell centred schemes,
    - can be adapted to highly heterogeneous media and different rocktypes.

- Parallel distributed code on polyhedral meshes (Cemracs project 07)
    - General polyhedral meshes
    - Different Finite Volume schemes
- Extension to more complex models with discontinuous capillary pressures
    - Dissolution (Black Oil)
    - Compositional

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