Residual Distribution

basics,

recents developments,

relations with other techniques

INTRODUCTION with historical perspective

$$\partial_t u + \nabla \cdot \mathcal{F}(u) = 0$$

In the 80's numerical techniques devised to provide high order monotone approximation to solutions of hyperbolic C.L.s

1. A. Harten (J.Comput.Phys., 1983) : TVD conditions
2. Goodman, LeVeque (Math.Comp.) : TVD in mulitD = first order

INTRODUCTION with historical perspective

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This has spawned a number of new approaches

- Monotonicity conditions
  1. ENO/WENO (Harten, Osher, Engquist, Chakravarthy, Shu)
  2. TVB conditions (Shu *Math.Comp.* 1987)
  3. Positive coefficient schemes (Spekreijse *Math.Comp.* 1987)
- Discretization frameworks
  1. Stabilized FE (or central) (Hughes, Morton, Ni, Lerat, Jameson)
  2. Discontinuous Galerkin (Cockburn and Shu, starting 1988)
  3. Roe's Fluctuation Splitting (starting 1986)

INTRODUCTION with historical perspective

$$\partial_t u + \nabla \cdot \mathcal{F}(u) = 0$$

Discontinuous Galerkin : *smart and elegant combination of existing tools (approximation, Galerkin projection, Riemann solvers, limiters) to generate automatically arbitrary higher order schemes*

An instant hit ...

INTRODUCTION with historical perspective

$$\partial_t u + \nabla \cdot \mathcal{F}(u) = 0$$

Fluctuation Splitting/Residual Distribution :

"A more **fundamental and robust** *approach [...] due to Roe (1986), is that of the "genuinely multidimensional" upwind schemes. These may be regarded as* **the true multi-D generalization** *of 1-D fluctuation splitting [...] These methods are best formulated on simplex-type (finite-element) grids and include* **newly developed, compact limiters** *for avoiding oscillations ..."*

INTRODUCTION with historical perspective

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Fluctuation Splitting/Residual Distribution :

"A more **fundamental and robust** *approach [...] due to Roe (1986), is that of the "genuinely multidimensional" upwind schemes. These may be regarded as* **the true multi-D generalization** *of 1-D fluctuation splitting [...] These methods are best formulated on simplex-type (finite-element) grids and include* **newly developed, compact limiters** *for avoiding oscillations ..."*

B. van Leer, (excerpt from *Upwind high resolution methods for compressible flow: from donor cell to residual distribution, Commun.Comput.Phys. 1(2), 2006*)

INTRODUCTION with historical perspective

$$\partial_t u + \nabla \cdot \mathcal{F}(u) = 0$$

Initial idea (P.L.Roe, *Num. Meth. Fluid Dyn. 1982*) : given values of u on the mesh, integral of $\nabla \cdot \mathcal{F}(u)$ over elements measures the error (fluctuation); decompose the fluctuation in signals allowing to evolve solution values to those solving the problem

INTRODUCTION with historical perspective

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- 80' and 90', under the lead of P.L. Roe, H. Deconinck :
  1. Genuinely multidimensional upwinding (scalar)
  2. Steady state hyperbolic decomposition
  3. Each scalar hyperbolic component discretized using MU technique

  Genuinely multiD upwind second order compact (nearest neighbor) second order, nonlinear, positive discretization. Very well adapted to steady supersonic, multidimensional Roe linearization, inexact decompositions in sub-critical case, no unsteady.

INTRODUCTION with historical perspective

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- 20 years later with contributions of R. Abgrall, T.J. Barth, D. Caraeni, M. Hubbard, C.W. Shu *et al.*
    1. Time dependent problems
    2. Conservation without Roe lienarization
    3. Higher (than second) accuracy
    4. More general data approximation (including discontinuous)

INTRODUCTION with historical perspective

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- 20 years later with contributions of R. Abgrall, T.J. Barth, D. Caraeni, M. Hubbard, C.W. Shu *et al.* The method has come to a form which is more correctly referred to as a *weighted residual method*, many of the initial *multidimensional upwind fluctuation splitting* ideas could not (so far) be retained ..

INTRODUCTION with historical perspective

$$\partial_t u + \nabla \cdot \mathcal{F}(u) = 0$$

Initial idea (P.L.Roe, *Num. Meth. Fluid Dyn. 1982*) : integral of $\nabla \cdot \mathcal{F}(u)$ over elements measures the error (fluctuation); decompose the fluctuation in signals allowing to evolve solution values to those solving the problem

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- 20 years later with contributions of R. Abgrall, T.J. Barth, D. Caraeni, M. Hubbard, C.W. Shu *et al.*

Despite so many contributions the method never really caught up with other approaches based on more sound mathematical foundations (DG), and it definitely has a much lower level of maturity. But some ideas have stuck.

Hybrid nature : in between finite element and finite volume. This allows easily to import/export ideas born in this framework to improve others and vice-versa ... keeping alive the interest in the method

INTRODUCTION with historical perspective

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Despite so many contributions the method never really caught up with other approaches based on more sound mathematical foundations (DG), and it definitely has a much lower level of maturity

What am I going to tell you ?

COURSE OUTLINE : Part I

1. Conservative FV discretization and fluctuations
2. Fluctuation splitting/residual distribution framework
3. Design principles
4. Limiters in reverse
5. Relations with other techniques

COURSE OUTLINE : Part II

1. Higher (than second) orders
2. Time dependent problems
3. Viscous problems
4. Free surface flows
5. Summary, perspectives

PART I OUTLINE

Finite Volume schemes and Fluctuations

Design criteria

Nonlinear schemes and limiters

Relations with other techniques

1
Conservative Finite Volume schemes and Fluctuations

Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Finite Volume schemes and Fluctuations in 1D

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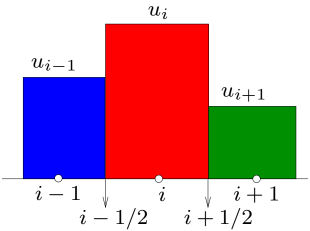
$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Conservative FV :

$$\Delta x_i \frac{du_i}{dt} + \widehat{\mathcal{F}}(u^L_{i+1/2}, u^R_{i+1/2}) - \widehat{\mathcal{F}}(u^L_{i-1/2}, u^R_{i-1/2}) = 0$$



Finite Volume schemes and Fluctuations in 1D

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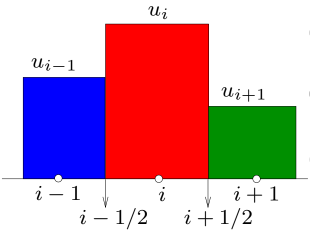

(consistent) $\widehat{\mathcal{F}}(u,u) = \mathcal{F}(u)$

(L-continuous) $\|\widehat{\mathcal{F}}(u,v) - \widehat{\mathcal{F}}(u,z)\| \leq \mathrm{K}_{\mathcal{F}}\|v-z\|$

(E-stable. Monotone, etc.)

Finite Volume schemes and Fluctuations in 1D

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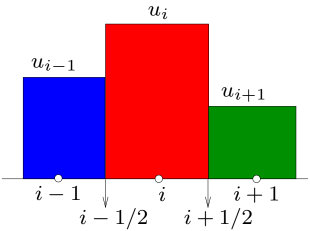

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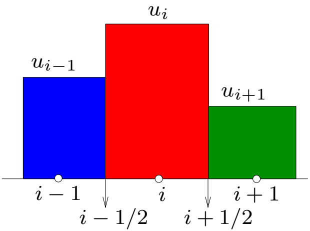

$$\Delta x_{i+1} \frac{du_{i+1}}{dt} + \widehat{\mathcal{F}}_{i+3/2} - \widehat{\mathcal{F}}_{i+1/2} = 0$$

$$\Delta x_{i-1} \frac{du_{i-1}}{dt} + \widehat{\mathcal{F}}_{i-1/2} - \widehat{\mathcal{F}}_{i-3/2} = 0$$

Finite Volume schemes and Fluctuations in 1D

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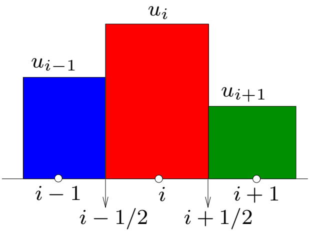


$$\Delta x_{i+1} \frac{du_{i+1}}{dt} + \widehat{\mathcal{F}}_{i+3/2} - \widehat{\mathcal{F}}_{i+1/2} = 0$$
$$\Delta x_{i-1} \frac{du_{i-1}}{dt} + \widehat{\mathcal{F}}_{i-1/2} - \widehat{\mathcal{F}}_{i-3/2} = 0$$

Conservation from flux cancelation at interfaces

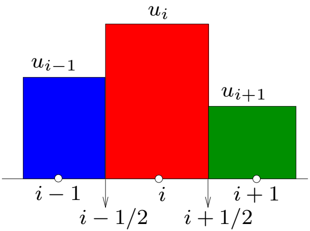
Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Conservative FV :

$$\Delta x_i \frac{du_i}{dt} + (\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}_i) + (\mathcal{F}_i - \widehat{\mathcal{F}}_{i-1/2}) = 0$$



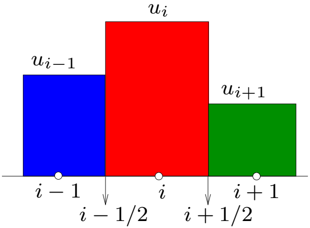
Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Conservative FV :

$$\Delta x_i \frac{du_i}{dt} + \overbrace{(\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}_i)}^{\phi_i^{i+1/2}} + \overbrace{(\mathcal{F}_i - \widehat{\mathcal{F}}_{i-1/2})}^{\phi_i^{i-1/2}} = 0$$



Finite Volume schemes and Fluctuations in 1D

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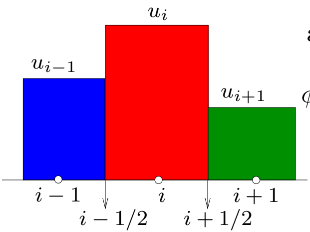

at $i+1/2$ conservation becomes :

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = (\widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}_i) + (\mathcal{F}_{i+1} - \widehat{\mathcal{F}}_{i+1/2})$$

Finite Volume schemes and Fluctuations in 1D

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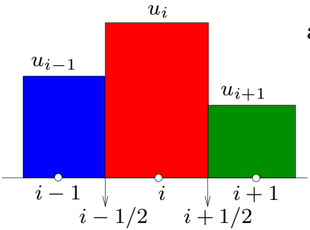

at $i+1/2$ conservation becomes :

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} = \mathcal{F}_{i+1} - \mathcal{F}_i := \phi^{i+1/2}$$

Finite Volume schemes and Fluctuations in 1D

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Conservative FV :

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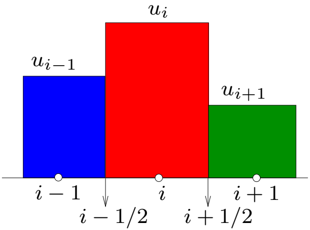

at $i-1/2$ conservation becomes :

$$\phi_i^{i-1/2} + \phi_{i-1}^{i-1/2} = \mathcal{F}_i - \mathcal{F}_{i-1} := \phi^{i-1/2}$$

Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Conservative FV :

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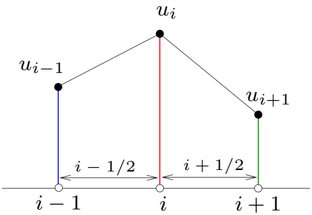

Finite Volume schemes and Fluctuations in 1D

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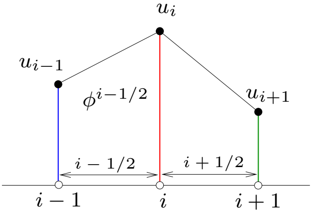

$$\phi^{i-1/2} := \int_{i-1}^{i} \partial_x \mathcal{F}, \qquad \text{(fluctuation)}$$

$$\phi^{i-1/2} = \mathcal{F}_i - \mathcal{F}_{i-1} \qquad \text{(as before ...)}$$

Finite Volume schemes and Fluctuations in 1D

Starting point : conservation law

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Conservative FV :

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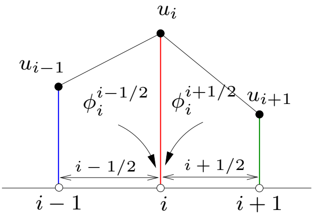

$$\phi^{i-1/2} := \int_{i-1}^{i} \partial_x \mathcal{F}\,, \qquad \text{(fluctuation)}$$

$$\phi_i^{i-1/2} = \mathcal{F}_i - \widehat{\mathcal{F}}_{i-1/2} \qquad \text{(splitting)}$$

$$\phi_{i-1}^{i-1/2} = \widehat{\mathcal{F}}_{i-1/2} - \mathcal{F}_{i-1} \quad \text{(splitting)}$$

Finite Volume schemes and Fluctuations in 1D

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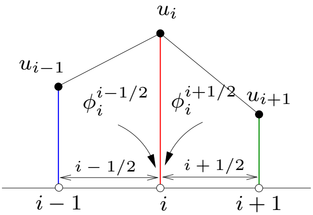


$$\phi^{i-1/2} := \int_{i-1}^{i} \partial_x \mathcal{F}\,, \qquad \phi^{i+1/2} := \int_{i}^{i+1} \partial_x \mathcal{F}$$

$$\phi_i^{i-1/2} = \mathcal{F}_i - \widehat{\mathcal{F}}_{i-1/2}\,, \qquad \phi_i^{i+1/2} = \widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}_i$$

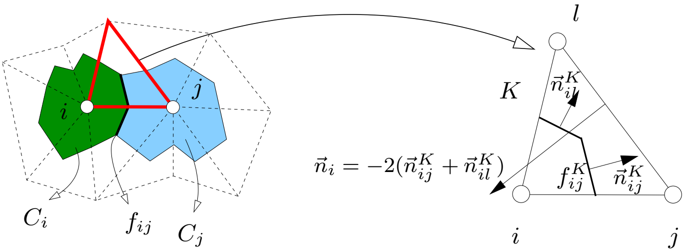
FV scheme in *Fluctuation splitting* form ... still the same guy though

Finite Volume schemes and Fluctuations in 2D

The multi-D case. Starting point : conservation law

$$\partial_t u + \nabla \cdot \mathcal{F}(u) = 0$$

Finite Volume schemes and Fluctuations in 2D

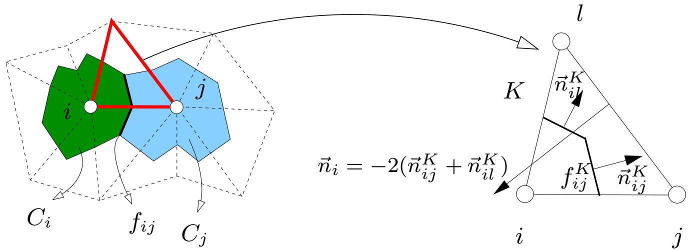


$$\vec{n}_i = -2(\vec{n}^K_{ij} + \vec{n}^K_{il})$$

The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_j \int_{f_{ij}} \widehat{\mathcal{F}} \cdot \hat{n}\, dl = 0$$

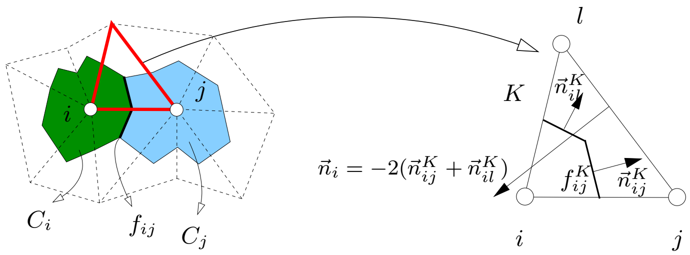
Finite Volume schemes and Fluctuations in 2D



The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K}\sum_{j\in K}\int_{f_{ij}^K} \widehat{\mathcal{F}}\cdot\hat{n}\,dl = 0$$

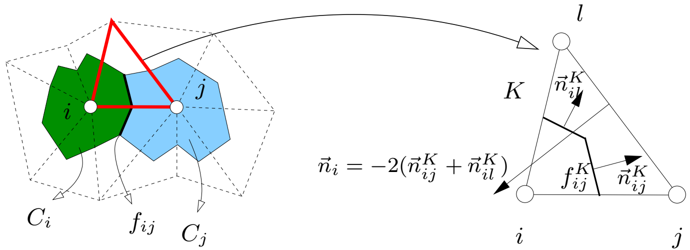
Finite Volume schemes and Fluctuations in 2D



The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K = 0$$

Finite Volume schemes and Fluctuations in 2D



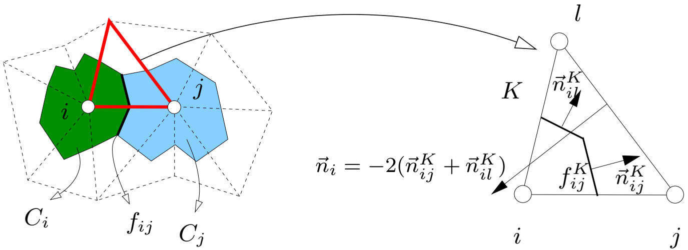
The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K = 0$$

Discrete conservation

$$\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K + \widehat{\mathcal{F}}_{ji}\cdot\vec{n}_{ji}^K = 0$$

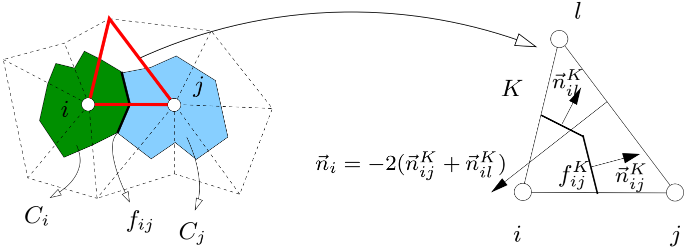
Finite Volume schemes and Fluctuations in 2D



Using the identity $\sum_K \sum_j \vec{n}_{ij}^K = 0$

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \sum_{j\in K} (\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i) \cdot \vec{n}_{ij}^K = 0$$

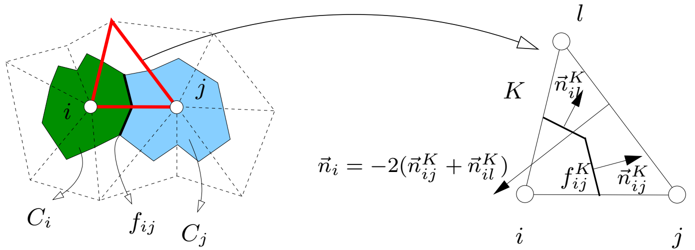
Finite Volume schemes and Fluctuations in 2D



The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \underbrace{\sum_{j\in K}(\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i)\cdot \vec{n}_{ij}^K}_{\phi_i^K} = 0$$

Finite Volume schemes and Fluctuations in 2D



The FV scheme reads

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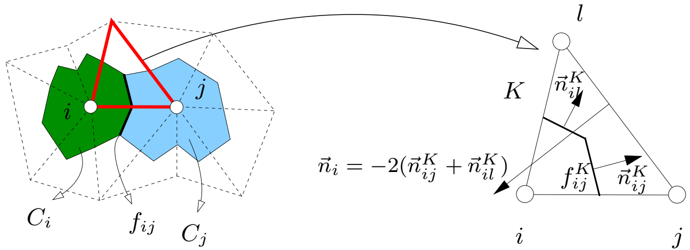
Finite Volume schemes and Fluctuations



The FV scheme reads

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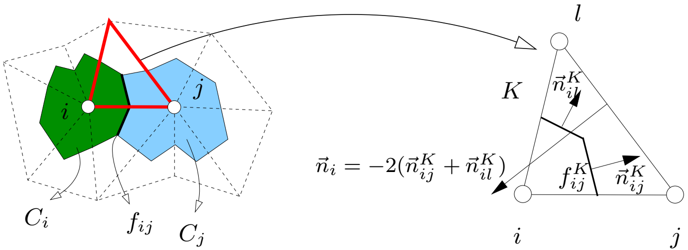
Discrete conservation

$$\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K + \widehat{\mathcal{F}}_{ji}\cdot\vec{n}_{ji}^K = 0 \Longrightarrow \sum_{j\in K}\phi_j^K = \frac{1}{2}\sum_{j\in K}\mathcal{F}_i\cdot\vec{n}_j := \phi^K$$

Finite Volume schemes and Fluctuations



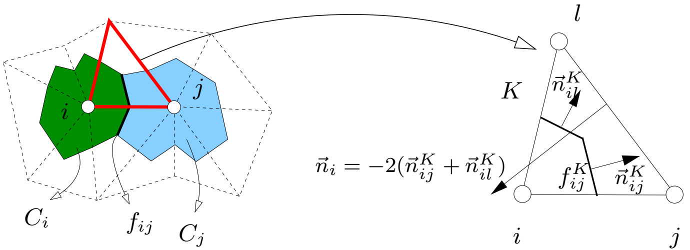
The FV scheme reads

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0\ , \quad \phi_i^K = \sum_{j\in K}(\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i)\cdot \vec{n}_{ij}^K$$

Discrete conservation ($\mathcal{F}_h$ continuous P^1 finite element approx.)

$$\sum_{j\in K} \phi_j^K = \phi^K = \int_K \nabla\cdot \mathcal{F}_h$$

Finite Volume schemes and Fluctuations

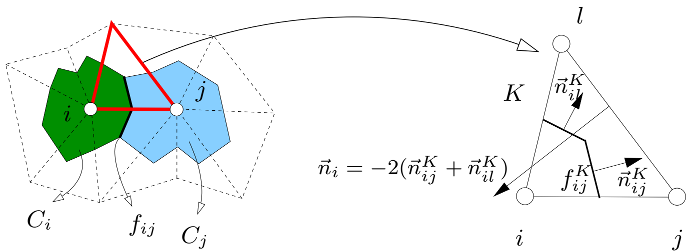


The FV scheme reads ($\mathcal{F}_h$ continuous P^1 finite element approx.)

$$\phi^K = \int_K \nabla \cdot \mathcal{F}_h \,, \qquad \overbrace{\sum_{j \in K} \phi_j^K = \phi^K}^{\text{Discrete conservation}}$$

$$|C_i| \frac{du_i}{dt} + \sum_{K|i \in K} \phi_i^K = 0 \,, \qquad \phi_i^K = \sum_{j \in K} (\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i) \cdot \vec{n}_{ij}^K$$

Finite Volume schemes and Fluctuations



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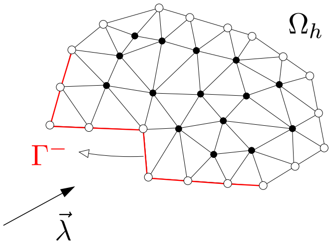
... but it's still the same guy .. !!

2
Residual Distribution
the framework

Residual Distribution framework

$$
\begin{aligned}
\nabla \cdot \mathcal{F}(u) &= 0 \qquad \text{in} \quad \Omega \\
u &= g \qquad \text{on} \quad \Gamma^- \\
\vec{\lambda}(u) &= \partial_u \mathcal{F}(u)
\end{aligned}
\qquad (1)
$$



Some notations...

- Consider Ω_h tesselation of Ω
- Unknowns (Degrees of Freedom, DoF) : $u_i \approx u(M_i)$
- $M_i \in \Omega_h$ a given set of nodes (vertices +other dofs)
- u_h : **continuous** polynomial interpolation $u_h = \sum_i \psi_i u_i$

Residual Distribution framework

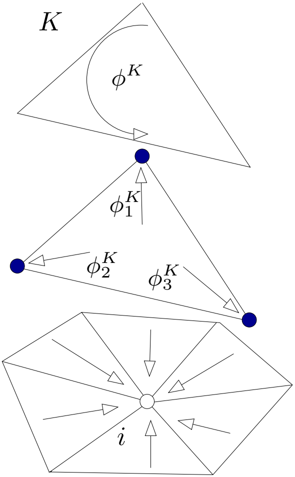
1. $\forall K \in \Omega_h$ compute : $\phi^K = \int_K \nabla \cdot \mathcal{F}_h(u_h)$

2. Distribution : $\phi^K = \sum_{i \in K} \phi_i^K$

   Distribution coeff.s : $\phi_i^K = \beta_i^K \phi^K$

3. Compute nodal values : solve algebraic system

$$\sum_{T|i\in T} \phi_i^K = 0, \quad \forall\, i \in \Omega_h \qquad (2)$$



Residual Distribution framework

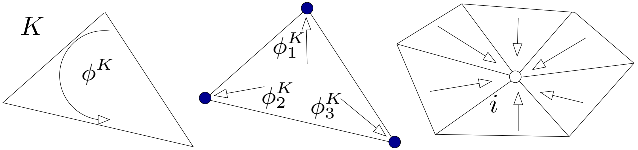
Seek the steady limit of

$$|C_i|\frac{du_i}{dt} + \sum_{K\,|\,i\in K} \phi_i^K \;\xrightarrow{t\to\infty}\; \sum_{K\,|\,i\in K} \phi_i^K = 0 \qquad (3)$$

The idea of Residual Distribution or Fluctuation Splitting

- Fluctuations & Signals (Roe, *Num.Meth.Fluid Dyn.*, 1982)
- From an initial guess, nodal values evolve to steady state due to signals "proportional" to <u>cell residuals</u> (Roe's fluctuation)

1 - Compute fluctuation **2 - Split** **3 - Gather signals**



4 - Evolve eq. ((3))

Structural conditions

Conservation. LW theorem : convergence (if ...) to weak solution ?

Stability. which form of stability (energy/entropy, equivalent algebraic condition, convergence ?), choice of ϕ_i^K

Accuracy. characterization of the error, choice of ϕ_i^K

Oscillations. monotonicity preserving schemes, choice of ϕ_i^K

Conservation

Conservation, LW theorem for RD and $\nabla \cdot \mathcal{F}(u) = 0$

Under some (standard) continuity assumptions on ϕ^K and ϕ_i^K the discrete solution u_h converges (if ..... !) to a weak solution of the continuous problem, provided that (Abgral, Barth *SISC*, 2002 ; Abgrall, Roe *J.Sci.Comp.*, 2003) :

$$\sum_{j \in K} \phi_j^K(u_h) = \phi^K(u_h) = \oint_{\partial K} \mathcal{F}_h(u_h) \cdot \hat{n}\, dl$$

for *some* continuous approximation of the flux $\mathcal{F}_h$.

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for *some* continuous approximation of the flux $\mathcal{F}_h$.

In practice, approach 1

Set $\mathcal{F}_h = \sum_{j \in K} \psi_j \mathcal{F}_j$ and integrate exactly.

This gives the P^1 element residual seen for the FV scheme

Conservation

Conservation, LW theorem for RD and $\nabla \cdot \mathcal{F}(u) = 0$

$$\sum_{j\in K} \phi_j^K(u_h) = \phi^K(u_h) = \oint_{\partial K} \mathcal{F}_h(u_h)\cdot \hat{n}\, dl$$

for *some* continuous approximation of the flux $\mathcal{F}_h$.

In practice, approach 2

Set $\mathcal{F}_h = \mathcal{F}(v_h)$ with v some set of variables, $v_h = \sum_j \psi_j v_j$, apply Gauss Formulae on ∂K (Csik, Ricchiuto, Deconinck, *J.Comput.Phys*, 2002)

$$\phi^K(u_h) = \sum_{f\in\partial K} \int_f \mathcal{F}_h(x)\cdot \hat{n}\, dl = \sum_{f\in\partial K} |f| \sum_{q=1}^{\mathrm{G}_p} \omega_q \mathcal{F}_h(x_q)\cdot \hat{n}_f$$

Conservation

Conservation, LW theorem for RD and $\nabla \cdot \mathcal{F}(u) = 0$

$$\sum_{j \in K} \phi_j^K(u_h) = \phi^K(u_h) = \oint_{\partial K} \mathcal{F}_h(u_h) \cdot \hat{n}\, dl$$

for *some* continuous approximation of the flux $\mathcal{F}_h$.

In practice, approach 3

Set $\mathcal{F}_h = \mathcal{F}(v_h)$ with v some set of variables, $v_h = \sum_j \psi_j v_j$, and integrate exactly of below truncation error (Deconinck, Struijs, Roe *Computers & Fluids*, 1993 ; Abgrall, Barth *SISC*, 2002)

$$\phi^K = \int_K \frac{\partial \mathcal{F}}{\partial v}(v_h) \cdot \nabla v_h\, dK$$

Conservation

Conservation, LW theorem for RD and $\nabla \cdot \mathcal{F}(u) = 0$

$$\sum_{j \in K} \phi_j^K(u_h) = \phi^K(u_h) = \oint_{\partial K} \mathcal{F}_h(u_h) \cdot \hat{n}\, dl$$

for *some* continuous approximation of the flux $\mathcal{F}_h$.

In practice

Schemes are more often written so that we recover at the end

$$\phi^K(u_h) = \sum_{f \in \partial K} \int_f \mathcal{F}_h(x) \cdot \hat{n}\, dl = \sum_{f \in \partial K} |f| \sum_{q=1}^{\mathrm{G}_p} \omega_q \mathcal{F}_h(x_q) \cdot \hat{n}_f$$

for some (edge) continuous polynomial reconstruction $\mathcal{F}_h(x)$ which remains one of the degrees of freedom of the method

Conservation

Conservation, LW theorem for RD and $\nabla \cdot \mathcal{F}(u) = 0$

$$\sum_{j \in K} \phi_j^K(u_h) = \phi^K(u_h) = \oint_{\partial K} \mathcal{F}_h(u_h) \cdot \hat{n}\, dl$$

for <u>*some*</u> continuous approximation of the flux $\mathcal{F}_h$.

This only guarantees that
if discontinuous solutions are approximated,
the correct jump (Rankine-Hugoniot) conditions are recovered

What about stability, accuracy, etc. ?

3
DESIGN CRITERIA

Accuracy, stability, and all that jazz

Design criteria

1. $\forall K \in \Omega_h$ compute :

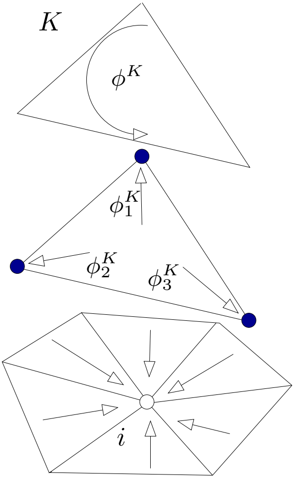
$$\phi^K = \int_K \nabla \cdot \mathcal{F}_h(u_h)$$

2. Distribution : $\phi^K = \sum_{i \in K} \phi_i^K$

   Distribution coeff.s : $\phi_i^K = \beta_i^K \phi^K$

3. Compute nodal values : solve algebraic system

$$|C_i| \frac{du_i}{dt} + \sum_{T|i \in T} \phi_i^K = 0, \quad t \to \infty \qquad (4)$$



What can we say about the stability of this method ?

First : what is stability ?

What can we say about the stability of this method ?

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Assume, for h fixed you do

$$u_i^{n+1} = u_i^n - \omega_i \sum_{K|i\in K} \phi_i^K(u_h^n)\,, \quad \omega_i = \frac{\Delta t}{|C_i|}$$

What can we say about the stability of this method ?

First : what is stability ?

More abstractly (ω a scalar, *e.g.* $\omega = \min_i \omega_i$) for h fixed you do

$$\mathrm{u}^{n+1} = \mathrm{u}^n - \omega(A_h\,\mathrm{u}^n - f)$$

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$$\mathrm{u}^{n+1} = \mathrm{u}^n - \omega(A_h\, \mathrm{u}^n - f)$$

A condition for convergence with $n \to \infty$ but h fixed

$$\|(\mathrm{I} - \omega A_h)\mathrm{u}\|^2 \leq r\|\mathrm{u}\|^2\,, \ \ \forall \mathrm{u} \text{ and with } r < 1$$

which is equivalent to

$$\mathrm{u}^t A_h \mathrm{u} \geq \frac{1-r}{2\omega}\|\mathrm{u}\|^2 + \frac{\omega}{2}\|A_h\,\mathrm{u}\|^2 \geq C_h\|\mathrm{u}\|^2 \geq 0 \quad \forall\, \mathrm{u}$$

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Coercivity ...

Stability and energy

Consider the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

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Semi-discrete counterpart

$$|C_i| \frac{du_i}{dt} + \sum_{K|i \in K} \phi_i^K = 0$$

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Energy budget

The equivalent of the quantity $\mathrm{u}^t A_h \mathrm{u}$ seen in the previous slides is

$$\mathrm{u}^t A_h \mathrm{u} \equiv \sum_{i \in \Omega_h} u_i \sum_{K|i \in K} \phi_i^K$$

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$$\mathrm{u}^t A_h \mathrm{u} \equiv \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}}$$

What is $\phi_K^{\mathcal{E}}$?

Stability and energy

Starting from

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

Energy budget

$$\sum_{i\in\Omega_h} |C_i| u_i \frac{du_i}{dt} + \sum_{K\in\Omega_h} \phi_K^{\mathcal{E}} = 0$$

Stability and energy

Starting from

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

Energy budget

$$\sum_{i\in\Omega_h} |C_i|\frac{d}{dt}\left(\frac{u_i^2}{2}\right) + \sum_{K\in\Omega_h} \phi_K^{\mathcal{E}} = 0$$

Stability and energy

Starting from

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

Energy budget

$$\int_{\Omega_h} \frac{d\mathcal{E}_h}{dt} + \sum_{K\in\Omega_h} \phi_K^{\mathcal{E}} = 0$$

with the *energy density*

$$\mathcal{E} = \frac{u^2}{2}$$

and with $\mathcal{E}_h = \sum_{i\in\Omega_h} \mathcal{E}_i\psi_i$ (piecewise linear)

Stability and energy

Saying that

$$0 \leq \mathrm{u}^t A_h \mathrm{u} \equiv \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}}$$

is equivalent to

Stability and energy

Saying that

$$0 < \mathrm{u}^t A_h \mathrm{u} \equiv \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}}$$

is equivalent to

Energy stability

$$\int\limits_{\Omega_h} \frac{d\mathcal{E}_h}{dt} = - \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}} \leq 0$$

with the *energy density*

$$\mathcal{E} = \frac{u^2}{2}$$

and with $\mathcal{E}_h = \sum_{i \in \Omega_h} \mathcal{E}_i \psi_i$ (piecewise linear)

Stability and energy

Saying that

$$0 < \mathrm{u}^t A_h \mathrm{u} \equiv \sum_{K \in \Omega_h} \phi_K^{\mathcal{E}}$$

is equivalent to

Energy stability (modulo boundary conditions)

$$\int_{\Omega_h} \frac{d\mathcal{E}_h}{dt} = - \int_{\partial\Omega_h} \mathcal{E}_h \, \vec{a} \cdot \hat{n} \, dl \; - \; \delta^{\mathcal{E}} \, , \quad \delta^{\mathcal{E}} \geq 0$$

what one would like is to find that

$$\phi_K^{\mathcal{E}} = \int_{\partial K} \mathcal{E}_h \, \vec{a} \cdot \hat{n} \, dl + \delta_K^{\mathcal{E}} \, , \quad \delta_K^{\mathcal{E}} \geq 0$$

Stability and upwinding

Consider again the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

A geometrical view of advection...

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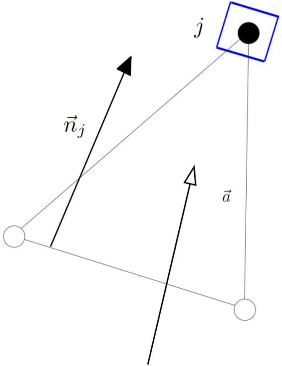

1-target triangle

The inlet region is an edge

1 node downstream : 1 target

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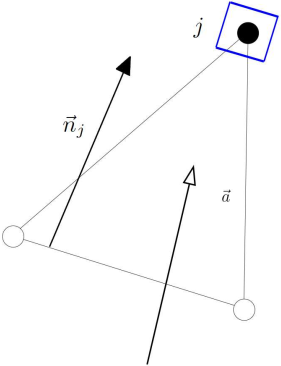
A geometrical view of advection...



1-target triangle

The inlet region is an edge

1 node downstream : 1 target

$$k_j = \frac{\vec{a} \cdot \vec{n}_j}{2} > 0$$

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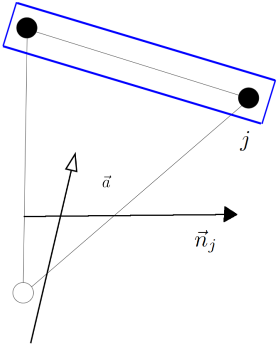


2-target triangle

The outlet region is an edge

2 nodes downstream : 2 targets

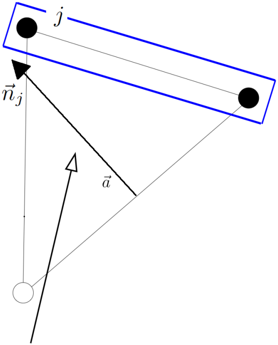
$$k_j = \frac{\vec{a} \cdot \vec{n}_j}{2} > 0$$

Stability and upwinding

Consider again the steady limit of

$$\partial_t u + \vec{a} \cdot \nabla u = 0$$

A geometrical view of advection...



2-target triangle

The outlet region is an edge

2 nodes downstream : 2 targets

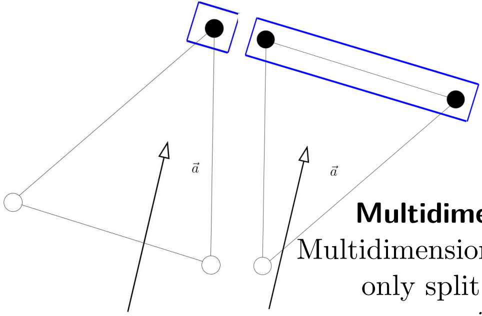
$$k_j = \frac{\vec{a} \cdot \vec{n}_j}{2} > 0$$

Stability and upwinding

Consider now the semi-discrete RD advection equation :

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

A geometrical view of advection...



Multidimensional Upwinding (MU)

Multidimensional Upwind (MU) schemes only split ϕ^K to downstream nodes, *i.e.* those for which $k_j > 0$.

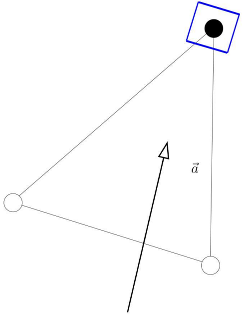
All MU schemes reduce to the same in the 1-target case.

Stability and upwinding

Consider now the semi-discrete RD advection equation :

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

A geometrical view of advection...



Multidimensional Upwinding (MU)

Multidimensional Upwind (MU) schemes

In 1-target elements

if $k_1 > 0$ (node 1 only node downstream)

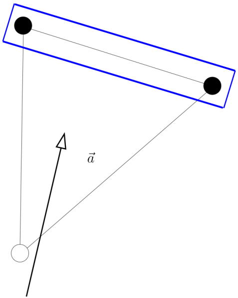
$$\phi_1^K = \phi^K \, , \; \phi_2^K = \phi_3^K = 0$$

Stability and upwinding

Consider now the semi-discrete RD advection equation :

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0$$

A geometrical view of advection...



Multidimensional Upwinding (MU)

Multidimensional Upwind (MU) schemes

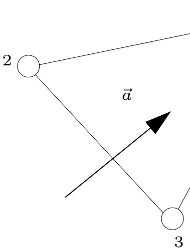
In 2-targets elements

if $k_1 < 0$ (node 1 only node upstream)

$$\phi_1^K = 0\,,\ \phi_2^K + \phi_3^K = \phi^K$$

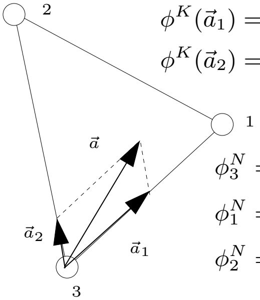
Stability and MU schemes

Example 1 : Roe's optimal N scheme



$$\phi_1^N = \phi^K$$

$$\phi_2^N = \phi_3^N = 0$$



$$\phi^K(\vec{a}_1) = \int_K \vec{a}_1 \cdot \nabla u_h$$

$$\phi^K(\vec{a}_2) = \int_K \vec{a}_2 \cdot \nabla u_h$$

$$\phi_3^N = 0$$

$$\phi_1^N = \phi^K(\vec{a}_1)$$

$$\phi_2^N = \phi^K(\vec{a}_2)$$

The formula (Roe Cranfield U.Tech.Rep., 1987 ; Roe, Sidilkover *SINUM*, 1992)

$$\phi_i^N = k_i^+(u_i - u_{in}) \, , \quad u_{in} = \frac{\sum\limits_{j\in K} k_j^- u_j}{\sum\limits_{j\in K} k_j^-}$$

Stability and MU schemes

Example 2 : the LDA scheme

The LDA scheme reads

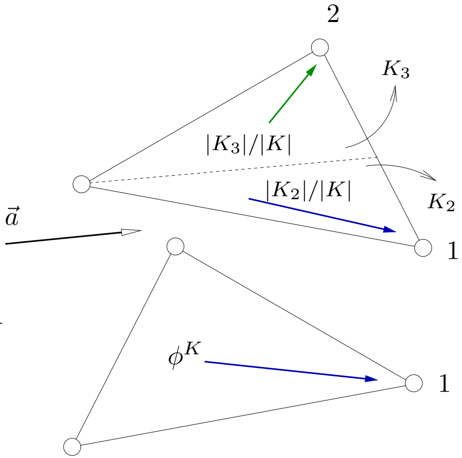
$$\phi_i^{\text{LDA}}(u_h) = \beta_i^{\text{LDA}} \phi^K(u_h)$$

where

$$\beta_i^{\text{LDA}} = \frac{k_i^+}{\sum\limits_{j \in K} k_j^+}$$

recalling that for the advection equation (u_h piecewise linear)

$$\phi^K(u_h) = \int\limits_K \vec{a} \cdot \nabla u_h$$



Stability and MU

The following properties can be easily shown :

1. MU schemes, 1-target (Deconinck, Ricchiuto *Enc.Comput.Mech.*, 2007)

$$\phi_K^{\mathcal{E}} = \int\limits_{\partial K} \mathcal{E}_h \, \vec{a} \cdot \hat{n} \, dl + \delta_K^{\mathcal{E}} \,, \qquad \delta_K^{\mathcal{E}} \geq 0$$

2. N scheme energy stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002)
3. LDA scheme, 2-targets (Deconinck, Ricchiuto *Enc.Comput.Mech.*, 2007)

$$\phi_{\text{LDA}}^{\mathcal{E}} = \underbrace{\Big(\sum_{j \in K} k_j^+\Big) \left(\frac{u_{out}^2}{2} - \frac{u_{in}^2}{2} \right)}_{\substack{\text{NRG balance} \\ \text{along streamline}}} + \delta_{\text{LDA}}^{\mathcal{E}} \,, \qquad \delta_{\text{LDA}}^{\mathcal{E}} \geq 0$$

Multidimensional upwinding does the job ...

Stability, upwinding, and dissipation

1. FV scheme (1st order upwind) NRG stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002), also E-flux schemes by (Osher *SINUM*, 1984)
2. Streamline upwind finite element scheme SUPG, (Hughes, Brooks *CMAME*, 1982) :

Stability, upwinding, and dissipation

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$$\int_{\Omega_h} \psi_i \nabla \cdot \mathcal{F}_h(u_h) + \sum_{K \in \Omega_h} \int_K \vec{a}(u_h) \cdot \nabla \psi_i \, \tau \, \vec{a}(u_h) \cdot \nabla u_h = 0$$

can be written as the RD scheme

$$\sum_{K | i \in K} \phi_i^K = 0$$

with

$$\phi_i^K = \int_K \psi_i \nabla \cdot \mathcal{F}_h(u_h) + \int_K \vec{a}(u_h) \cdot \nabla \psi_i \, \tau \, \vec{a}(u_h) \cdot \nabla u_h$$

Stability, upwinding, and dissipation

1. FV scheme (1st order upwind) NRG stable (Barth, NASA 1996 ; Abgrall, Barth *SISC*, 2002), also E-flux schemes by (Osher *SINUM*, 1984)
2. Streamline upwind finite element scheme SUPG :

$$\phi_i^{\mathrm{SUPG}} = \int\limits_K \psi_i \vec{a} \cdot \nabla u_h + \int\limits_K \vec{a} \cdot \nabla \psi_i \, \tau \, \vec{a} \cdot \nabla u_h$$

one easily checks that since $\sum_j \psi_j = 1$ and $\sum_j \nabla \psi_j = 0$

$$\sum_{j \in K} \phi_j^{\mathrm{SUPG}} = \int\limits_K \vec{a} \cdot \nabla u_h = \phi^K(u_h)$$

Stability, upwinding, and dissipation

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2. Streamline upwind finite element scheme (SUPG) :

$$\phi^{\mathcal{E}}_{\text{SUPG}} = \oint\limits_{\partial K} \frac{u_h^2}{2}\,\vec{a}\cdot\hat{n}\,dl + \underbrace{\int\limits_{K} \vec{a}\cdot\nabla u_h\,\tau\,\vec{a}\cdot\nabla u_h}_{\substack{\text{Streamline}\\ \text{dissipation } \geq 0}}$$

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3. Lax-Friedrich's/Rusanov scheme

$$\phi_i^{\text{LF}} = \int_K \psi_i\vec{a}\cdot\nabla u_h + \alpha_{\text{LF}}\sum_{j\in K}(u_i - u_j)$$

Stability, upwinding, and dissipation

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3. Lax-Friedrich's/Rusanov scheme

$$\phi^{\mathcal{E}}_{\text{LF}} = \oint\limits_{\partial K} \frac{u_h^2}{2}\,\vec{a}\cdot\hat{n}\,dl + \frac{\alpha_{\text{LF}}}{3}\sum_{i,j\in K}(u_i - u_j)^2$$

Stability, upwinding, and dissipation

Upwinding has beneficial effect in terms of energy stability

Design criteria : what is the truncation error ?

- By Taylor expansion : no way (unless meshes with particular structure are considered)
- Error analysis based on variational form :
    1. which variational form ?
    2. NRG stability not enough, no coercivity no tools for analysis
- Idea : use 'weak' form to define error (consistency estimate)

Design criteria : what is the truncation error ?

Idea : use 'weak' form to define an *integral truncation error*

$$\int_\Omega \nabla\varphi \cdot \mathcal{F}(u)dx + \text{BCs} = 0 \longleftrightarrow \int_\Omega \nabla\varphi \cdot \mathcal{F}_h(u_h)dx + \text{BCs} = \varepsilon_h$$

with u a smooth exact (classical) solution

This gives a consistency estimate..

What is ε_h ?

Design criteria, consistency analysis

What do we have ... ?

Design criteria, consistency analysis

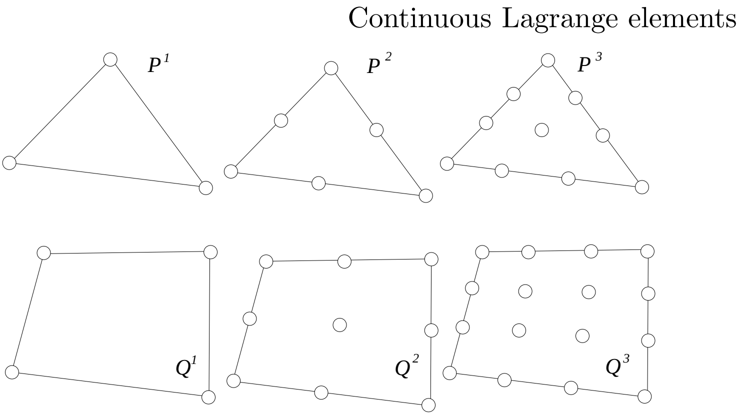
What do we have ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \mathcal{F}(w) = \partial_u \mathcal{F}(w) \cdot \nabla w = 0$
2. $w - w_h = O(h^{k+1})$, $\mathcal{F}(w) - \mathcal{F}_h(w_h) = O(h^{k+1})$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)
3. $\nabla(w - w_h) = O(h^k)$, $\nabla \cdot (\mathcal{F}(w) - \mathcal{F}_h(w_h)) = O(h^k)$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)

with w_h a continuous polynomial approximation of degree k (*e.g* standard Lagrange elements)

Design criteria, consistency analysis



Design criteria, consistency analysis

What do we do ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \mathcal{F}(w) = \partial_u \mathcal{F}(w) \cdot \nabla w = 0$
2. $w - w_h = O(h^{k+1})$, $\mathcal{F}(w) - \mathcal{F}_h(w_h) = O(h^{k+1})$ in L^2 from approximation theory, see *e.g.* (Ern, Guermond Springer, 2004)
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Take the steady RD scheme

$$\sum_{K|i \in K} \phi_i^K(u_h) = 0$$

approximating $\nabla \cdot \mathcal{F}$ in node i

Design criteria, consistency analysis

What do we do ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \mathcal{F}(w) = \partial_u \mathcal{F}(w) \cdot \nabla w = 0$
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Formally replace the nodal values of u_h, computed by the scheme, with those of the exact solution w, exaclty as done in finite difference TE analysis

Design criteria, consistency analysis

What do we do ... ?

Consider

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We obtain

$$\sum_{K|i \in K} \phi_i^K(w_h) \neq 0$$

since of course the nodal values of the exact solution w <u>do not</u> verify the discrete equations

Design criteria, consistency analysis

What do we do ... ?

Consider

1. $w \in H^{k+1}$ smooth solution : $\nabla \cdot \mathcal{F}(w) = \partial_u \mathcal{F}(w) \cdot \nabla w = 0$
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Given φ a $C_0^r(\Omega)$ class function, r large enough, define

$$\epsilon_h := \sum_{i \in \Omega_h} \varphi_i \sum_{K | i \in K} \phi_i^K(w_h)$$

A global measure of how much the discrete equations differ from the continuous one

Design criteria, consistency analysis

What do we do ... ?

Estimate ϵ_h (Abgrall, Roe *J.Sci.Comp.*, 2003 ; Ricchiuto, Abgrall, Deconinck *J.Comput.Phys*, 2007)

$$\epsilon_h = \sum_{K\in\Omega_h}\sum_{i\in K}\varphi_i\phi_i^K(w_h) = \epsilon_a + \epsilon_d$$

$$\epsilon_a = \underbrace{-\int_{\Omega_h}\nabla\varphi_h\cdot\left(\mathcal{F}_h(w_h) - \mathcal{F}(w)\right)}_{\text{approximation error}}$$

$$\varphi_h = \sum_{K\in\Omega_h}\sum_{j\in K}\psi_j\varphi_j$$

Design criteria, consistency analysis

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$$\epsilon_d = \underbrace{\sum_{K\in\Omega_h}\sum_{i,j\in K}\frac{\varphi_i-\varphi_j}{n^K_{\mathrm{DoF}}}(\phi_i^K(w_h)-\phi_i^{\mathrm{G}}(w_h))}_{\text{distribution error}}$$

$$\phi_i^{\mathrm{G}}(w_h) = \int_K \psi_i\nabla\cdot(\mathcal{F}_h(w_h)-\mathcal{F}(w)) \quad \text{(Galerkin proj.)}$$

Design criteria, consistency analysis

What do we do ... ?

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Design criteria, consistency analysis

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We get easily the estimates

$$\|\epsilon_a\| = \| \int_{\Omega_h} \nabla \varphi_h \cdot (\mathcal{F}_h(w_h) - \mathcal{F}(w)) \| \leq C'_a \, h^{k+1}$$

Design criteria, consistency analysis

What do we do ... ?

Estimate ϵ_h (Abgrall, Roe *J.Sci.Comp.*, 2003 ; Ricchiuto, Abgrall, Deconinck *J.Comput.Phys*, 2007)

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We get easily the estimates

$$\|\phi^{\mathrm{G}}(w_h)\| = \|\int_K \psi_i \nabla \cdot (\mathcal{F}_h(w_h) - \mathcal{F}(w))\| \leq C''_a\, h^{k+2}$$

$$\|\varphi_i - \varphi_j\| \leq h\|\nabla\varphi\| \leq C\, h$$

Design criteria, consistency analysis

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We get easily the estimates

$$\begin{aligned} \|\epsilon_d\| &= \| \sum_{K \in \Omega_h} \sum_{i,j \in K} \frac{\varphi_i - \varphi_j}{n^K_{\text{DoF}}} (\phi_i^K(w_h) - \phi_i^{\text{G}}(w_h)) \| \\ &\le C_{\Omega_h} h^{-2} \times C\, h \times (\sup_K \sup_{i \in K} \|\phi_i^K(w_h)\| + C''_a\, h^{k+2}) \end{aligned}$$

Design criteria, consistency analysis

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We get easily the estimates

$$\|\epsilon_d\| \leq C'\, h^{-1} \mathrm{sup}_K \sup_{i \in K} \|\phi_i^K(w_h)\| + C_a'''\, h^{k+1}$$

Design criteria, consistency analysis

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$$\|\epsilon_h\| \leq C_a\, h^{k+1} + C'\, h^{-1} \text{sup}_K \sup_{i\in K} \|\phi_i^K(w_h)\|$$

Design criteria, consistency analysis

What do we do ... ?

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$$\|\epsilon_h\| \leq C_a\, h^{k+1} + C'\, h^{-1}\sup_K \sup_{i\in K} \|\phi_i^K(w_h)\|$$

For a polynomial approximation of degree k,
a sufficient condition to have a $\|\epsilon_h\| \leq C\, h^{k+1}$ is (in 2d)

$$\phi_i^K(w_h) = \mathcal{O}(h^{k+2})\,, \quad \forall\, K_h\,,\, \forall\, i \in K$$

A local TE condition ..

Design criteria, high order schemes

For a polynomial approximation of degree k,
a sufficient condition to have a $\|\epsilon_h\| \leq C\, h^{k+1}$ is (in 2d)

$$\phi_i^K(w_h) = \mathcal{O}(h^{k+2}), \quad \forall\, K_h\,,\ \forall\, i \in K$$

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Design criteria, high order schemes

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3. $\nabla(w - w_h) = O(h^k)$, $\nabla \cdot (\mathcal{F}(w) - \mathcal{F}_h(w_h)) = O(h^k)$

High order prototype 1, Petrov-Galerkin

$$\phi_i^K(u_h) = \int\limits_K \omega_i^K \nabla \cdot \mathcal{F}_h(u_h)\,, \quad \|\omega_i^K\| < C < \infty$$

Design criteria, high order schemes

For a polynomial approximation of degree k,
a sufficient condition to have a $\|\epsilon_h\| \leq C\, h^{k+1}$ is (in 2d)

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3. $\nabla(w - w_h) = O(h^k)$, $\nabla \cdot (\mathcal{F}(w) - \mathcal{F}_h(w_h)) = O(h^k)$

High order prototype 2, accuracy preserving RD

$$\phi_i^K(u_h) = \beta_i^K \int\limits_K \nabla \cdot \mathcal{F}_h(u_h) = \beta_i^K \phi^K(u_h)\,, \quad \|\beta_i^K\| < C < \infty$$

High order schemes, examples

LDA scheme (P^1) elements

$$\beta_i^{\mathrm{LDA}} = k_i^+ \Big(\sum_{j \in K} k_j^+ \Big)^{-1}$$

SUPG

$$\omega_i^K = \psi_i + \vec{a}(u_h) \cdot \nabla \psi_i \, \tau \,, \quad \vec{a}(u_h) = \partial_u \mathcal{F}(u_h)$$

4
LIMITERS
using them in reverse

Nonlinear high order schemes

So far we have

1. A "stability" criterion requiring an upwind bias (other stabilization strategies mentioned later if time ..)
2. An accuracy (consistency) criterion requiring bounded weights in the residual splitting

How about discontinuity capturing ?

Discontinuity capturing : positivity

$$|C_i|\frac{du_i}{dt} = -\sum_{K|i\in K}\phi_i^K$$

Positive coefficient scheme (Spekreijse, *Math.Comp.* 49, 1987)

A scheme for which

$$\phi_i^K = \sum_{\substack{j\in K\\ j\neq i}} c_{ij}^K(u_i - u_j) \quad \text{with} \quad c_{ik}^K \geq 0$$

s said to be LED (Local Extremum Diminishing)

Discontinuity capturing : positivity

$$|C_i|\frac{u_i^{n+1}-u_i^n}{\Delta t} = -\sum_{K|i\in K}\sum_{\substack{j\in K\\ j\neq i}} c_{ij}^K(u_i^n - u_j^n)$$

Positive coefficient scheme (Spekreijse, *Math.Comp.* 49, 1987)

When combined with Explicit Euler time integration (or with another boundedness preserving time integration scheme) LED leads to

$$u_i^{n+1} = \sum_j \overline{c}_{ij} u_j^n$$

where

$$\overline{c}_{ij} \geq 0\,, \quad \sum_j \overline{c}_{ij}^K = 1 \quad \text{provided} \quad \frac{\Delta t}{|C_i|}\sum_j c_{ij} \leq 1$$

Discontinuity capturing : positivity

$$|C_i|\frac{u_i^{n+1}-u_i^n}{\Delta t} = -\sum_{K|i\in K}\sum_{\substack{j\in K\\ j\neq i}} c_{ij}^K(u_i^n-u_j^n)$$

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where

$$\overline{c}_{ij} \geq 0\,,\quad \sum_j \overline{c}_{ij}^K = 1 \quad \text{provided} \quad \frac{\Delta t}{|C_i|}\sum_j c_{ij} \leq 1$$

In this case the scheme is said (by abuse of language) to be <u>positive</u>

Discontinuity capturing : positivity

$$u_i^{n+1} = \sum_j \overline{c}_{ij} u_j^n$$

with

$$\overline{c}_{ij} \geq 0\,, \quad \sum_j \overline{c}_{ij}^K = 1$$

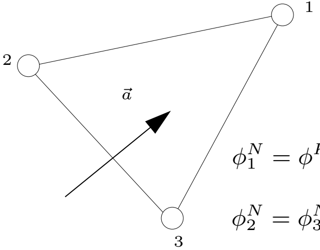
Positive coefficient scheme (Spekreijse, *Math.Comp.* 49, 1987)

A positive scheme verifies the discrete max principle

$$\min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$$

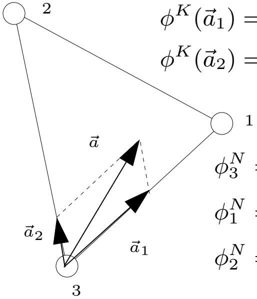
Positive schemes : examples

Example 1 : Roe's optimal N scheme



$$\phi_1^N = \phi^K$$

$$\phi_2^N = \phi_3^N = 0$$



$$\phi^K(\vec{a}_1) = \int_K \vec{a}_1 \cdot \nabla u_h$$

$$\phi^K(\vec{a}_2) = \int_K \vec{a}_2 \cdot \nabla u_h$$

$$\phi_3^N = 0$$

$$\phi_1^N = \phi^K(\vec{a}_1)$$

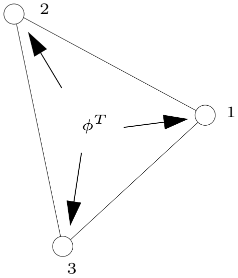
$$\phi_2^N = \phi^K(\vec{a}_2)$$

The formula (Roe Cranfield U.Tech.Rep., 1987 ; Roe, Sidilkover *SINUM*, 1992)

$$\phi_i^N = k_i^+(u_i - u_{in}), \quad u_{in} = \frac{\sum_{j \in K} k_j^- u_j}{\sum_{j \in K} k_j^-}$$

Nonlinear high order schemes

Example 2 : Lax-Friedrich's distribution



$$\phi_i^{\mathrm{LF}} = \int_K \psi_i \nabla \cdot \mathcal{F}_h + \alpha_{\mathrm{LF}} \sum_{j \in K} (u_i - u_j)$$

for positivity (scalar case)

$$\alpha_{\mathrm{LF}} \geq h_K \sup_{x \in K} \|\partial_u \mathcal{F}(u_h(x))\|$$

Nonlinear high order schemes

Bad news ... (Godunov)

All linear positive (LED) schemes are first order accurate ...

Nonlinear high order schemes

Where does the limiter come in

Recall that one prototype of a high order scheme is

$$\phi_i^K(u_h) = \beta_i^K \phi^K(u_h)\,, \quad \|\beta_i^K\| \leq C < \infty$$

Nonlinear high order schemes

Where does the limiter come in

Recall that one prototype of a high order scheme is

$$\phi_i^K(u_h) = \beta_i^K \phi^K(u_h)\,, \quad \|\beta_i^K\| \leq C < \infty$$

For linear positive coefficient schemes

$$\phi_i^{\mathrm{P}}(u_h) = \sum_{j\in K} c_{ij}^K(u_i - u_j)\,, \quad c_{ij}^K \geq 0$$

Formally we have

$$\beta_i^{\mathrm{P}}(u_h) = \frac{\sum\limits_{j\in K} c_{ij}^K(u_i - u_j)}{\phi^K(u_h)} \quad \text{in general unbounded !}$$

Nonlinear high order schemes

Where does the limiter come in

The idea : apply a limiter function to bound the distribution coefficient

Nonlinear high order schemes

Where does the limiter come in

The idea : apply a limiter function to bound the distribution coefficient

$$\beta_i^{\rm LP}(u_h) = \frac{\psi\big(\beta_i^{\rm P}(u_h)\big)}{\sum\limits_{j\in K}\psi\big(\beta_j^{\rm P}(u_h)\big)}$$

The scaling on the denominator guarantees that $\sum\limits_j \beta_i^{\rm LP} = 1$

What are the conditions on the limiter function $\psi(\cdot)$?

Nonlinear high order schemes

Where does the limiter come in

Linear positive coefficient schemes

$$\phi_i^{\mathrm{P}}(u_h) = \sum_{j\in K} c_{ij}^K (u_i - u_j)\,, c_{ij}^K \geq 0\ ; \quad \beta_i^{\mathrm{P}}(u_h) = \frac{\phi_i^{\mathrm{P}}(u_h)}{\phi^K(u_h)}\ \text{unbounded}$$

$$\beta_i^{\mathrm{LP}}(u_h) = \frac{\psi\big(\beta_i^{\mathrm{P}}(u_h)\big)}{\sum\limits_{j\in K} \psi\big(\beta_j^{\mathrm{P}}(u_h)\big)}\ \text{limited distribution coefficient}$$

Nonlinear high order schemes

Where does the limiter come in

Linear positive coefficient schemes

$$\phi_i^{\mathrm{P}}(u_h) = \sum_{j\in K} c_{ij}^K (u_i - u_j)\,,\, c_{ij}^K \geq 0\,; \quad \beta_i^{\mathrm{P}}(u_h) = \frac{\phi_i^{\mathrm{P}}(u_h)}{\phi^K(u_h)} \text{ unbounded}$$

$$\beta_i^{\mathrm{LP}}(u_h) = \frac{\psi\big(\beta_i^{\mathrm{P}}(u_h)\big)}{\sum\limits_{j\in K} \psi\big(\beta_j^{\mathrm{P}}(u_h)\big)} \text{ limited distribution coefficient}$$

Provided $\psi(r) \geq 0$ and $\dfrac{\psi(r)}{r} \geq 0$ we have

$$\phi_i^{\mathrm{LP}}(u_h) = \beta_i^{\mathrm{LP}} \phi^K = \underbrace{\frac{\beta_i^{\mathrm{LP}}}{\beta_i^{\mathrm{P}}}}_{\gamma_i^{\mathrm{P}} \geq 0} \phi_i^{\mathrm{P}} = \sum_{j\in K} c_{ij}^{\mathrm{LP}} (u_i - u_j)\,, \quad c_{ij}^{\mathrm{LP}} = \gamma_i^{\mathrm{P}} c_{ij}^K \geq 0!$$

High order schemes

High order RD scheme

1. Compute cell residual $\phi^K = \oint\limits_{\partial K} \mathcal{F}_h(u_h) \cdot dl$
2. Compute linear positive distribution $\phi_i^{\mathrm{P}} = \sum_j c_{ij}^K (u_i - u_j)$
3. Limit $\beta_i^{\mathrm{P}} = \phi_i^{\mathrm{P}} / \phi^K \;\rightarrow\; \beta_i^{\mathrm{LP}} = \psi(\beta_i^{\mathrm{P}}) / \big(\sum_j \psi(\beta_j^{\mathrm{P}}) \big)$
4. Distribute cell residual $\phi_i^K = \beta_i^{\mathrm{LP}} \phi^K$
5. Evolve $|C_i| \dfrac{du_i}{dt} = - \sum\limits_{K | i \in K} \phi_i^K$ until steady state

High order schemes

High order RD scheme

1. Compute cell residual $\phi^K = \oint\limits_{\partial K} \mathcal{F}_h(u_h) \cdot dl$

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3. Limit $\beta_i^{\mathrm{P}} = \phi_i^{\mathrm{P}} / \phi^K \ \rightarrow \ \beta_i^{\mathrm{LP}} = \psi(\beta_i^{\mathrm{P}}) / \big(\sum_j \psi(\beta_j^{\mathrm{P}}) \big)$

4. Distribute cell residual $\phi_i^K = \beta_i^{\mathrm{LP}} \phi^K$

5. Evolve $|C_i| \dfrac{du_i}{dt} = - \sum\limits_{K | i \in K} \phi_i^K$ until steady state

The simplest possible choice for $\psi(\cdot)$ is

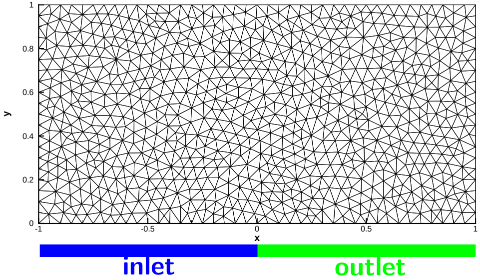
$$\psi(r) = \max(0,\, r)$$

Examples

Rotational advection

Scalar example : $\vec{a} \cdot \nabla u = 0$ with $\vec{a} = (y, 1 - x)$ and bcs

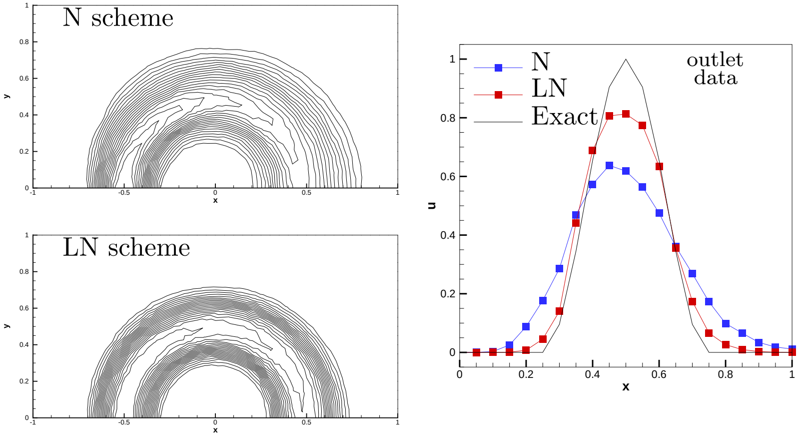
$$u_{\text{in}} = \begin{cases} \cos(2\pi(x+0.5))^2 & \text{if} \quad x \in [-0.75, -0.25] \\ 0 & \text{otherwise} \end{cases}$$



Examples (cont'd)

Rotational advection

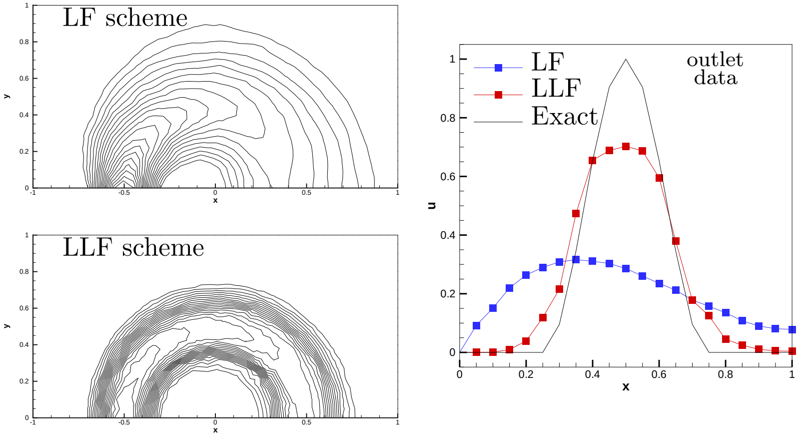
N and Limited N (LN) schemes



Examples (cont'd)

Rotational advection

LF and Limited LF (LLF) schemes

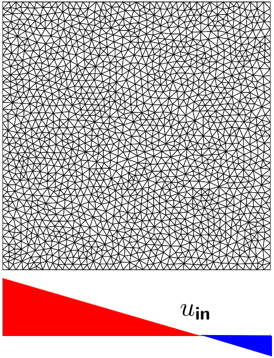


Examples (cont'd)

Burger's equation

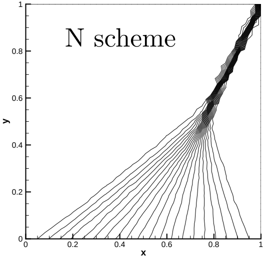
Scalar example : $\nabla \cdot \mathcal{F}(u) = 0$ with $\mathcal{F}(u) = (u,\ \frac{u^2}{2})$ and bcs

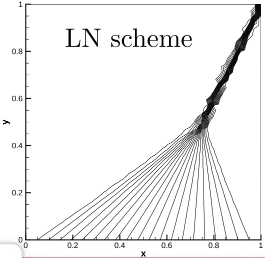
$$u(x, y = 0) = 1.5 - 2x$$

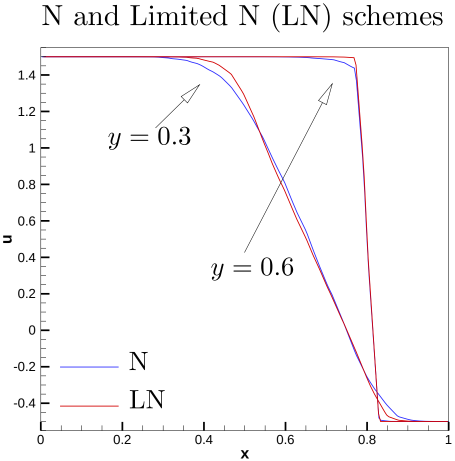


Examples (cont'd)

Burger's equation

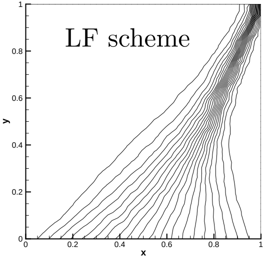


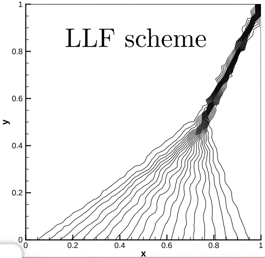


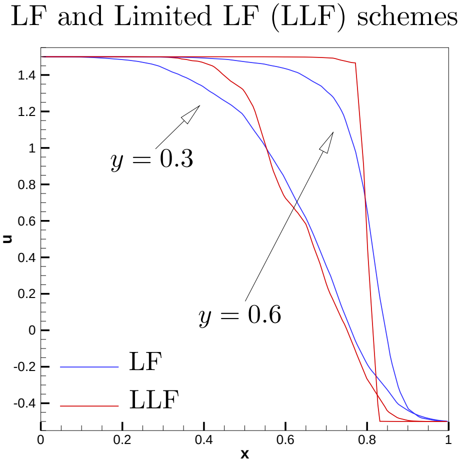


Examples (cont'd)

Burger's equation







Remarks on extension to systems

Historical perspective

Two approaches (Roe *J.Comput.Phys*, 1986 ; Nishikawa, Rad, Roe AIAA Conf. 2001) and (van der Weide, Deconinck *Comput.Fluid Dyn.*, Wiley 1996)

1. Local projection (wave decomposition) of the *continuous PDE* to obtain (possibly decoupled) scalar equations discretized independently
2. Formal matrix generalization in which the scalar flux vector is replaced by a tensor and the $k_i = \vec{a} \cdot \vec{n}_i / 2$ coefficients become matrix flux Jacobians

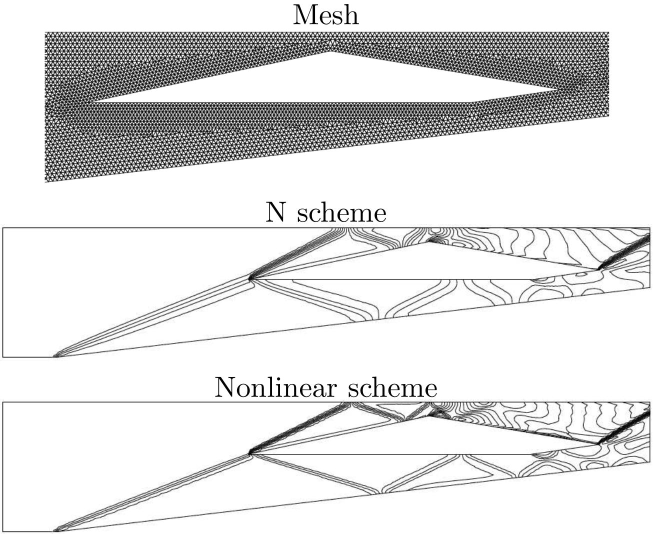
Remarks on extension to systems

Practical implementation

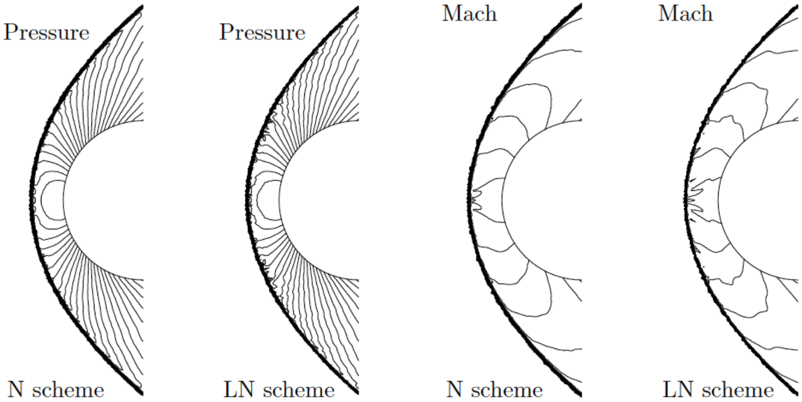
Hybrid of the two (Abgrall, Mezine *J.Comput.Phys*, 2004 ; Ricchiuto, Csik, Deconinck *J.Comput.Phys*, 2005) :

- Matrix formulation for linear first order schemes
- Projection onto characteristic directions to obtain scalar residuals to work with for the limiting procedure (similar to FV limiting on characteristic var.s)

Example 1 : Mach 3.6 scramjet inlet (Euler, perfect gas)



Example 2 : Mach 10 bow shock (Euler, perfect gas)



6
ON THE RELATIONS WITH
FEM, FV, DG, WENO FD, etc. etc.

Relations with other techniques

Continuous, Stabilized Finite Elements

By nature of the underlying approximation, these methods bear close resemblance to stabilized continuous Galerkin methods as *e.g.* the SUPG of (Hughes, Brooks *CMAME*, 1982) (bcs omitted)

$$\int_{\Omega_h} \psi_i \nabla \cdot \mathcal{F}_h(u_h) + \sum_{K \in \Omega_h} \int_K \vec{a}(u_h) \cdot \nabla \psi_i \, \tau \, \vec{a}(u_h) \cdot \nabla u_h = 0$$

which, as seen, can be written as the RD scheme

$$\sum_{K|i \in K} \phi_i^K = 0 \;\; \text{with} \;\; \phi_i^K = \int_K \psi_i \nabla \cdot \mathcal{F}_h(u_h) + \int_K \vec{a}(u_h) \cdot \nabla \psi_i \, \tau \, \vec{a}(u_h) \cdot \nabla u_h$$

Relations with other techniques

Continuous, Stabilized Finite Elements

More generally, a Petrov-Galerkin method with test space spanned by functions $\{\omega_i\}_{i\in\Omega_h}$ such that $\forall K \in \Omega_h$

$$\sum_{j\in K} \omega_i\big|_K = \sum_{j\in K} \omega_i^K = 1$$

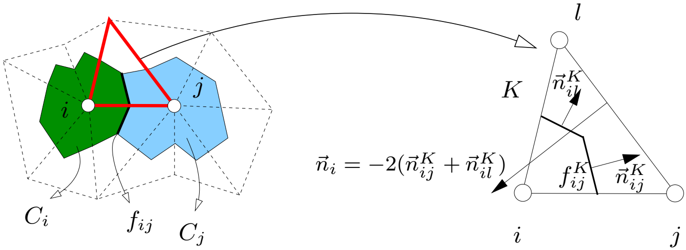
can be recast as a RD scheme

$$\sum_{K|i\in K} \phi_i^K = 0 \;\text{ with }\; \phi_i^K = \int_K \omega_i^K \nabla\cdot\mathcal{F}_h(u_h)$$

Relations with other techniques : RD and FV

Which numerical flux defines your conservative statement ?

The first relation we have already seen : FV schemes can be written such that they "sit" in an element



$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0\;, \qquad \phi_i^K = \sum_{j\in K}(\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i)\cdot \vec{n}_{ij}^K$$

Relations with other techniques : RD and FV

Which numerical flux defines your conservative statement ?

The first relation we have already seen : FV schemes can be written such that they "sit" in an element

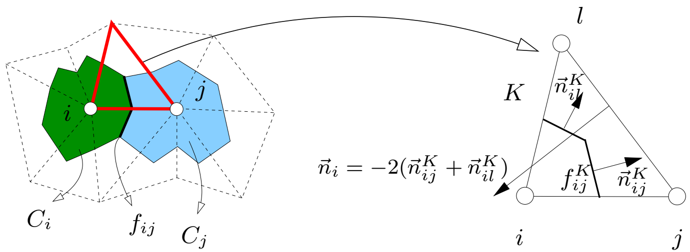
Here conservation is expressed by the FV flux function $\widehat{\mathcal{F}}$

$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K} \phi_i^K = 0 \,, \quad \phi_i^K = \sum_{j\in K}(\widehat{\mathcal{F}}_{ij} - \mathcal{F}_i)\cdot \vec{n}_{ij}^K$$

Relations with other techniques : RD and FV

Which numerical flux defines your conservative statement ?

A different view is to recast RD as FV on the median dual

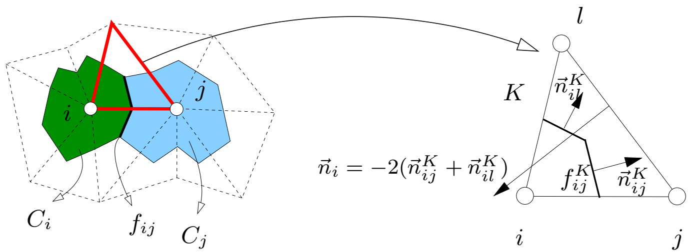


$$|C_i|\frac{du_i}{dt} + \sum_{K|i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K = 0$$

Relations with other techniques : RD and FV

Which numerical flux defines your conservative statement ?

A different view is to recast RD as FV on the median dual



What definition of $\widehat{\mathcal{F}}_{ij}$ such

$$|C_i|\frac{du_i}{dt} = -\sum_{K|i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K$$

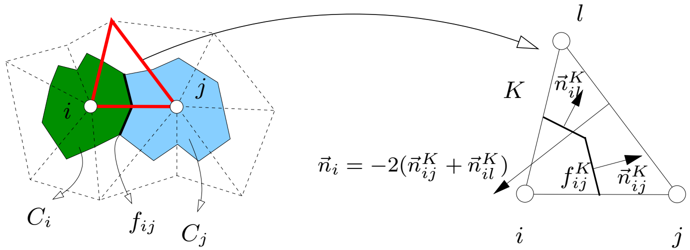
$$\sum_{j\in K}\widehat{\mathcal{F}}_{ij}\cdot\vec{n}_{ij}^K = \beta_i^K\phi^K$$

for a given $\phi_i^K = \beta_i^K\phi^K$

Relations with other techniques : RD and FV

Which numerical flux defines your conservative statement ?

A different view is to recast RD as FV on the median dual



$$|C_i|\frac{du_i}{dt} = -\sum_{K|i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}\cdot\vec{n}^K_{ij}$$

What numerical flux $\widehat{\mathcal{F}}^{\rm RD}_{ij}$ defines local conservation on the median dual cell for RD ?

Relations with other techniques : RD and FV

Which numerical flux defines your conservative statement ?

A different view is to recast RD as FV on the median dual.

$$|C_i|\frac{du_i}{dt} = -\sum_{K|i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}^{\mathrm{RD}}\cdot\vec{n}_{ij}^K$$

$$\sum_{j\in K}\widehat{\mathcal{F}}_{ij}^{\mathrm{RD}}\cdot\vec{n}_{ij}^K = \beta_i^K\Phi^K\,,\ \forall\, i$$

What numerical flux $\widehat{\mathcal{F}}_{ij}^{\mathrm{RD}}$ defines local conservation on the median dual cell for RD ?
Answer in (Abgrall, 2012)

$$\widehat{\mathcal{F}}_{ij}^{\mathrm{RD}}\cdot\vec{n}_{ij} = \Psi_i - \Psi_j \ \text{ with }\ \Psi_i = \beta_i^K\phi^K - \mathcal{F}_i\cdot\frac{\vec{n}_i}{2}$$

Relations with other techniques : RD and FV

Which numerical flux defines your conservative statement ?

A different view is to recast RD as FV on the median dual.

$$|C_i|\frac{du_i}{dt} = -\sum_{K|i\in K}\sum_{j\in K}\widehat{\mathcal{F}}_{ij}^{\mathrm{RD}}\cdot\vec{n}_{ij}^K$$

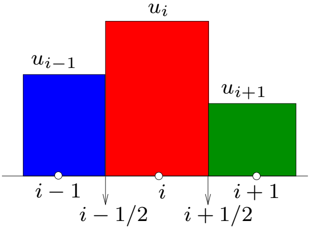
What numerical flux $\widehat{\mathcal{F}}_{ij}^{\mathrm{RD}}$ defines local conservation on the median dual cell for RD ? Answer in (Abgrall, 2012)

$$\widehat{\mathcal{F}}_{ij}^{\mathrm{RD}}\cdot\vec{n}_{ij} = \Psi_i - \Psi_j \;\text{ with }\; \Psi_i = \beta_i^K\phi^K - \mathcal{F}_i\cdot\frac{\vec{n}_i}{2}$$

Consistent 3-states numerical flux function

This is still the RD scheme we start with

Relations with other techniques : RD and FV



FV fluxes from RD schemes

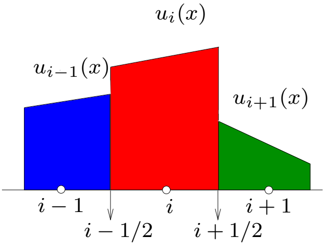
Starting point : conservation law

$$\partial_t u + \partial_x \mathcal{F}(u) = 0$$

Conservative FV :

$$\Delta x_i \frac{du_i}{dt} + \widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2} = 0$$

Relations with other techniques : RD and FV



$$\widehat{\mathcal{F}}_{i+1/2} = \widehat{\mathcal{F}}(u_i(x_{i+1/2}), u_{i+1}(x_{i+1/2}))$$

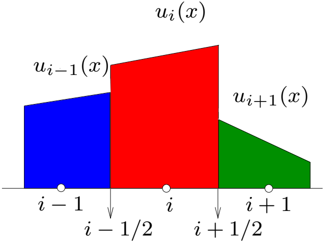
$$\widehat{\mathcal{F}}_{i-1/2} = \widehat{\mathcal{F}}(u_{i-1}(x_{i-1/2}), u_i(x_{i-1/2}))$$

FV fluxes from RD schemes

Conservative FV :

$$\Delta x_i \frac{du_i}{dt} + \widehat{\mathcal{F}}_{i+1/2} - \widehat{\mathcal{F}}_{i-1/2} = 0$$

Relations with other techniques : RD and FV



$$\Delta\mathcal{F}_i = \mathcal{F}(u_i(x_{i+1/2}) - \mathcal{F}(u_i(x_{i-1/2}))$$

$$\phi_i^{i+1/2} = \widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i(x_{i+1/2}))$$

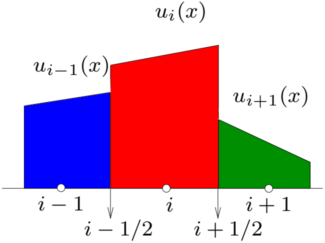
$$\phi_i^{i+1/2} = \mathcal{F}(u_{i+1}(x_{i+1/2})) - \widehat{\mathcal{F}}_{i+1/2}$$

FV fluxes from RD schemes

Conservative FV, reformulation :

$$\Delta x_i \frac{du_i}{dt} + \Delta\mathcal{F}_i + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$

Relations with other techniques : RD and FV



$$\Delta \mathcal{F}_i = \mathcal{F}(u_i(x_{i+1/2}) - \mathcal{F}(u_i(x_{i-1/2}))$$

$$\phi_i^{i+1/2} = \widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i(x_{i+1/2}))$$

$$\phi_i^{i+1/2} = \mathcal{F}(u_{i+1}(x_{i+1/2})) - \widehat{\mathcal{F}}_{i+1/2}$$

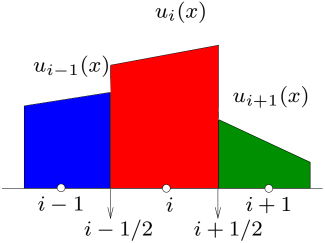
FV fluxes from RD schemes

Conservative FV, reformulation :

$$\Delta x_i \frac{du_i}{dt} + \Delta \mathcal{F}_i + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$

$$\Delta x_{i+1} \frac{du_{i+1}}{dt} + \Delta \mathcal{F}_{i+1} + \phi_{i+1}^{i+3/2} + \phi_{i+1}^{i+1/2} = 0$$

Relations with other techniques : RD and FV



$$\Delta\mathcal{F}_i = \mathcal{F}(u_i(x_{i+1/2}) - \mathcal{F}(u_i(x_{i-1/2}))$$

$$\phi_i^{i+1/2} = \widehat{\mathcal{F}}_{i+1/2} - \mathcal{F}(u_i(x_{i+1/2}))$$

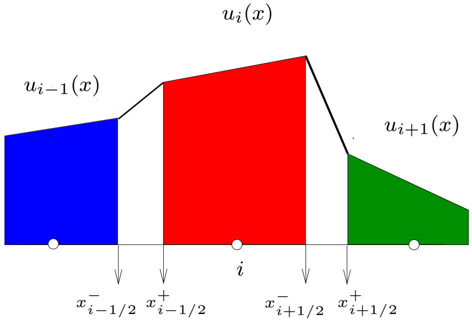
$$\phi_i^{i+1/2} = \mathcal{F}(u_{i+1}(x_{i+1/2})) - \widehat{\mathcal{F}}_{i+1/2}$$

FV fluxes from RD schemes

Conservative FV, reformulation :

$$\phi_i^{i+1/2} + \phi_{i+1}^{i+1/2} := \phi^{i+1/2} = \mathcal{F}(u_{i+1}(x_{i+1/2})) - \mathcal{F}(u_i(x_{i+1/2}))$$

Relations with other techniques : RD and FV

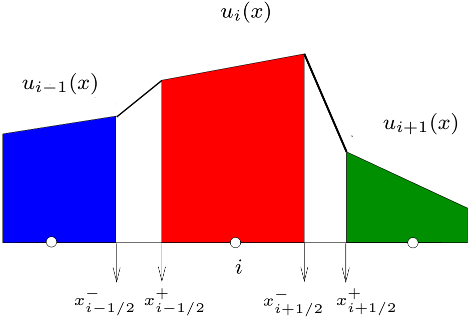


FV fluxes from RD schemes

Integrate over each control volume : no need for numerical flux (continuity through ghost elements)

In each ghost element apply any RD scheme which will provide a definition for the numerical flux as $x^+_{i\pm1/2} - x^-_{i\pm1/2} \to 0$

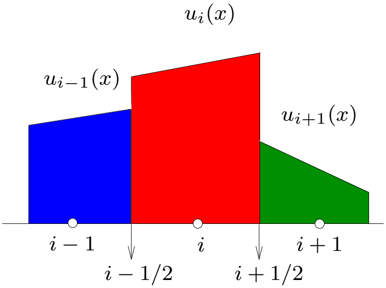
Relations with other techniques : RD and FV



FV fluxes from RD schemes

$$\Delta x_i \frac{u_i^{n+1} - u_i^n}{\Delta t} + \Delta \mathcal{F}_i + \beta_i^{i+1/2} \phi^{i+1/2} + \beta_i^{i-1/2} \phi^{i+1/2} = 0$$

Relations with other techniques : RD and FV



$$\widehat{\mathcal{F}}_{i+1/2} = \widehat{\mathcal{F}}(u_i(x_{i+1/2}), u_{i+1}(x_{i+1/2}))$$

$$\widehat{\mathcal{F}}_{i-1/2} = \widehat{\mathcal{F}}(u_{i-1}(x_{i-1/2}), u_i(x_{i-1/2}))$$

FV fluxes from RD schemes

$$\Delta x_i \frac{u_i^{n+1} - u_i^n}{\Delta t} + \overline{\mathcal{F}}^{\mathrm{RD}}_{i+1/2} - \overline{\mathcal{F}}^{\mathrm{RD}}_{i-1/2} = 0$$

$$\overline{\mathcal{F}}^{\mathrm{RD}}_{i+1/2} = \mathcal{F}(u_i(x_{i+1/2})) + \beta_i^{i+1/2}\phi^{i+1/2}$$

$$\overline{\mathcal{F}}^{\mathrm{RD}}_{i-1/2} = \mathcal{F}(u_i(x_{i-1/2})) - \beta_i^{i-1/2}\phi^{i-1/2}$$

Relations with other techniques

Other examples

In (Chou, Shu *J.Comput.Phys*, 2006 ; Chou, Shu *J.Comput.Phys*, 2006) the authors propose a WENO Finite Difference scheme consisting of a RD technique using nodal WENO reconstructions instead of Lagrange approximation. The RD formulation permits to keep the simplicity of the WENO FD approach, while allowing high accurate solutions on non-smooth cartesian meshes

Relations with other techniques

Other examples

In (Abgrall, Shu *Comm.Compu.Phys*, 2009) DG schemes are recast as RD. The key is defining the fluctuation including a numerical flux $\widehat{\mathcal{F}}$ as

$$\phi^K = \oint_{\partial K} \widehat{\mathcal{F}} \cdot \hat{n} \, dl$$

A preliminary construction of a hybrid DG-RD nonlinear scheme is proposed

Relations with other techniques

Other examples

Residual distribution schemes based on discontinuous approximation explored in (Hubbard *J.Comput.Phys*, 2008 ; Abgrall *Adv.Appl.Math.Mech*, 2010 ; Hubbard, Ricchiuto *Computers & Fluids*, 2011).

As before, the key is defining the fluctuation including a numerical flux $\widehat{\mathcal{F}}$ as

$$\phi^K = \oint_{\partial K} \widehat{\mathcal{F}} \cdot \hat{n}\, dl$$

RD techniques used to generate nonlinear schemes. The advantages of the discontinuous approximation are retained

Relations with other techniques

Important points

1. RD as a general framework to study weighted residual discretizations on general meshes
2. RD as a means of constructing non-oscillatory schemes
3. RD to define FV numerical fluxes, interesting applications in presence of source terms (see Part II)

.. to be continued ...

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