Two-Level Domain Decomposition Methods

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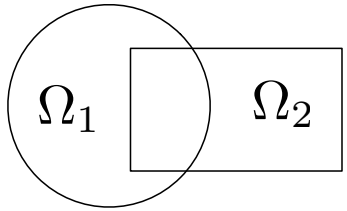
Outline

The original Schwarz Method (H.A. Schwarz, 1870)



$$-\Delta(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1$$
$$u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega$$
$$u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}.$$

$$-\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2$$
$$u_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega$$
$$u_2^{n+1} = u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.$$

Parallel algorithm, converges but very slowly, overlapping subdomains only.

The parallel version is called **Jacobi Schwarz method (JSM)**.

Motivation: pro and cons of direct solvers

Complexity of the Gauss factorization

Gauss	$d = 1$	$d = 2$	$d = 3$
dense matrix	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$
using band structure	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^{7/3})$
using sparsity	$\mathcal{O}(n)$	$\mathcal{O}(n^{3/2})$	$\mathcal{O}(n^2)$

Different sparse direct solvers

- PARDISO (http://www.pardiso-project.org)
- SUPERLU (http://crd.lbl.gov/~xiaoye/SuperLU)
- SPOOLES (www.netlib.org/linalg/spooles/spooles.2.2.html)
- MUMPS (http://graal.ens-lyon.fr/MUMPS/)
- UMFPACK (http://www.cise.ufl.edu/research/sparse/umfpack)

Why iterative solvers?

Limitations of direct solvers

In practice all direct solvers work well until a certain barrier:

- **two-dimensional problems** (10^6 unknowns)
- **three-dimensional problems** (10^5 unknowns).

Beyond, the factorization cannot be stored in memory any more.

To summarize:

- below a certain size, direct solvers are chosen.
- beyond the critical size, iterative solvers are needed.

Why domain decomposition?

Natural iterative/direct trade-off

- Parallel processing is the only way to have faster codes, new generation processors are parallel: dual, quadri core.
- Large scale computations need for an ”artificial” decomposition
- Memory requirements, direct solvers are too costly.
- Iterative solvers are not robust enough.

New iterative/direct solvers are welcome : these are domain decomposition methods

In some situations, the decomposition is natural

- Moving domains (rotor and stator in an electric motor)
- Strongly heterogeneous media
- Different physics in different subdomains

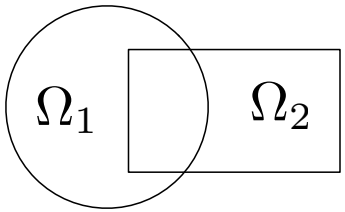
Linear Algebra from the End User point of view

Direct	DDM	Iterative
Cons: Memory Difficult to || Pros: Robustness	Pro: Flexible Naurally ||	Pros: Memory Easy to || Cons: Robustness
solve(MAT,RHS,SOL)	Few black box routines Few implementations of efficient DDM	solve(MAT,RHS,SOL)

Multigrid methods: very efficient but may lack robustness, not always applicable (Helmholtz type problems, complex systems) and difficult to parallelize.

Outline

The original Schwarz Method (H.A. Schwarz, 1870)



$$-\Delta(u) = f \quad \text{in } \Omega$$
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Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1$$
$$u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega$$
$$u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}.$$

$$-\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2$$
$$u_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega$$
$$u_2^{n+1} = u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.$$

Parallel algorithm, converges but very slowly, overlapping subdomains only.

The parallel version is called **Jacobi Schwarz method (JSM)**.

Continuous ASM and RAS - I

The algorithm acts on the local functions $(u_i)_{i=1,2}$.
To make things global, we need:

- extension operators, E_i, s.t. for a function $w_i : \Omega_i \mapsto \mathbb{R}$, $E_i(w_i) : \Omega \mapsto \mathbb{R}$ is the extension of w_i by zero outside Ω_i.
- partition of unity functions $\chi_i : \Omega_i \mapsto \mathbb{R}$, $\chi_i \geq 0$ and $\chi_i(x) = 0$ for $x \in \partial\Omega_i$ and s.t.

$$w = \sum_{i=1}^{2} E_i(\chi_i \, w_{|\Omega_i}) \, .$$

Let u^n be an approximation to the solution to the global Poisson problem and u^{n+1} is computed by solving first local subproblems and then gluing them together.

Continuous ASM and RAS - II

Local problems to solve

$$\begin{aligned} -\Delta(u_i^{n+1}) &= f \quad \text{in} \quad \Omega_i \\ u_i^{n+1} &= 0 \quad \text{on} \quad \partial\Omega_i \cap \partial\Omega \\ u_i^{n+1} &= u^n \quad \text{on} \quad \partial\Omega_i \cap \overline{\Omega}_{3-i}. \end{aligned}$$

Two ways to "glue" solutions

- Using the partition of unity functions
  Restricted Additive Schwarz (RAS)

$$u^{n+1} := \sum_{i=1}^{2} E_i(\chi_i \, u_i^{n+1}) \, .$$

- Not based on the partition of unity **Additive Schwarz (ASM)**

$$u^{n+1} := \sum_{i=1}^{2} E_i(u_i^{n+1}) \, .$$

Block Jacobi methods - I

Let us consider a linear system:

$$AU = F$$

with a matrix A of size $m \times m$, a right handside $F \in \mathbb{R}^m$ and a solution $U \in \mathbb{R}^m$ where m is an integer. Let D be the diagonal of A, the Jacobi algorithm reads:

$$DU^{n+1} = DU^n + (b - AU^n),$$

or equivalently,

$$U^{n+1} = U^n + D^{-1}(b - AU^n) = U^n + D^{-1} r^n,$$

where r^n is the residual of the equation.

Block Jacobi methods - II

We now define a block Jacobi algorithm. The set of indices $\{1, \ldots, m\}$ is partitioned into two sets

$$\mathcal{N}_1 := \{1, \ldots, m_s\} \text{ and } \mathcal{N}_2 := \{m_s + 1, \ldots, m\}.$$

Let $U_1 := U_{|\mathcal{N}_1}$, $U_2 := U_{|\mathcal{N}_2}$ and similarly $F_1 := F_{|\mathcal{N}_1}$, $F_2 := F_{|\mathcal{N}_2}$. The linear system has the following block form:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

where $A_{ij} := A_{|\mathcal{N}_i \times \mathcal{N}_j}$, $1 \leq i, j \leq 2$.

Block Jacobi methods - III

The block-Jacobi algorithm reads:

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \end{pmatrix} = \begin{pmatrix} F_1 - A_{12}\, U_2^n \\ F_2 - A_{21}\, U_1^n \end{pmatrix}. \qquad (1)$$

Let $U^n = (U_1^n, U_2^n)^T$, algorithm (1) becomes

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} U^{n+1} = F - \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} U^n. \qquad (2)$$

On the other hand, it can be rewritten equivalently

$$\begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \end{pmatrix} = \begin{pmatrix} U_1^n \\ U_2^n \end{pmatrix} + \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} r_1^n \\ r_2^n \end{pmatrix} \qquad (3)$$

where

$$r^n := F - AU^n\,, r_i^n := r^n_{|\mathcal{N}_i},\ i = 1, 2\,.$$

Block-Jacobi compact form

In order to have a more compact form, let us introduce

- R_1 the restriction operator from $\mathcal{N}$ into $\mathcal{N}_1$
- R_2 the restriction operator from $\mathcal{N}$ into $\mathcal{N}_2$.

The transpose operator R_1^T is an extension operator from $\mathcal{N}_1$ into $\mathcal{N}$ and the same holds for R_2^T.
Notice that $A_{ii} = R_i A R_i^T$.

Block-Jacobi in compact form

$$U^{n+1} = U^n + (R_1^T (R_1 A R_1^T)^{-1} R_1 + R_2^T (R_2 A R_2^T)^{-1} R_2) r^n \,. \quad (4)$$

where

$$r^n := F - AU^n \,, r_i^n := r^n_{|\mathcal{N}_i}, \; i = 1, 2 \,.$$

Schwarz algorithms as block Jacobi methods - I

Let $\Omega := (0,1)$ and consider the following BVP

$$-\Delta u = f \text{ in } \Omega$$
$$u(0) = u(1) = 0\,.$$

discretized by a three point finite difference scheme on the grid $x_j := j\,h$, $1 \leq j \leq m$ where $h := 1/(m+1)$.
Let $u_j \simeq u(x_j)$, $f_j := f(x_j)$, $1 \leq j \leq m$ and the discrete problem

$$AU = F,\ U = (u_j)_{1 \leq j \leq m},\ F = (f_j)_{1 \leq j \leq m}.$$

where $A_{j\,j} := 2/h^2$ and $A_{j\,j+1} = A_{j+1\,j} := -1/h^2$.
Let domains $\Omega_1 := (0, (m_s+1)\,h)$ and $\Omega_2 := (m_s\,h, 1)$ define an overlapping decomposition with a minimal overlap of width h.

Schwarz algorithms as block Jacobi methods - II

The discretization of the **JSM** for domain Ω_1 reads

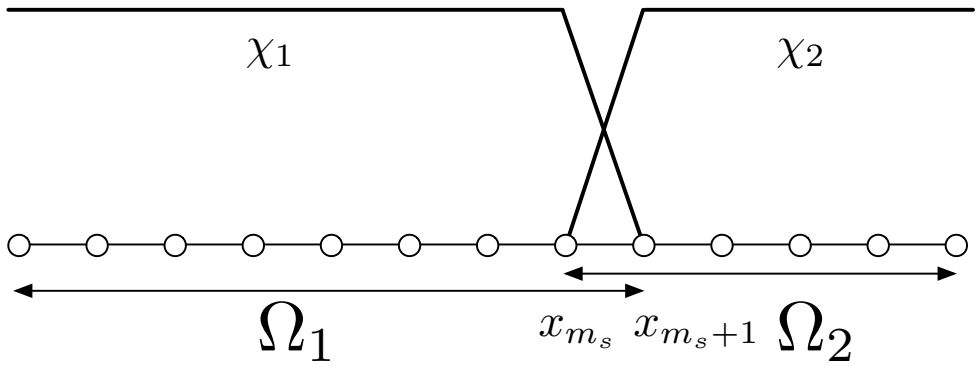
$$\begin{cases} -\dfrac{u_{1,j-1}^{n+1} - 2u_{1,j}^{n+1} + u_{1,j+1}^{n+1}}{h^2} = f_j, \ 1 \leq j \leq m_s \\ u_{1,0}^{n+1} = 0 \\ u_{1,m_s+1}^{n+1} = u_{2,m_s+1}^{n} \end{cases}.$$

Solving for $U_1^{n+1} = (u_{1,j}^{n+1})_{1 \leq j \leq m_s}$ corresponds to solving a Dirichlet boundary value problem in subdomain Ω_1 with Dirichlet data taken from the other subdomain at the previous step. Then, U_1^{n+1} and U_1^{n+1} satisfy

$$\begin{aligned} A_{11} U_1^{n+1} + A_{12} U_2^{n} &= F_1, \\ A_{22} U_2^{n+1} + A_{21} U_1^{n} &= F_2 . \end{aligned}$$

Schwarz as block Jacobi methods - III

The discrete counterpart of the extension operator E_1 (resp. E_2) is defined by $E_1(U_1) = (U_1, 0)^T$ (resp. $E_2(U_2) = (0, U_2)^T$).



then $E_1(U_1) + E_2(U_2) = E_1(\chi_1 U_1) + E_2(\chi_2 U_2) = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$.

When the overlap is minimal, **the discrete counterparts of the three Schwarz methods are equivalent to the same block Jacobi algorithm.**

Continuous level

- Ω and an overlapping decomposition $\Omega = \cup_{i=1}^{N} \Omega_i$.
- A function $u : \Omega \to \mathbb{R}$.
- Restriction of $u : \Omega \to \mathbb{R}$ to Ω_i, $1 \leq i \leq N$.
- The extension E_i of a function $\Omega_i \to \mathbb{R}$ to a function $\Omega \to \mathbb{R}$.
- Partition of unity functions χ_i, $1 \leq i \leq N$.

Discrete level

- A set of d.o.f. $\mathcal{N}$ and a decomposition $\mathcal{N} = \cup_{i=1}^{N} \mathcal{N}_i$.
- A vector $U \in \mathbb{R}^{\#\mathcal{N}}$.
- The restriction $R_i\, U$ where $U \in \mathbb{R}^{\#\mathcal{N}}$ and R_i is a rectangular $\#\mathcal{N}_i \times \#\mathcal{N}$ boolean matrix.
- Extension R_i^T.
- Diagonal matrices with positive entries, of size $\#\mathcal{N}_i \times \#\mathcal{N}_i$ s. t. $Id = \sum_{i=1}^{N} R_i^T\, D_i\, R_i$.

Restrictions operators

Let $\mathcal{T}_h$ be a mesh of a domain Ω and u_h some discretization of a function u which is the solution of an elliptic Dirichlet BVP. This yields a linear algebra problem

$$\text{Find } U \in \mathbb{R}^{\#\mathcal{N}} \text{s.t. } A\,U = F\,.$$

Define the restriction operator $r_i = E_i^T$:

$$r_i : \quad u_h \mapsto u_{h|\Omega_i}$$

Let R_i be the boolean matrix corresponding to the restriction operator r_i:

$$R_i := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \end{bmatrix}$$

$$R_i : \ \mathbb{R}^{\#\mathcal{N}} \longmapsto \mathbb{R}^{\#\mathcal{N}_i}$$

Partition of unity

We have

$$R_i : \mathbb{R}^{\#\mathcal{N}} \longmapsto \mathbb{R}^{\#\mathcal{N}_i}$$

and the transpose is a prolongation operator

$$R_i^T : \mathbb{R}^{\#\mathcal{N}_i} \longmapsto \mathbb{R}^{\#\mathcal{N}} .$$

The local Dirichlet matrices are given by

$$A_i := R_i A R_i^T .$$

We also need a kind of partition of unity defined by matrices D_i

$$D_i : \mathbb{R}^{\#\mathcal{N}_i} \longmapsto \mathbb{R}^{\#\mathcal{N}_i}$$

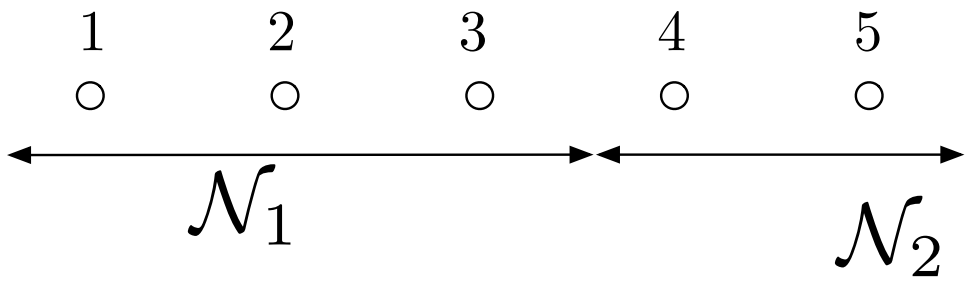
so that we have:

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id$$

Two subdomain case: 1d algebraic setting

Let $\mathcal{N} := \{1, \ldots, 5\}$ be partitioned into

$$\mathcal{N}_1 := \{1, 2, 3\} \text{ and } \mathcal{N}_2 := \{4, 5\}.$$



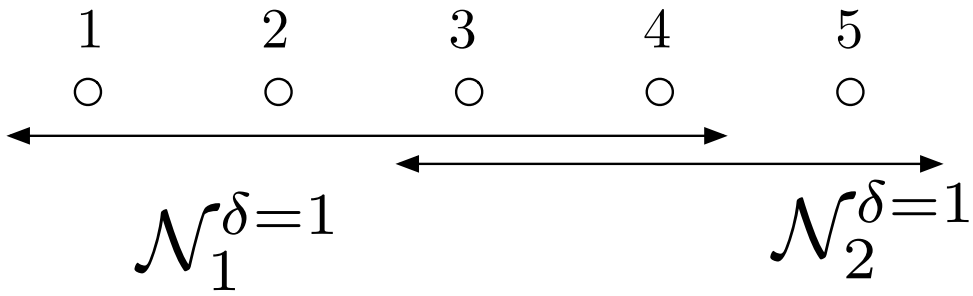
Then, matrices R_1, R_2, D_1 and D_2 are:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider now the case overlapping case

$$\mathcal{N}_1^{\delta=1} := \{1,2,3,4\} \text{ and } \mathcal{N}_2^{\delta=1} := \{3,4,5\}\,.$$



Then, matrices R_1, R_2, D_1, D_2 are:

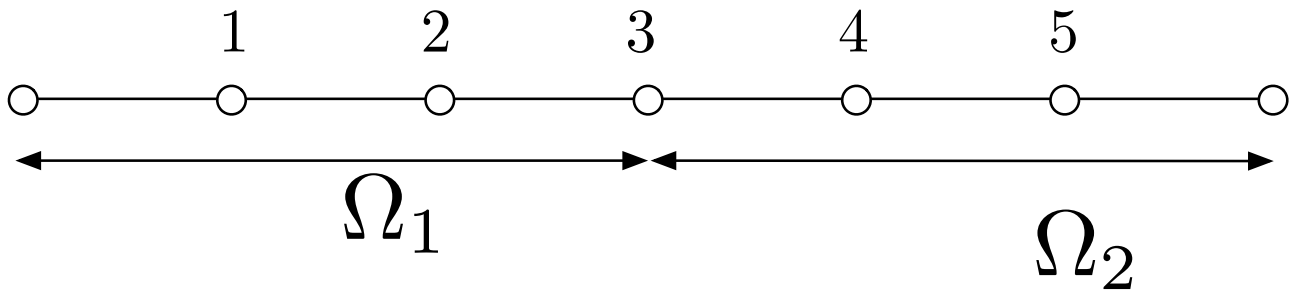
$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Two subdomain case: 1d finite element decomposition

Partition of the 1D mesh corresponds to an ovr. decomp. of $\mathcal{N}$:

$$\mathcal{N}_1 := \{1, 2, 3\} \text{ and } \mathcal{N}_2 := \{3, 4, 5\}\,.$$



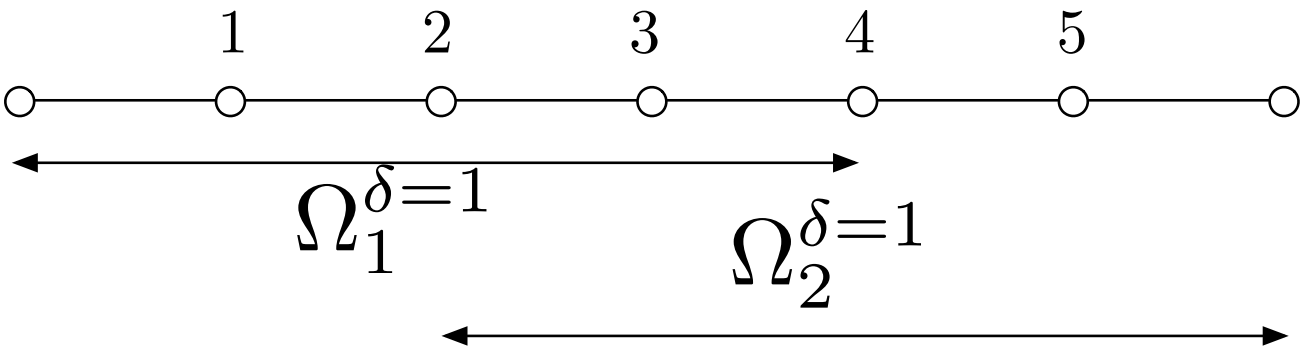
Then, matrices R_1, R_2, D_1, D_2 are:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consider now the situation of an overlapping partition.

$$\mathcal{N}_1^{\delta=1} := \{1,2,3,4\} \text{ and } \mathcal{N}_2^{\delta=1} := \{2,3,4,5\}\,.$$



Then, matrices R_1, R_2, D_1, D_2 are:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multi-D and many subdomains: General procedure

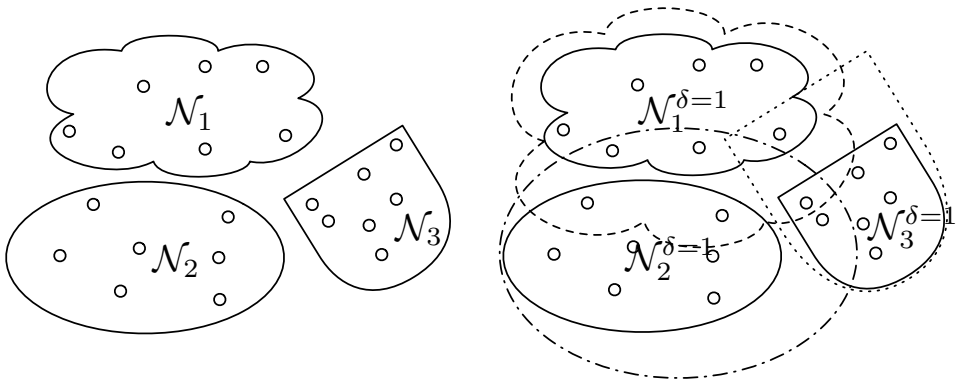
The set of indices $\mathcal{N}$ can be partitioned by an automatic graph partitioner such as **METIS** or **SCOTCH**.

- From the input matrix A, a connectivity graph is created.
- Two indices $i, j \in \mathcal{N}$ are connected if the matrix coefficient $A_{ij} \neq 0$.
- Even if matrix A is not symmetric, the connectivity graph is symmetrized.
- Algorithms that find a good partitioning of highly unstructured graphs are used.
- This distribution must be done so that the number of elements assigned to each processor is roughly the same (balance the computations among the processors).
- The number of adjacent elements assigned to different processors is minimized (minimize the communication between different processors).

Multi-D algebraic setting

Let us consider a partition into N subsets

$$\mathcal{N} := \cup_{i=1}^{N} \mathcal{N}_i, \quad \mathcal{N}_i \cap \mathcal{N}_j = \emptyset \text{ for } i \neq j\,.$$



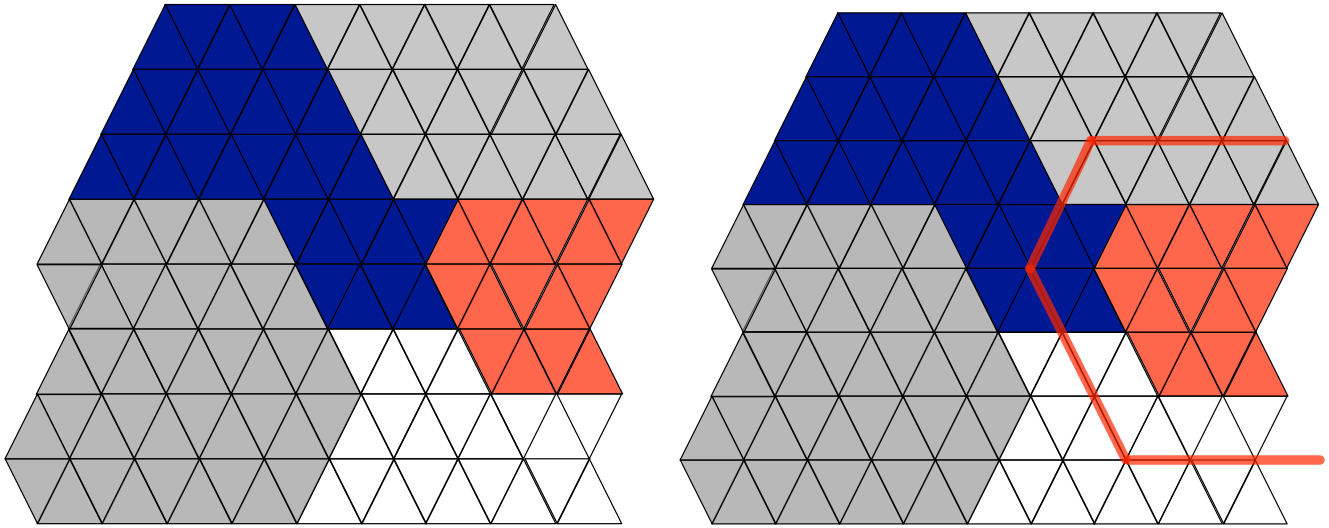
Extend each subset $\mathcal{N}_i$ with its direct neighbors to form $\mathcal{N}_i^{\delta=1}$.
Let R_i be the restriction matrix from set $\mathcal{N}$ to the subset $\mathcal{N}_i^{\delta=1}$ and D_i be a diagonal matrix of size $\#\mathcal{N}_i^{\delta=1} \times \#\mathcal{N}_i^{\delta=1}$, $1 \leq i \leq N$ such that for

$$\mathcal{M}_j := \{1 \leq i \leq N | \; j \in \mathcal{N}_i^{\delta=1}\}\,.$$

and $j \in \mathcal{N}_i^{\delta=1}$, we define $(D_i)_{jj} := 1/\#\mathcal{M}_j$.

Multi-D algebraic finite element decomposition

In a FE setting, the computational domain is the union of elements of the finite element mesh $\mathcal{T}_h$.



It is possible to create overlapping subdomains resolved by the finite element meshes:

$$\Omega_i = \bigcup_{\tau \in \mathcal{T}_{i,h}} \tau \quad \text{for } 1 \leq i \leq N\,. \tag{5}$$

Let $\{\phi_k\}_{k\in\mathcal{N}}$ be a basis of the finite element space. For $1 \leq i \leq N$, we define

$$\mathcal{N}_i := \{k \in \mathcal{N} : \text{supp}(\phi_k) \cap \Omega_i \neq \emptyset\}.$$

For all degree of freedom $k \in \mathcal{N}$, let

$$\mu_k := \#\{j : 1 \leq j \leq N \text{ and } \text{supp}(\phi_k) \cap \Omega_j \neq \emptyset\}.$$

Let R_i be the restriction matrix from set $\mathcal{N}$ to the subset $\mathcal{N}_i$ and D_i be a diagonal matrix of size $\#\mathcal{N}_i \times \#\mathcal{N}_i$, $1 \leq i \leq N$. Then, for $k \in \mathcal{N}_i$, we define $(D_i)_{kk} := 1/\mu_k$.

Algebraic formulation - JSM

Define local unknowns $U_i := R_i\, U$ for $i = 1, \ldots, N$ and local right handside $F_i := R_i\, F$.

$$\begin{aligned} R_i\, A\, U &= R_i\, A\, R_i^T\, (R_i\, U) + R_i\, A\, (Id - R_i^T\, R_i) U = F_i \\ &= R_i\, A\, R_i^T\, U_i + R_i\, A\, (Id - R_i^T\, R_i) \sum_{j=1}^{N} R_j^T\, D_j\, R_j\, U \\ &= R_i\, A\, R_i^T\, U_i + \sum_{j=1}^{N} R_i\, A\, (Id - R_i^T\, R_i) R_j^T\, D_j U_j \end{aligned}$$

Notice that $(Id - R_i^T\, R_i) R_i^T\, R_i = 0$ so we have

$$R_i\, A\, R_i^T\, U_i + \sum_{j \neq i} R_i\, A\, (Id - R_i^T\, R_i) R_j^T\, D_j U_j = F_i \qquad (6)$$

Algebraic formulation - JSM

Let us define the block matrix $\widetilde{A}$ (extended matrix)

$$(\widetilde{A})_{ii} := R_i\, A\, R_i^T,\ (\widetilde{A})_{ij} := R_i\, A\, (Id - R_i^T\, R_i) R_j^T\, D_j,\ 1 \leq i \neq j \leq N$$

Define (extended) unknown vector and right-hand side

$$\begin{aligned}\widetilde{U} &:= (U_1, \ldots, U_i, \ldots, U_N)^T \in \mathbb{R}^{\sum_{i=1}^N \#\mathcal{N}_i},\\ \widetilde{F} &:= (R_1\, F, \ldots, R_i\, F, \ldots, R_N\, F)^T \in \mathbb{R}^{\sum_{i=1}^N \#\mathcal{N}_i}.\end{aligned}$$

Let $(M_{JSM})_{ii} := (\widetilde{A})_{ii} = R_i\, A\, R_i^T$. The block Jacobi method applied to the (extended) system

$$\widetilde{A}\, \widetilde{U} = \widetilde{F}$$

is the JSM:

$$\widetilde{U}^{n+1} = \widetilde{U}^n + M_{JSM}^{-1} \widetilde{r}^n\,, \widetilde{r}^n := \widetilde{F} - \widetilde{A}\, \widetilde{U}^n. \tag{7}$$

Algebraic formulation - RAS and ASM

As for (RAS), we give the following definition

$$M_{RAS}^{-1} := \sum_{i=1}^{N} R_i^T \, D_i \, (R_i \, A \, R_i^T)^{-1} \, R_i \tag{8}$$

so that the iterative RAS algorithm reads:

$$U^{n+1} = U^n + M_{RAS}^{-1} r^n \, , r^n := F - A \, U^n.$$

For (ASM), we give the following definition

$$M_{ASM}^{-1} := \sum_{i=1}^{N} R_i^T \, (R_i \, A \, R_i^T)^{-1} \, R_i \tag{9}$$

so that the iterative ASM algorithm reads:

$$U^{n+1} = U^n + M_{ASM}^{-1} r^n \, .$$

Geometrical analysis in 1d

Let $L > 0$ and the domain $\Omega = (0, L)$ be decomposed into two subodmains $\Omega_1 := (0, L_1)$ and $\Omega_2 := (l_2, L)$ with $l_2 \leq L_1$. The error $e_i^n := u_i^n - u_{|\Omega_i}$, $i = 1, 2$ satisfies

$$\begin{array}{l} -\dfrac{d^2 e_1^{n+1}}{dx^2} = 0 \quad \text{in } (0, L_1) \\ e_1^{n+1}(0) = 0 \\ e_1^{n+1}(L_1) = e_2^n(L_1) \end{array} \quad \text{then,} \quad \begin{array}{l} -\dfrac{d^2 e_2^{n+1}}{dx^2} = 0 \quad \text{in } (l_2, L) \\ e_2^{n+1}(l_2) = e_1^{n+1}(l_2) \\ e_2^{n+1}(L) = 0\,. \end{array} \tag{10}$$

Thus the errors are affine functions in each subdomain:

$$e_1^{n+1}(x) = e_2^n(L_1)\,\frac{x}{L_1} \text{ and } e_2^{n+1}(x) = e_1^{n+1}(l_2)\,\frac{L - x}{L - l_2}\,.$$

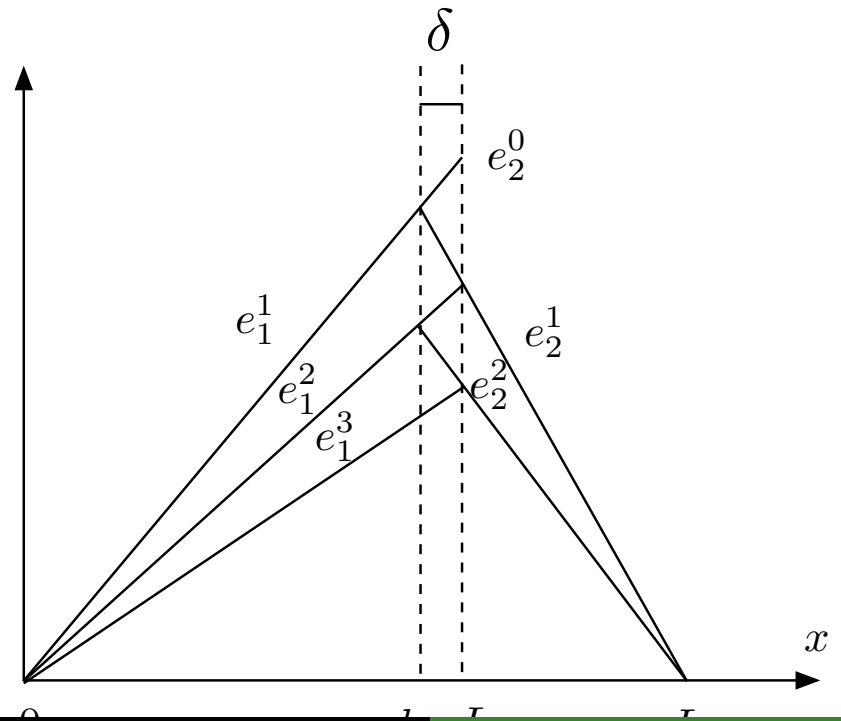
Thus, we have

$$e_2^{n+1}(L_1) = e_1^{n+1}(l_2)\,\frac{L-L_1}{L-l_2} = e_2^{n}(L_1)\,\frac{l_2}{L_1}\,\frac{L-L_1}{L-l_2}\,.$$

Let $\delta := L_1 - l_2$ denote the size of the overlap, we have

$$e_2^{n+1}(L_1) = \frac{l_2}{l_2+\delta}\,\frac{L-l_2-\delta}{L-l_2}\,e_2^{n}(L_1) = \frac{1-\delta/(L-l_2)}{1+\delta/l_2}\,e_2^{n}(L_1)\,.$$

It is clear that $\delta > 0$ is sufficient and necessary to have convergence.



Fourier analysis in 2d - I

Let $\mathbb{R}^2$ decomposed into two half-planes $\Omega_1 = (-\infty, \delta) \times \mathbb{R}$ and $\Omega_2 = (0, \infty) \times \mathbb{R}$ with an overlap of size $\delta > 0$ and the problem

$$\begin{aligned} (\eta - \Delta)(u) &= f \quad \text{in} \quad \mathbb{R}^2, \\ u &\quad \text{is bounded at infinity}, \end{aligned}$$

By linearity, the errors $e_i^n := u_i^n - u|_{\Omega_i}$ satisfy the JSM $f = 0$:

$$\begin{aligned} (\eta - \Delta)(e_1^{n+1}) &= 0 \quad \text{in} \quad \Omega_1 \\ e_1^{n+1} &\quad \text{is bounded at infinity} \\ e_1^{n+1}(\delta, y) &= e_2^n(\delta, y), \end{aligned} \tag{11}$$

$$\begin{aligned} (\eta - \Delta)(e_2^{n+1}) &= 0 \quad \text{in} \quad \Omega_2 \\ e_2^{n+1} &\quad \text{is bounded at infinity} \\ e_2^{n+1}(0, y) &= e_1^n(0, y). \end{aligned} \tag{12}$$

Fourier analysis in 2d - II

By taking the partial Fourier transform of the equation in the y direction we get:

$$\left(\eta - \frac{\partial^2}{\partial x^2} + k^2\right)(\hat{e}_1^{n+1}(x,k)) = 0 \quad \text{in} \quad \Omega_1.$$

For a given k, the solution

$$\hat{e}_1^{n+1}(x,k) = \gamma_+^{n+1}(k)\exp(\lambda^+(k)x) + \gamma_-^{n+1}(k)\exp(\lambda^-(k)x).$$

must be bounded at $x = -\infty$. This implies

$$\hat{e}_1^{n+1}(x,k) = \gamma_+^{n+1}(k)\exp(\lambda^+(k)x)$$

and similarly,

$$\hat{e}_2^{n+1}(x,k) = \gamma_-^{n+1}(k)\exp(\lambda^-(k)x)$$

Fourier analysis in 2d - III

From the interface conditions we get

$$\gamma_+^{n+1}(k) = \gamma_-^n(k)\ \exp(\lambda^-(k)\delta),\ \gamma_-^{n+1}(k) = \gamma_+^n(k)\exp(-\lambda^+(k)\delta).$$

Combining these two and denoting $\lambda(k) = \lambda^+(k) = -\lambda^-(k)$, we get for $i = 1, 2$,

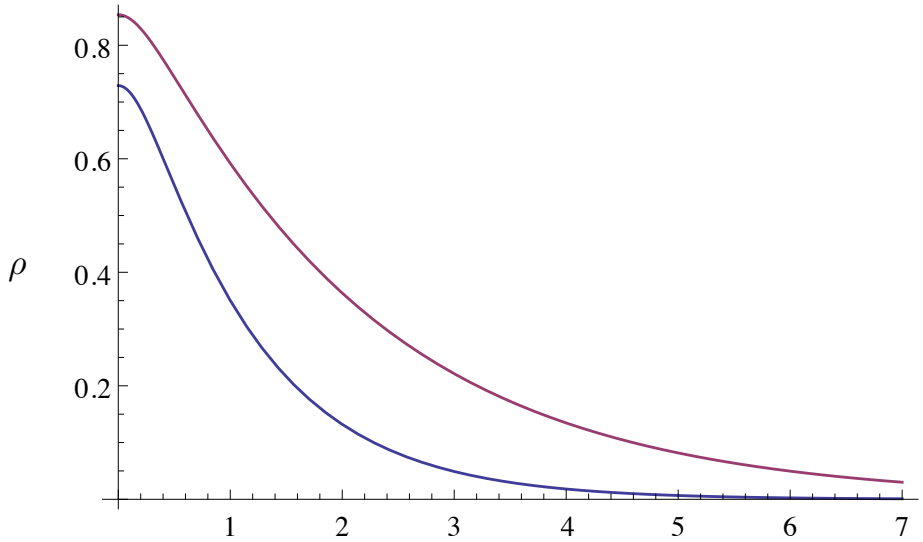
$$\gamma_\pm^{n+1}(k) = \rho(k;\alpha,\delta)^2\,\gamma_\pm^{n-1}(k)$$

with ρ the convergence rate given by:

$$\rho(k;\alpha,\delta) = \exp(-\lambda(k)\delta), \tag{13}$$

where $\lambda(k) = \sqrt{\eta + k^2}$.

Fourier analysis in 2d - IV



Remark

We have the following properties:

- *For all $k \in \mathbb{R}$, $\rho(k) < \exp(-\sqrt{\eta}\,\delta) < 1$ so that $\gamma_i^n(k) \to 0$ uniformly as n goes to infinity.*
- *$\rho \to 0$ as k tends to infinity, high frequency modes of the error converge very fast.*
- *When there is no overlap ($\delta = 0$), $\rho = 1$ and there is stagnation of the method.*

About FreeFem++ (survival kit)

`FreeFem++` allows a very simple and natural way to solve a great variety of variational problems (FEM, DG).

It is possible to have access to the underlying linear algebra such as the stiffness or mass matrices.

Tutorial: `http://www.cmap.polytechnique.fr/spip.php?article239`.

A very detailed documentation of `FreeFem++` is available on the official website `http://www.freefem.org/ff++`

`http://www.freefem.org/ff++/ftp/freefem++doc.pdf`

Let a homogeneous Dirichlet boundary value problem for a Laplacian defined on a unit square $\Omega =]0,1[^2$:

$$\begin{cases} -\Delta u = f & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases} \tag{14}$$

The variational formulation of the problem

$$\text{Find } u \in H_0^1(\Omega) := \{ w \in H^1(\Omega) \ : \ w = 0, \ \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \}$$

such that

$$\int_\Omega \nabla u . \nabla v dx - \int_\Omega f \, v \, dx = 0, \forall v \in H_0^1(\Omega) \, .$$

Feature of `Freefem++`: penalization of Dirichlet BC. Let *TGV* (*Très Grande Valeur* in French) be a very large value, the above variational formulation is approximated by
Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u . \nabla v dx + TGV \int_{\cup_{i=1,\ldots,4}\Gamma_i} u \, v - \int_\Omega f v \, dx = 0, \forall v \in H^1(\Omega) \, .$$

The following FreeFem++ script is solving this problem

```
// Number of mesh points in x and y directions
int Nbnoeuds=10;
```

The text after // symbols are comments ignored by the FreeFem++ language.
Each new variable must be declared with its type (here int designs integers).

```
//Mesh definition
mesh Th=square(Nbnoeuds,Nbnoeuds,[x,y]);
```

The function square returns a structured mesh of the square, the sides of the square are labelled from 1 to 4 in trigonometrical sense.

Define the function representing the right hand side

```
// Function of x and y
func f=x*y;
```

and the P_1 finite element space `Vh` over the mesh `Th`.

```
// Finite element space on the mesh Th
fespace Vh(Th,P1);
//uh and vh are of type Vh
Vh uh,vh;
```

The functions u_h and v_h belong to the P_1 finite element space V_h which is an approximation to $H^1(\Omega)$.

```
// variational problem definition
problem heat(uh,vh,solver=LU)=
        int2d(Th)(dx(uh)*dx(vh)+dy(uh)*dy(vh))
        -int2d(Th)(f*vh)
        +on(1,2,3,4,uh=0);
```

The keyword `problem` allows the definition of a variational problem (without solving it)

$$\int_\Omega \nabla u_h . \nabla v_h dx + TGV \int_{\cup_{i=1,\ldots,4}\Gamma_i} u_h\, v_h - \int_\Omega f v_h dx = 0, \forall v_h \in V_h\,.$$

where TGV is equal to 10^{30}.
The parameter `solver` sets the method that will be used to solve the resulting linear system. To solve the problem we need

```
//Solving the problem
heat;
// Plotting the result
plot(uh,wait=1);
```

The `Freefem++` script can be saved with your favourite text editor (e.g. under the name `heat.edp`). In order to execute the script write the shell command

```
FreeFem++ heat.edp
```

The result will be displayed in a graphic window.

Solve Neumann or Fourier boundary conditions such as

$$\begin{cases} -\Delta u + u = f & \text{dans } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{sur } \Gamma_1 \\ u = 0 & \text{sur } \Gamma_2 \\ \frac{\partial u}{\partial n} + \alpha u = g & \text{sur } \Gamma_3 \cup \Gamma_4 \end{cases} \tag{15}$$

The new variational formulation consists in determining $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h . \nabla v_h dx + \int_{\Gamma_3 \cup \Gamma_4} \alpha u_h v_h + TGV \int_{\Gamma_2} u_h . v_h$$
$$- \int_{\Gamma_3 \cup \Gamma_4} g v_h - \int_\Omega f v_h dx = 0, \forall v_h \in V_h.$$

The `Freefem++` definition of the problem

```
problem heat(uh,vh)=
int2d(Th)(dx(uh)*dx(vh)+dy(uh)*dy(vh))
+int1d(Th,3,4)(alpha*uh*vh)
-int1d(Th,3,4)(g*vh)
-int2d(Th)(f*vh)
+on(2,uh=0);
```

In order to use some **linear algebra** package, we need the matrices. The keyword `varf` allows the definition of a variational formulation

```
varf heat(uh,vh)=
int2d(Th)(dx(uh)*dx(vh)+dy(uh)*dy(vh))
+int1d(Th,3,4)(alpha*uh*vh)
-int1d(Th,3,4)(g*vh)
-int2d(Th)(f*vh)
+on(2,uh=0);
matrix Aglobal; // stiffness sparse matrix
Aglobal = heat(Vh,Vh,solver=UMFPACK);// UMFPACK solver
Vh rhsglobal;  //right hand side vector
rhsglobal[] = heat(0,Vh);
```

Here `rhsglobal` is a FE function and the associated vector of d.o.f. is `rhsglobal[]`.
The linear system is solved by using UMFPACK

```
// Solving the problem by a sparse LU sover
uh[] = Aglobal^-1*rhsglobal[];
```

Decomposition into overlapping domains I

Suppose we want a decomposition of a rectangle Ω into nn$\times$mm domains with approximately nloc points in one direction.

```
int nn=4,mm=4;
int npart= nn*mm;
int nloc = 20;
real allong = 1;
allong = real(nn)/real(mm);

mesh Th=square(nn*nloc*allong,mm*nloc,[x*allong,y]);
fespace Vh(Th,P1);
fespace Ph(Th,P0);
Ph  part;
Ph xx=x,yy=y;
part = int(xx/allong*nn)*mm + int(yy*mm);
plot(part,fill=1,value=1,wait=1,ps="decompunif.eps");
```

For arbitrary decompositions, use METIS or SCOTCH.

```
int nn=4,mm=4;
int npart= nn*mm;
int nloc = 20;
real allong = 1;
allong = real(nn)/real(mm);

mesh Th=square(nn*nloc*allong,mm*nloc,[x*allong,y]);
fespace Vh(Th,P1);
fespace Ph(Th,P0);
Ph  part;
bool withmetis = 1;
if(withmetis)  // Metis partition
  {
    load "metis";
    int[int] nupart(Th.nt);
    metisdual(nupart,Th,npart);
    for(int i=0;i<nupart.n;++i)
      part[][i]=nupart[i];
  }
plot(part,fill=1,value=1,wait=1,ps="decompMetis.eps");
```

Decomposition into overlapping domains II

To build the overlapping decomposition and the associated algebraic call the routine `SubdomainsPartitionUnity`.
Output:

- overlapping meshes `aTh[i]`
- the restriction/interpolation operators `Rih[i]` from the local finite element space `Vh[i]` to the global one `Vh`
- the diagonal local matrices `Dih[i]` from the partition of unity.

```
include "createPartition.edp";
include "decompMetis.edp";

// overlapping partition
int sizeovr = 3;
mesh[int] aTh(npart);   // sequence of ovr. meshes
matrix[int] Rih(npart); // local restriction operators
matrix[int] Dih(npart); // partition of unity operators

SubdomainsPartitionUnity(Th,part[],sizeovr,aTh,Rih,Dih);
```

RAS and ASM: global data

We first need to define the global data.

```
// Solve Dirichlet subproblem Delta (u) = f
// u = 1 on the global boundary
int[int] chlab=[1,1  ,2,1  ,3,1  ,4,1  ];
Th=change(Th,refe=chlab);

macro Grad(u) [dx(u),dy(u)]          // EOM
func f = 1;                          // right hand side
func g = 1;                          // Dirichlet data

// global problem
Vh rhsglobal,uglob;
varf vaglobal(u,v) =
    int2d(Th)(Grad(u)'*Grad(v))
    +on(1,u=g) + int2d(Th)(f*v);
matrix Aglobal;
Aglobal = vaglobal(Vh,Vh,solver = UMFPACK);  // matrix
rhsglobal[] = vaglobal(0,Vh);                // rhs
uglob[] = Aglobal^-1*rhsglobal[];
```

RAS and ASM: local data

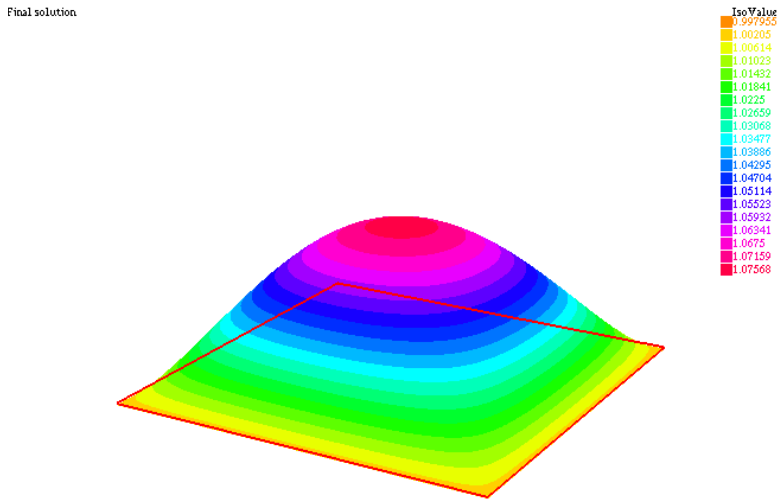
And then the local problems

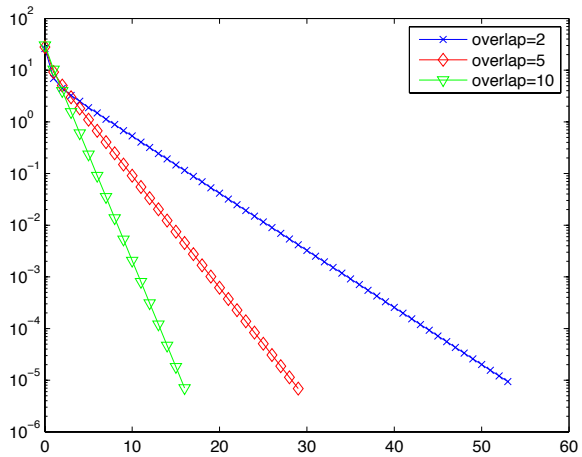
```
// overlapping partition
int sizeovr = 4;
mesh[int] aTh(npart);                  // overlapping meshes
matrix[int] Rih(npart);                // restriction operators
matrix[int] Dih(npart);                // partition of unity
SubdomainsPartitionUnity(Th,part[],sizeovr,aTh,Rih,Dih);

matrix[int] aA(npart);                 // Dirichlet matrices
for(int i = 0;i<npart;++i)
  {
    cout << " Domain :" << i << "/" << npart << endl;
    matrix aT = Aglobal*Rih[i]';
    aA[i] =  Rih[i]*aT;
    set(aA[i],solver = UMFPACK);// direct solvers
  }
```

RAS and ASM : Schwarz iteration

Convergence

Convergence history of the RAS solver for different values of the overlapping parameter.





Note that this convergence, not very fast even in a simple configuration of 4 subdomains.
The iterative version of ASM does not converge. For this reason, the ASM method is always used a preconditioner for a Krylov method such as CG, GMRES or BiCGSTAB.

Outline

Fixed point method

Consider a well-posed but difficult to solve linear system

$$A\,x = b$$

and B an "easy to invert" matrix of the same size than A. A possible iterative method is a fixed point algorithm

$$x^{n+1} = x^n + B^{-1}(b - A\,x^n)$$

and x is a fixed point of the operator:

$$x \longmapsto x + B^{-1}(b - A\,x)\,.$$

Let $r_0 := b - Ax^0$ and $C := B^{-1}\,A$, a direct computation shows that we have:

$$x^n = \sum_{i=0}^{n} (I_d - C)^i\, B^{-1} r_0 + x^0\,. \tag{16}$$

We have convergence iff the spectral radius of the matrix $I_d - C$ is smaller than one.

Why Krylov methods I

Consider now a preconditioned Krylov applied to the linear system:

$$B^{-1} A x = B^{-1} b$$

Let us denote x^0 an initial guess and $r^0 := B^{-1} b - C x^0$ the initial residual. Then $y := x - x^0$ solves

$$C y = r^0 .$$

The basis for Krylov methods is the following

Lemma

Let C be an invertible matrix of size $N \times N$.
Then, there exists a polynomial $\mathcal{P}$ of degree $p < N$ such that

$$C^{-1} = \mathcal{P}(C) .$$

Proof.

Let be a minimal polynomial of C of degree $d \leq N$:

$$\mathcal{M}(X) := \sum_{i=0}^{d} a_i X^i$$

We have $\sum_{i=0}^{d} a_i C^i = 0$ and there is no non zero polynomial of lower degree that annihilates C. Thus, a_0 cannot be zero since

$$C \sum_{i=1}^{d} a_i C^{i-1} = 0 \Rightarrow \sum_{i=1}^{d} a_i C^{i-1} = 0 .$$

Then, $\sum_{i=0}^{d-1} a_{i+1} X^i$ would be an annihiling polynomial of C of degree lower than d. This implies

$$I_d + C \sum_{i=1}^{d} \frac{a_i}{a_0} C^{i-1} = 0 \Rightarrow C^{-1} := -\sum_{i=1}^{d} \frac{a_i}{a_0} C^{i-1} .$$

Coming back to the linear system, we have

$$x = x^0 + \sum_{i=1}^{d} (-\frac{a_i}{a_0}) C^{i-1} r^0 .$$

Thus, it makes sense to introduce Krylov spaces, $\mathcal{K}^n(C, r^0)$

$$\mathcal{K}^n(C, r^0) := Span\{r^0, Cr^0, \ldots, C^{n-1} r^0\}, \; n \geq 1.$$

to seek y^n an approximation to y.
Example: The CG methods applies to symmetric positive definite (SPD) matrices and minimizes the A^{-1}-norm of the residual when solving $Ax = b$:

$$CG \left\{ \begin{array}{l} \text{Find } y^n \in \mathcal{K}^n(A, r^0) \text{ such that} \\ \|A y^n - r^0\|_{A^{-1}} = \min\limits_{w \in \mathcal{K}^n(A, r^0)} \|A w - r^0\|_{A^{-1}} . \end{array} \right.$$

A detailed analysis reveals that $x^n = y^n + x_0$ can be obtained by the quite cheap recursion formula:

for $i = 1, 2, \ldots$ **do**
  $\rho_{i-1} = (r_{i-1}, r_{i-1})_2$
  if $i = 1$ **then**
    $p_1 = r_0$
  else
    $\beta_{i-1} = \rho_{i-1} / \rho_{i-2}$
    $p_i = r_{i-1} + \beta_{i-1} p_{i-1}$
  end if
  $q_i = A p_{i-1}$
  $\alpha_i = \dfrac{\rho_{i-1}}{(p_i, q_i)_2}$
  $x_i = x_{i-1} + \alpha_i p_i$
  $r_i = r_{i-1} - \alpha_i q_i$
  check convergence; continue if necessary
end for

By solving an optimization problem:

$$GMRES \left\{ \begin{array}{l} \text{Find } y^n \in \mathcal{K}^n(C, r^0) \text{ such that} \\ \|C\, y^n - r^0\|_2 = \min\limits_{w \in \mathcal{K}^n(C, r^0)} \|C\, w - r^0\|_2 \end{array} \right.$$

a preconditioned Krylov solve will generate an optimal x_K^n in

$$\mathcal{K}^n(C, B^{-1} r_0) := x_0 + Span\{B^{-1} r_0,\ C\, B^{-1} r_0, \ldots, C^{n-1}\, B^{-1} r_0\}\,.$$

This minimization problem is of size n. When n is small w.r.t. N, its solving has a marginal cost. Thus, x_K^n has a computing cost similar to that of x^n. But, since $x^n \in \mathcal{K}^n(B^{-1}A, B^{-1} r_0)$ as well but with "frozen" coefficients, we have that x_n is less optimal (actually much much less) than x_K^n.

Schwarz methods as preconditioners

In the previous Krylov methods we can use as preconditioner

- RAS (in conjunction with BiCGStab or GMRES)

$$B^{-1} := M_{RAS}^{-1} = \sum_{i=1}^{N} R_i^T D_i \left(R_i A R_i^T\right)^{-1} R_i$$

- ASM (in a CG methods)

$$B^{-1} := M_{ASM}^{-1} = \sum_{i=1}^{N} R_i^T \left(R_i A R_i^T\right)^{-1} R_i$$

Preconditioner in CG

We use

- M_{ASM}^{-1} as a preconditioner
- a Krylov method: conjugate gradient since M_{ASM}^{-1} and A are symmetric.

At iteration m the error for the PCG method is bounded by:

$$||\bar{x} - x_m||_{M_{ASM}^{-\frac{1}{2}} A M_{ASM}^{-\frac{1}{2}}} \leq 2 \left[\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right]^m ||\bar{x} - x_0||_{M_{ASM}^{-\frac{1}{2}} A M_{ASM}^{-\frac{1}{2}}}.$$

where κ is the condition number of $M_{ASM}^{-1} A$ and $\bar{x}$ is the exact solution.

The CG with the ASM preconditioner becomes:

for $i = 1, 2, \ldots$ **do**
  $\rho_{i-1} = (r_{i-1}, M_{ASM}^{-1} r_{i-1})_2$
  if $i = 1$ **then**
    $p_1 = M_{ASM}^{-1} r_0$
  else
    $\beta_{i-1} = \rho_{i-1} / \rho_{i-2}$
    $p_i = M_{ASM}^{-1} r_{i-1} + \beta_{i-1} p_{i-1}$
  end if
  $q_i = A p_{i-1}$
  $\alpha_i = \dfrac{\rho_{i-1}}{(p_i, q_i)_2}$
  $x_i = x_{i-1} + \alpha_i p_i$
  $r_i = r_{i-1} - \alpha_i q_i$
  check convergence; continue if necessary
end for

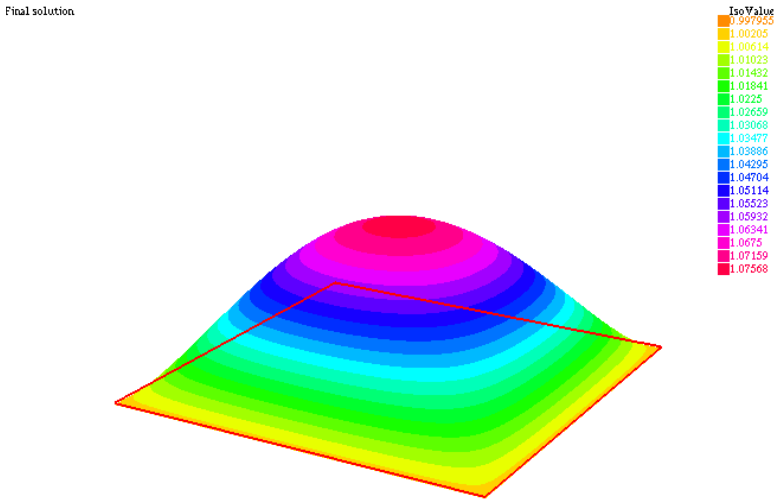
The action of the global operator is given by

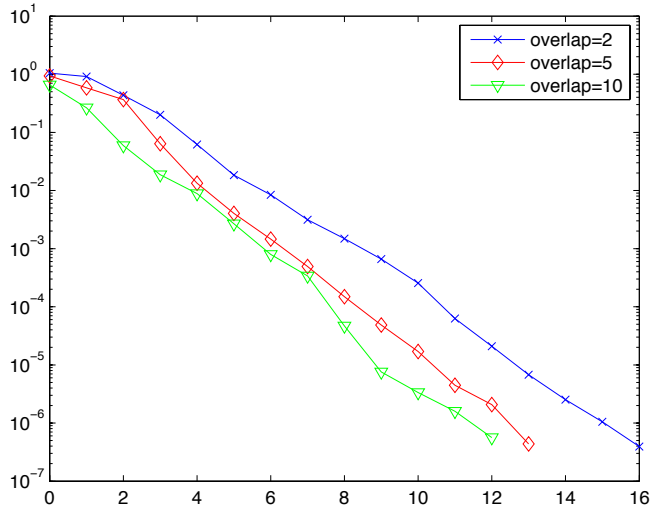
```
Vh Ax;
func real[int] A(real[int] &x)                    // A*u
{
  Ax[]= Aglobal*x;
  return Ax[];
}
```

The preconditioning method can be Additive Schwarz (ASM)

```
func real[int] Mm1(real[int] &l)
{
   s = 0;
   for(int i=0;i<npart;++i)
     {
       mesh Thi = aTh[i];
       real[int] bi = Rih[i]*l;          // restricts rhs
       real[int] ui = aA[i] ^-1 * bi;    // local solves
       s[] += Rih[i]'*ui;                // prolongation
      }
   return s[];
}
```

The Krylov method applied in this case is the CG. The performance is now less sensitive to the overlap size.





We can also use RAS as a preconditioner, by taking into account the partition of unity

```
func real[int] Mm1(real[int] &l)
{
   s = 0;
   for(int i=0;i<npart;++i)
     {
       mesh Thi = aTh[i];
       real[int] bi = Rih[i]*l;          // restricts rhs
       real[int] ui = aA[i] ^-1 * bi; // local solves
       bi = Dih[i]*ui;
       s[] += Rih[i]'*bi;                // prolongation
     }
   return s[];
}
```

this time in conjuction with BiCGStab since we deal with non-symmetric problems.

Outline

Helmholtz Equation

We want to solve

$$\begin{aligned} -\omega^2 u - \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The convergence rate of the classical Schwarz method is:

$$\rho = e^{-\sqrt{-\omega^2 + k^2}\,\delta}$$

One possible Improvement: other interface conditions

(P.L. Lions, 1988)

$$
\begin{aligned}
&-\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1, \\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\
&(\frac{\partial}{\partial n_1} + \alpha)(u_1^{n+1}) = (-\frac{\partial}{\partial n_2} + \alpha)(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2},
\end{aligned}
$$

(n_1 and n_2 are the outward normal on the boundary of the subdomains)

$$
\begin{aligned}
&-\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2, \\
&u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega \\
&(\frac{\partial}{\partial n_2} + \alpha)(u_2^{n+1}) = (-\frac{\partial}{\partial n_1} + \alpha)(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.
\end{aligned}
$$

with $\alpha > 0$. Overlap is not necessary for convergence. Extended to the Helmholtz equation (B. DesprÃ¨s, 1991) a.k.a FETI 2 LM (Two-Lagrange Multiplier) Method, 1998.

Convergence with second order interface conditions

$$(\frac{\partial}{\partial n_i} + \alpha - \frac{\partial}{\partial \tau}\gamma\frac{\partial}{\partial \tau})$$

Proof of convergence valid for a problem discretized by a finite volume scheme. At the continuous level we consider the following problem.

$$\begin{aligned} &\eta(\mathbf{x})u - div(\kappa(\mathbf{x})\nabla u) = f \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $\eta(\mathbf{x}), \kappa(\mathbf{x}) > C > 0$. The domain is decomposed into N subdomains $(\Omega_i)_{1 \leq i \leq N}$ without overlap.
Let Γ_{ij} denote the interface $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$.

Convergence

The interface condition is

$$\kappa(\mathbf{x})\frac{\partial u_i^{n+1}}{\partial n_i} + \alpha_{ij}(\mathbf{x})u_i^{n+1} - \frac{\partial}{\partial \tau_i}(\gamma_{ij}(\mathbf{x})\frac{\partial u_i^{n+1}}{\partial \tau_i}) = -\kappa(\mathbf{x})\frac{\partial u_j^{n}}{\partial n_j} + \alpha_{ij}(\mathbf{x})u_j^{n} - \frac{\partial}{\partial \tau_j}(\gamma_{ij}(\mathbf{x})\frac{\partial u_j^{n}}{\partial \tau_j}) \text{ on } \Gamma_{ij}.$$

with

$$\alpha_{ij}(\mathbf{x}) = \alpha_{ji}(\mathbf{x}) \geq \alpha_0 > 0,$$

$$\gamma(\mathbf{x})_{ij} = \gamma(\mathbf{x})_{ji} \geq 0 \text{ et } \gamma_{ij}(\mathbf{x}) = 0 \text{ sur } \partial\Gamma_{ij}$$

Let us denote

$$\Lambda_{ij} = \alpha_{ij}(\mathbf{x}) - \frac{\partial}{\partial \tau_i}(\gamma_{ij}(\mathbf{x})\frac{\partial}{\partial \tau_i}), \quad \mathbf{x} \in \Gamma_{ij}.$$

Lemma

The algorithm converges in H^1 (discrete norm).

Optimal Interface Conditions

(Hagstrom, 1988)
Constant coefficient Advection-Diffusion equation on a domain decomposed into two subdomains.

$$
\begin{aligned}
&(\vec{a}\nabla - \nu\Delta)(u_1^{n+1}) = f \quad \text{in } \Omega_1,\\
&u_1^{n+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\\
&(\frac{\partial}{\partial n_1} + \mathcal{B}_1)(u_1^{n+1}) = (-\frac{\partial}{\partial n_2} + \mathcal{B}_1)(u_2^n) \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2},
\end{aligned}
$$

$$
\begin{aligned}
&(\vec{a}\nabla - \nu\Delta)(u_2^{n+1}) = f \quad \text{in } \Omega_2,\\
&u_2^{n+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega\\
&(\frac{\partial}{\partial n_2} + \mathcal{B}_2)(u_2^{n+1}) = (-\frac{\partial}{\partial n_1} + \mathcal{B}_2)(u_1^n) \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.
\end{aligned}
$$

where $\mathcal{B}_i$, $i = 1, 2$ are defined via a Fourier transform along the interface

Convergence in two iterations

Optimal Interface Conditions

Let us consider the problem

$$\begin{aligned} &\mathcal{L}_i(P_i) = f \quad \text{in } \Omega_i, \quad i = 1, 2 \\ &P_1 = P_2 \quad \text{on } \Gamma_{12}, \\ &\kappa_1 \frac{\partial P_1}{\partial n_1} + \kappa_2 \frac{\partial P_2}{\partial n_2} = 0 \quad \text{on } \Gamma_{12}. \end{aligned}$$

where

$$\mathcal{L}_i = \eta_i - div(\kappa_i \nabla \dot{)}$$

Could be as well Fluid/Structure interaction or Plate/beam coupling.

Optimal Interface Conditions

Let

$$u_i = \kappa_i \nabla P_i$$

Let us consider a Schwarz type method:

$$\begin{aligned} &\mathcal{L}_1(P_1^{n+1}) = f \quad \text{in } \Omega_1 \\ &P_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \\ &u_1^{n+1}.\vec{n}_1 + \mathcal{B}_1(P_1^{n+1}) \\ &= -u_2^n.\vec{n}_2 + \mathcal{B}_1(P_2^n) \quad \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} &\mathcal{L}_2(P_2^{n+1}) = f \quad \text{in } \Omega_2 \\ &P_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\ &u_2^{n+1}.\vec{n}_2 + \mathcal{B}_2(P_2^{n+1}) \\ &= -u_1^n.\vec{n}_1 + \mathcal{B}_2(P_1^n) \quad \text{on } \Gamma_2 \end{aligned}$$

We take

$$\mathcal{B}_1 = DtN_2.$$

and have convergence in two iterations.

Optimal Interface Conditions

We introduce the DtN (Dirichlet to Neumann) map (a.k.a. Steklov-Poincaré):

$$\text{Let } P_0 : \Gamma_{12} \to \mathbb{R} \tag{17}$$
$$DtN_2(P_0) \equiv \kappa_2 \frac{\partial}{\partial n_2}(P)_{|\Gamma_{12}}$$

where n_2 is the outward normal to $\Omega_2 \backslash \bar{\Omega}_1$ and P satisfies the following boundary value problem:

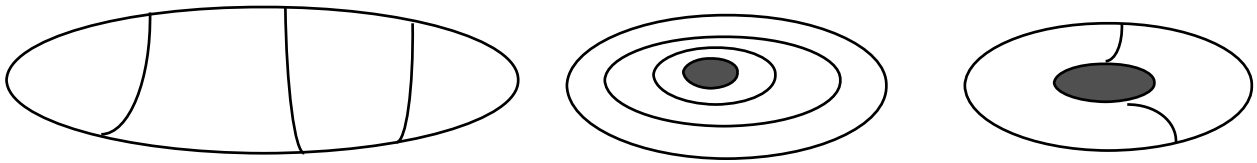
$$\mathcal{L}(P) = 0 \text{ in } \Omega_2$$
$$P = 0 \text{ on } \partial\Omega_2 \backslash \Gamma_i$$
$$P = P_0 \text{ on } \Gamma_{12}.$$

We take

$$\mathcal{B}_1 = DtN_2.$$

Optimal Interface Conditions

(Rogier, de Sturler and Nataf, 1993)
The result can be generalized to variable coefficients operators and a decomposition of the domain Ω in more than two subdomains. For the following geometries,



one can define interface conditions such as to have convergence in a number of iterations equals to the number of subdomains.
For arbitrary decompositions, negative conjectures have been formulated (F. Nier, *Séminaire X-EDP*, 1998).

Analytical CIO in the scalar case

The Steklov-PoincarÃ© *DtN* is not a partial differential operator:

1. it is non local
2. its explicit form is not known in the general case

It is approximated by a partial differential operator

$$DtN \simeq \alpha_{opt} - \frac{\partial}{\partial \tau}(\gamma_{opt}\frac{\partial}{\partial \tau})$$

trying to minimize the convergence rate of the algorithm.
We speak of optimized of order 2 (opt2) interface conditions
If we take $\gamma = 0$ and optimize only with respect to α, we speak of optimized of order 0 (opt0)

A model problem

$$\mathcal{L}(u) := \eta u - \Delta u = f \text{ in } \mathbb{R}^2, \quad \eta > 0$$

The plane $\mathbb{R}^2$ is divided into two half-planes with an overlap of size $\delta \geq 0$ and the algorithm writes:

$$\begin{aligned}
&\mathcal{L}(u_1^{n+1}) = f \quad \text{in } \Omega_1 :=]-\infty, \delta[\times \mathbb{R}, \\
&(\frac{\partial}{\partial n_1} + \alpha)(u_1^{n+1}) = (-\frac{\partial}{\partial n_2} + \alpha)(u_2^n) \quad \text{at } x = \delta \\
&\mathcal{L}(u_2^{n+1}) = f \quad \text{in } \Omega_2 :=]0, \infty[\times \mathbb{R}, \\
&(\frac{\partial}{\partial n_2} + \alpha)(u_2^{n+1}) = (-\frac{\partial}{\partial n_1} + \alpha)(u_1^n) \quad \text{at } x = 0
\end{aligned}$$

A Fourier analysis leads to the following convergence rate (k is the dual variable):

$$\rho(k; \delta, \alpha) = \left| \frac{\sqrt{\eta + k^2} - \alpha}{\sqrt{\eta + k^2} + \alpha} \right| e^{-\sqrt{\eta + k^2}\,\delta}$$

Optimizing the interface condition

In the physical space:

$$\rho(\delta,\alpha) = \max_{|k|\leq 1/h} \left| \frac{\sqrt{\eta + k^2} - \alpha}{\sqrt{\eta + k^2} + \alpha} \right| e^{-\sqrt{\eta + k^2}\,\delta}$$

When there is no overlap ($\delta = 0$):

- if α is h independent, then $\rho \simeq 1 - C^t\, h$
- if α varies like $1/h$, then $\rho \simeq 1 - C^t\, h$
- if α solves the min-max problem:

$$\rho(\alpha_{opt}) := \min_{\alpha>0} \max_{|k|\leq 1/h} \rho(k;0,\alpha)$$

  then α_{opt} varies like $1/\sqrt{h}$ and $\rho_{opt} \simeq 1 - C^t \sqrt{h}$

With overlap

- Classical Schwarz: $\alpha = \infty$, $\rho_{Scwharz} > \rho_\alpha$, $\forall \alpha$
- Optimization for small h with $\delta = C\, h$, (Gander, SISC, 2006)

$1/\Delta y$	10	20	40	80
$\alpha_{\text{opt}}^{\text{sc}}$	6	7	10	16
$\alpha = 1$	27	51	104	231

Table: Number of iterations for different values of the mesh size and two possible choices for α, no overlap

Application: the Helmholtz Equation

M. Gander, F. Nataf, F. Magoulès
SIAM J. Sci. Comp., 2002.

We want to solve

$$-\omega^2 u - \Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

The relaxation algorithm is : $(u_1^p, u_2^p) \to (u_1^{p+1}, u_2^{p+1})$ with $(i \neq j, i = 1, 2)$

$$(-\omega^2 - \Delta)(u_i^{p+1}) = f \quad \text{in } \Omega_i$$
$$(\frac{\partial}{\partial n_i} + \mathcal{S})(u_i^{p+1}) = (-\frac{\partial}{\partial n_j} + \mathcal{S})(u_j^p) \quad \text{on } \Gamma_{ij}.$$
$$u_i^{p+1} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega$$

The operator $\mathcal{S}$ has the form

$$\mathcal{S} = \alpha - \gamma \frac{\partial^2}{\partial \tau^2} \qquad \alpha, \gamma \in \mathbb{C}$$

Application: the Helmholtz Equation

By choosing carefully the coefficients α and γ, it is possible to optimize the convergence rate of the iterative method which in the Fourier space is given by

$$\rho(k;\alpha,\gamma) \equiv \begin{cases} \left| \dfrac{I\sqrt{\omega^2-k^2}-(\alpha+\gamma k^2)}{I\sqrt{\omega^2-k^2}+(\alpha+\gamma k^2)} \right| & \text{if } |k|<\omega \quad (I^2=-1) \\ \left| \dfrac{\sqrt{k^2-\omega^2}-(\alpha+\gamma k^2)}{\sqrt{k^2-\omega^2}+(\alpha+\gamma k^2)} \right| & \text{if } |k|>\omega \end{cases}$$

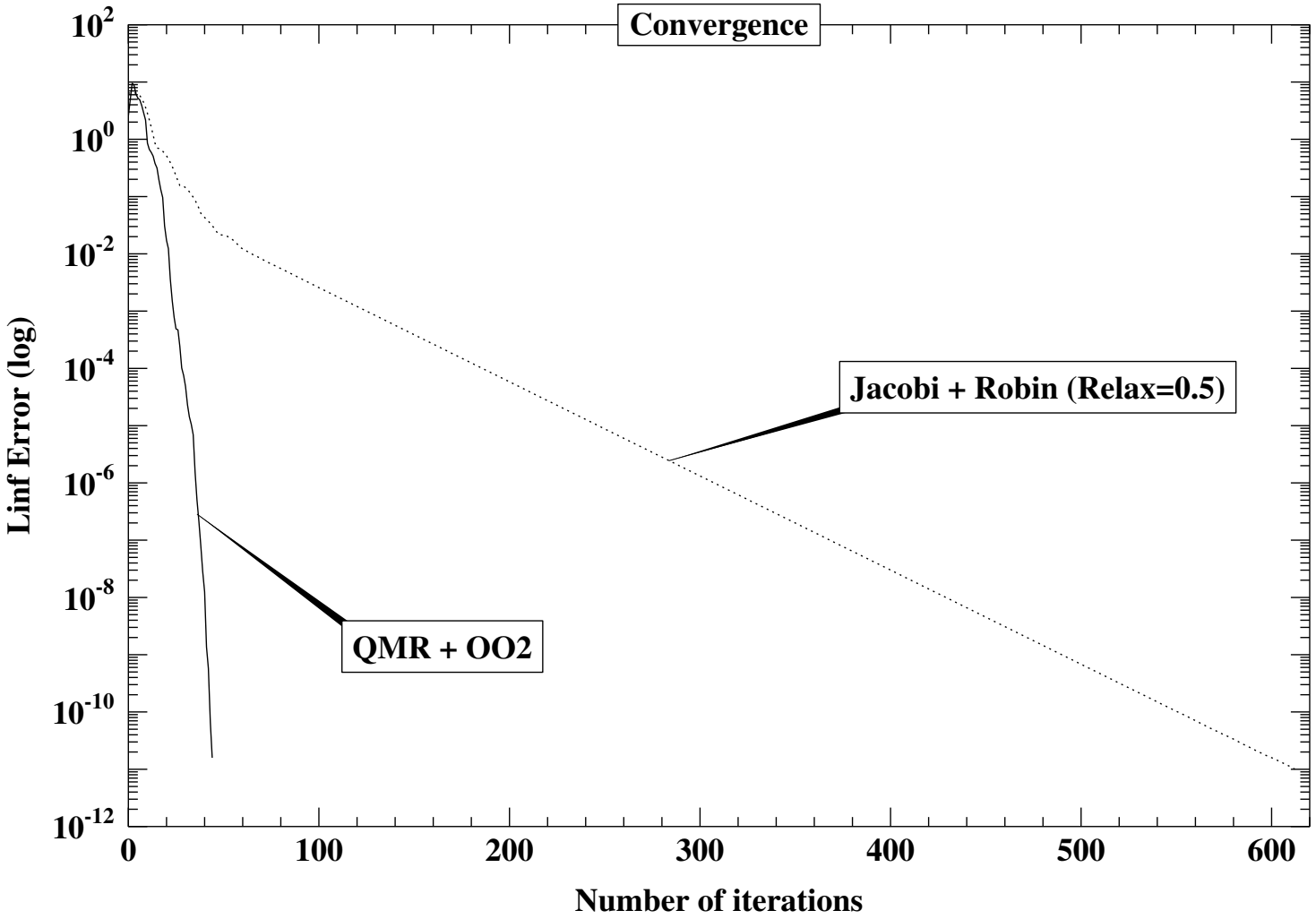
Finally, we get analytic formulas for α and γ (h is the mesh size):

$$\alpha_{opt} = \alpha(\omega, h) \text{ and } \gamma_{opt} = \gamma(\omega, h),$$

Moreover, a Krylov method (GC, GMRES, BICGSTAB, ...) replaces the fixed point algorithm.

The Helmholtz Equation – Numerical Results

Waveguide: Optimized Schwarz method with QMR compared to ABC0 ($\partial_n + I\omega$) with relaxation on the interface



Chevalier and N., 1997

Discretization of the two-field formulation

A direct discretization would require the computation of the normal derivatives along the interfaces in order to evaluate the right handsides. We introduce two new variables

$$\lambda^1 = -\frac{\partial u_2}{\partial n_2} + \mathcal{S}(u_2) \text{ and } \lambda^2 = -\frac{\partial u_1}{\partial n_1} + \mathcal{S}(u_1).$$

The algorithm reads now

$$-\Delta u_1^{n+1} + \omega^2 u_1^{n+1} = f \text{ in } \Omega_1$$

$$\frac{\partial u_1^{n+1}}{\partial n_1} + \mathcal{S}(u_1^{n+1}) = \lambda^{1\,n} \text{ on } \Gamma_{12}$$

$$-\Delta u_2^{n+1} + \omega^2 u_2^{n+1} = f \text{ in } \Omega_2$$

$$\frac{\partial u_2^{n+1}}{\partial n_2} + \mathcal{S}(u_2^{n+1}) = \lambda^{2\,n} \text{ on } \Gamma_{12}$$

$$\lambda^{1\,n+1} = -\lambda^{2\,n} + (\mathcal{S} + \mathcal{S})(u_2^{n+1}(\lambda^{1\,p}, f))$$

$$\lambda^{2\,n+1} = -\lambda^{1\,n} + (\mathcal{S} + \mathcal{S})(u_1^{n+1}(\lambda^{2\,p}, f)).$$

Finite Element Discretization

A finite element discretization leads to the following linear system:

$$\begin{aligned} \lambda^1 &= -\lambda^2 + (S+S)B^2u^2 \\ \lambda^2 &= -\lambda^1 + (S+S)B^1u^1 \\ \widetilde{K}^1u^1 &= f^1 + {B^1}^T\lambda^1 \\ \widetilde{K}^2u^2 &= f^2 + {B^2}^T\lambda^2 \end{aligned} \tag{18}$$

where B^1 (resp. B^2) is the trace operator of domain Ω^1 (resp. Ω^2) on the interface Γ_{12}. Matrix $\widetilde{K}^i,\ i=1,2$ arises from the discretization of the local Helmholtz subproblems along with the interface condition $\partial_n + \alpha - \gamma\partial_{\tau\tau}$.

$$\widetilde{K}^i = K^i - \omega^2 M^i + {B^i}^T(\alpha M_{\Gamma_{12}} + \gamma K_{\Gamma_{12}})B^i \tag{19}$$

where K^i is the stiffness matrix, M^i the mass matrix, $M_{\Gamma_{12}}$ is the interface mass matrix and $K_{\Gamma_{12}}$ is the interface stiffness matrix.

More precisely, the interface mass matrix $M_{\Gamma_{12}}$ and the interface stiffness matrix $K_{\Gamma_{12}}$ are defined by

$$[M_{\Gamma_{12}}]_{lm} = \int_{\Gamma_{12}} \phi_l \phi_m d\xi \quad \text{and} \quad [K_{\Gamma_{12}}]_{lm} = \int_{\Gamma_{12}} \nabla_\tau \phi_l \nabla_\tau \phi_m d\xi \quad (20)$$

where ϕ_l et ϕ_m are the basis functions associated to nodes l and m on the interface Γ_{12} and $\nabla_\tau \phi$ is the tangential component of $\nabla \phi$ on the interface.
We have

$$S = \alpha M_{\Gamma_{12}} + \gamma K_{\Gamma_{12}}.$$

The substructured linear system of the two-field formulation has the form

$$F\lambda = d \tag{21}$$

where $\lambda = (\lambda^1, \lambda^2)$, F is a matrix and d is the right handside

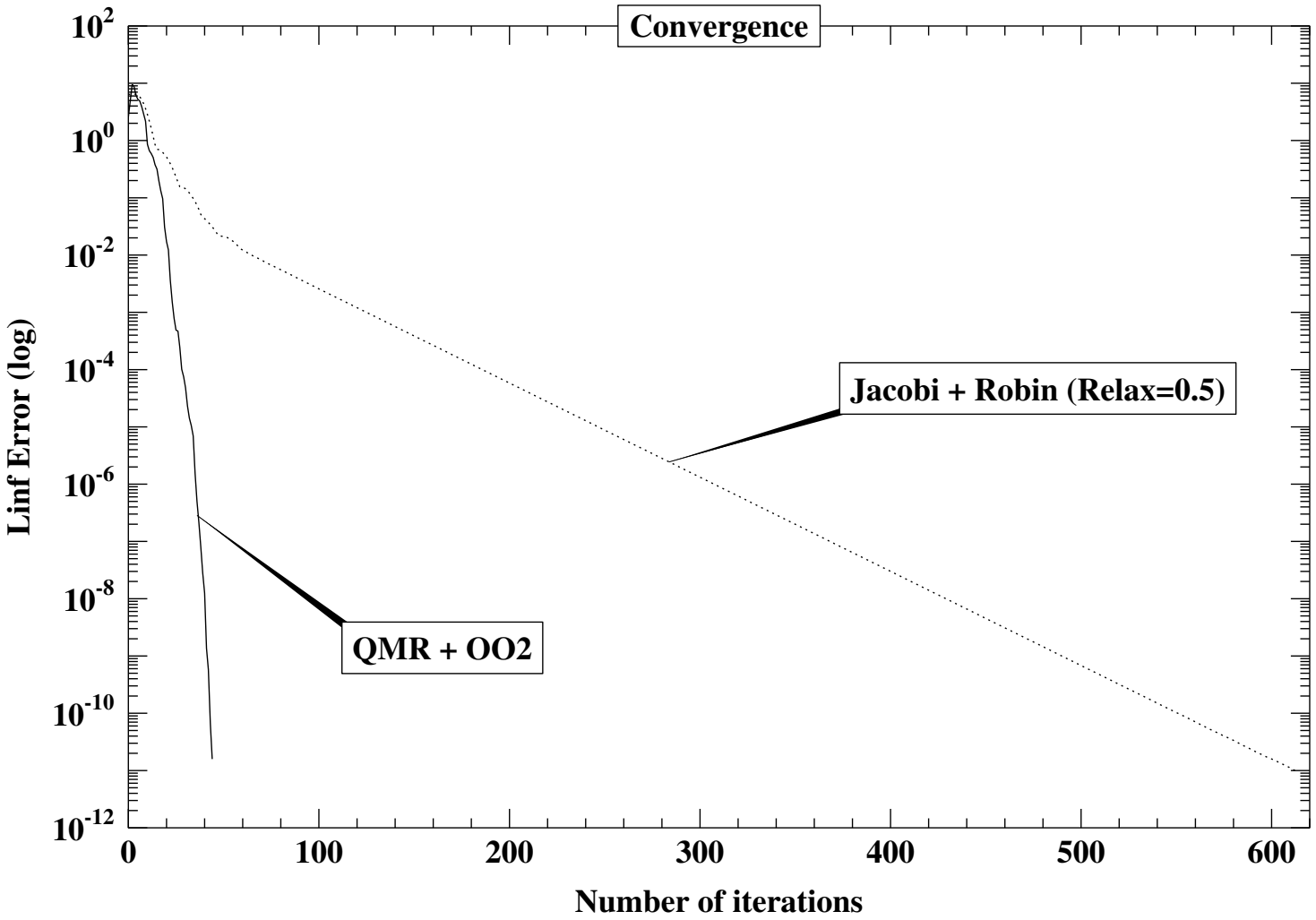
$$F = \begin{bmatrix} I & I - (S+S)B^2\widetilde{K}^{2^{-1}}B^{2^T} \\ I - (S+S)B^1\widetilde{K}^{1^{-1}}B^{1^T} & I \end{bmatrix}$$

$$d = \begin{bmatrix} (S+S)B^1\widetilde{K}^{1^{-1}}f^1 \\ (S+S)B^2\widetilde{K}^{2^{-1}}f^2 \end{bmatrix}$$

The linear system is solved by a Krylov type method, here the ORTHODIR algorithm. The matrix vector product amounts to solving a subproblem in each subdomain and to send interface data between subdomains.

General Interface Conditions for the Helmholtz Equation
Numerical Results

Waveguide: Optimized Schwarz method with QMR and ABC0 ($\partial_n + I\omega$) with relaxation on the interface



General Interface Conditions for the Helmholtz Equation
Numerical Results

Acoustic in a Car : Iteration Counts for various interface conditions

N_s	ABC 0	ABC 2	Optimized
2	16 it	16 it	9 it
4	50 it	52 it	15 it
8	83 it	93 it	25 it
16	105 it	133 it	34 it

ABC 0: Absorbing Boundary Conditions of Order 0 ($\partial_n + I\omega$)
ABC 2: Absorbing Boundary Conditions of Order 2 ($\partial_n + I\omega - 1/(2I\omega)\partial_{y^2}$)
Optimized: Optimized Interface Conditions

Maxwell equations

The same ideas apply to Maxwell's equations

Després, ; Joly, ; Roberts, A domain decomposition method for the harmonic Maxwell equations. Iterative methods in linear algebra , 1992.

Dolean, ; Gander, ; Gerardo-Giorda, Optimized Schwarz methods for Maxwell's equations. SISC, 2009

They are currently used in electromagnetic simulations:
LEE Jin-Fa - Ohio State University, ECE Department, USA:

Z. Peng, K. H. Lim, and J. F. Lee, Computations of Electromagnetic Wave Scattering from Penetrable Composite Targets using a Surface Integral Equation Method with Multiple Traces, IEEE T. ANTENNA PROPAG., 2012.

Z. Peng, K. H. Lim, and J. F. Lee, Non-conformal Domain Decomposition Methods for Solving Large Multi-scale Electromagnetic Scattering Problems, Proceeding of IEEE, 2012.

Optimal Interface Condition at the matrix level

When a finite element method, for instance, is used it yields a linear system of the form $AU = F$, where F is a given right-hand side and U is the set of unknowns.
Corresponding to a domain decomposition, the set of unknowns U is decomposed into interior nodes of the subdomains U_1 and U_2, and to unknowns, U_Γ, associated to the interface Γ.
This leads to a block decomposition of the linear system

$$\begin{pmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_\Gamma \\ U_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_\Gamma \\ F_2 \end{pmatrix}. \tag{22}$$

Optimal Interface Condition at the matrix level

The DDM method reads:

$$\begin{pmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} + S_2 \end{pmatrix} \begin{pmatrix} U_1^{n+1} \\ U_{\Gamma,1}^{n+1} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_\Gamma + S_2 U_{\Gamma,2}^n - A_{\Gamma 2} U_2^n \end{pmatrix} \quad (23)$$

$$\begin{pmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} + S_1 \end{pmatrix} \begin{pmatrix} U_2^{n+1} \\ U_{\Gamma,2}^{n+1} \end{pmatrix} = \begin{pmatrix} F_2 \\ F_\Gamma + S_1 U_{\Gamma,1}^n - A_{\Gamma 1} U_1^n \end{pmatrix} \quad (24)$$

where

$$S_1 = -A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$$

and

$$S_2 = -A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$$

Convergence in two iterations

Approximate Interface Condition at the matrix level

The matrices $S_1 = -A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$ and $S_2 = -A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$ are full interface matrices ($\Gamma \times \Gamma$).

Cons

- Costly to compute
- The subdomain matrix is partly full

Approximate S_1 and S_2 by sparse matrices

1. e.g. via sparse approximations to A_{ii}^{-1}: SPAI
2. via local Schur complements on successive reduced "outer" domains ($\gamma \times \delta$), "patches", (Roux et al., 2006)

The first approach gives mild results. The second one is not better than using an overlap of depth δ but is cheaper.

Many layers

Very anisotropic and heterogeneous media, $\kappa_M/\kappa_m = 10^7$

Table: Gmres solve with $TOL = 10^{-6}$

	Cond. Nb	Iter	$\|e\|$
RAS (Schwarz)	$1.99\,10^6$	37	$3.6\,10^{-2}$
Patch	$5.29\,10^5$	15	$6.1\,10^{-5}$
Best IC	2.1	9	$1.5\,10^{-7}$

Condition number of Patch method is very bad but only one eigenvalue is very small, thus iteration count is good.

Outline

Weak scalability

How to evaluate the efficiency of a domain decomposition?

Weak scalability – definition

"How the solution time varies with the number of processors for a fixed problem size per processor."

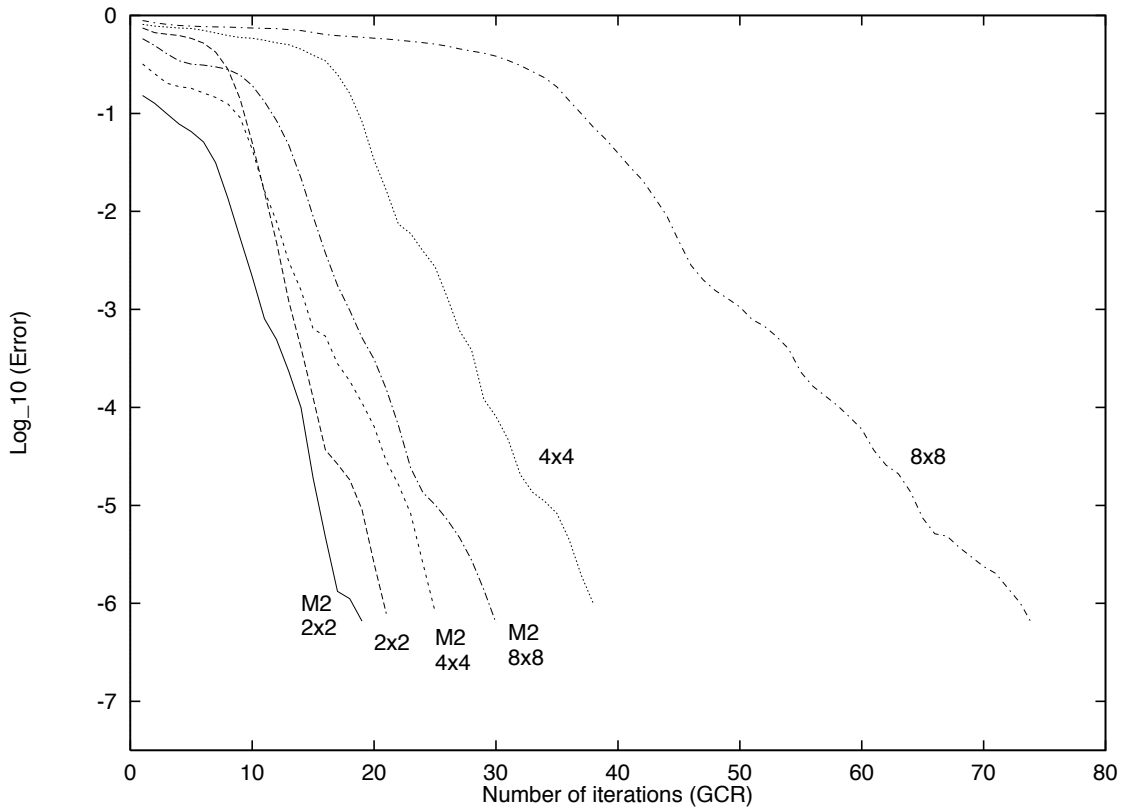
It is not achieved with the one level method

Number of subdomains	8	16	32	64
ASM	18	35	66	128

The iteration number increases linearly with the number of subdomains in one direction.

Convergence curves- more subdomains

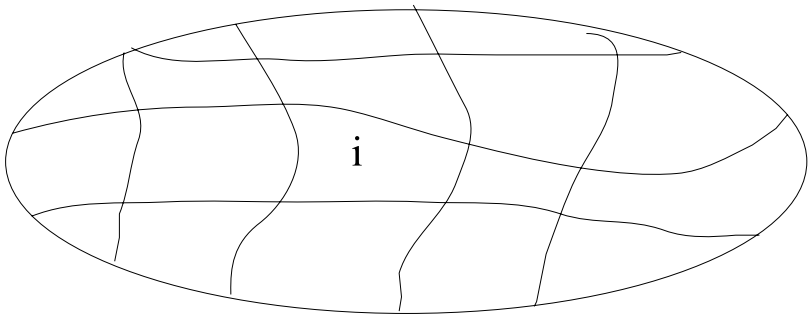
Plateaus appear in the convergence of the Krylov methods.



How to achieve scalability

Stagnation corresponds to a few very low eigenvalues in the spectrum of the preconditioned problem. They are due to the lack of a global exchange of information in the preconditioner.

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$



The mean value of the solution in domain i depends on the value of f on all subdomains.
A classical remedy consists in the introduction of a coarse grid problem that couples all subdomains. This is closely related to deflation technique classical in linear algebra (see Nabben and Vuik's papers in SIAM J. Sci. Comp, 200X).

Adding a coarse grid

We add a coarse space correction (*aka* second level)
Let V_H be the coarse space and z be a basis, $V_H = \operatorname{span} z$,
writing $R_0 = Z^T$ we define the two level preconditioner as:

$$M_{ASM,2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i.$$

The Nicolaides approach is to use the kernel of the operator as a coarse space, this is the constant vectors, in local form this writes:

$$Z := (R_i^T D_i R_i \mathbf{1})_{1 \leq i \leq N}$$

where D_i are chosen so that we have a partition of unity:

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id.$$

Theoretical convergence result

Theorem (Widlund, Sarkis)

Let $M_{ASM,2}^{-1}$ *be the two-level additive Schwarz method:*

$$\kappa(M_{ASM,2}^{-1} A) \leq C \left(1 + \frac{H}{\delta}\right)$$

where δ *is the size of the overlap between the subdomains and* H *the subdomain size.*

This does indeed work very well

Number of subdomains	8	16	32	64
ASM	18	35	66	128
ASM + Nicolaides	20	27	28	27

Idea of the proof (Upper bound)

Lemma

If each point in Ω belongs to at most k_0 of the subdomains Ω_j, then the largest eigenvalue of $M_{ASM,2}^{-1} A$ satisfies

$$\lambda_{max}(M_{ASM,2}^{-1} A) \leq k_0 + 1.$$

Assumption (Stable decomposition)

There exists a constant C_0, such that every $u \in V$ admits a decomposition $u = \sum_{i=0}^{N} R_i^T u_i$, $u_i \in V_i$, $i = 0, \ldots, N$ that satisfies:

$$\sum_{i=0}^{N} \tilde{a}_i(u_i, u_i) \leq C_0^2 a(u, u).$$

Idea of the proof (Lower bound)

Theorem

If every $v \in V$ admits a C_0-stable decomposition (with uniform C_0), then the smallest eigenvalue of $M_{AS,2}^{-1} A$ satisfies

$$\lambda_{min}(M_{ASM,2}^{-1} A) \geq C_0^{-2}.$$

Therefore, the condition number of the two-level Schwarz preconditioner can be bounded by

$$\kappa(M_{ASM,2}^{-1} A) \leq C_0^2 (k_0 + 1).$$

Deflation and Coarse grid correction

Let A be a SPD matrix, we want to solve

$$Ax = b$$

with a preconditioner M (for example the Schwarz method).
Let Z be a rectangular matrix so that the "bad eigenvectors" belong to the space spanned by its columns. Define

$$P := I - AQ, \quad Q := ZE^{-1}Z^T, \quad E := Z^T AZ,$$

Examples of coarse grid preconditioners

$$\mathcal{P}_{A-DEF2} := P^T M^{-1} + Q, \quad \mathcal{P}_{BNN} := P^T M^{-1} P + Q \text{ (Mandel, 1993)}$$

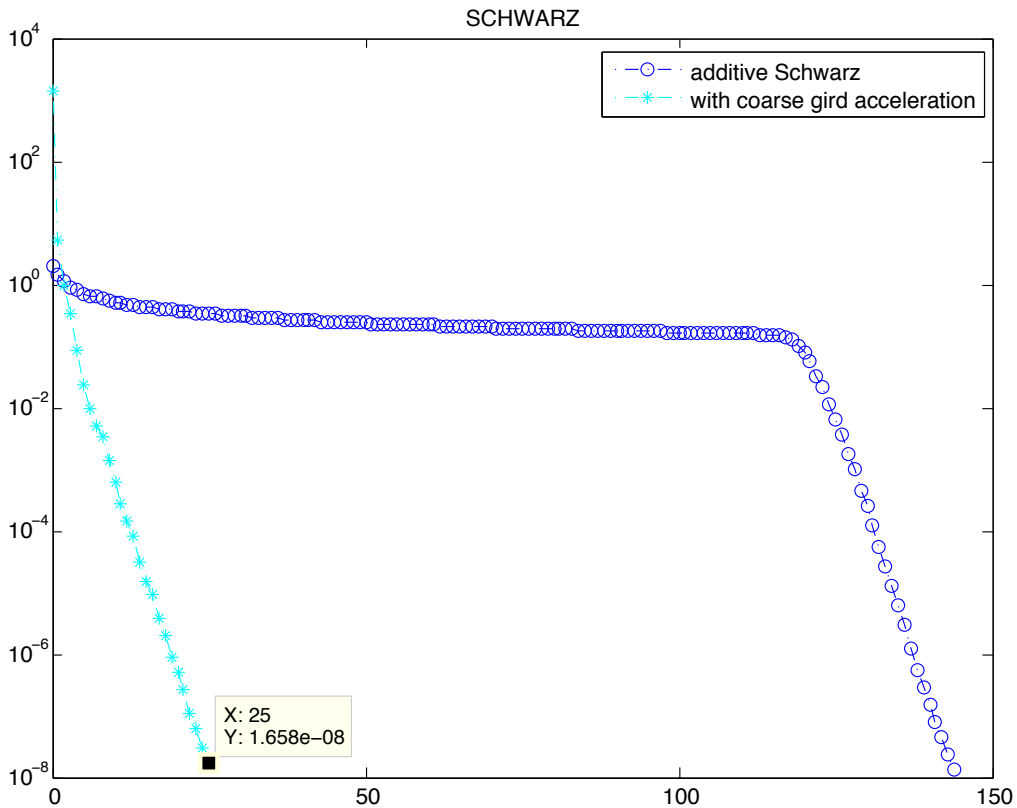
Some properties: $QAZ = Z$, $P^T Z = 0$ and $P^T Q = 0$.
Let r_n be the residual at step n of the algorithm: $Z^T r_n = 0$.

How to choose Z?

Coarse grid correction for smooth problems

For a Poisson like problem, Nicolaides (1987), Sarkis (2002).
Let $(\chi_i)_{1\leq i\leq N}$ denote a partition of unity :

$$Z = \begin{bmatrix} \chi_1 & 0 & \cdots & 0 \\ \vdots & \chi_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \chi_N \end{bmatrix}$$



Coarse grid implementation - I

It is enough to replace the Schwarz preconditioner by P_{BNN} as follows. First build $E = Z^T AZ$

```
Vh[int]  Z(npart);
for(int i=0;i<npart;++i)
{ Z[i]=1.;
real[int] zit = Rih[i]*Z[i][];
real[int] zitemp = Dih[i]*zit;
Z[i][]=Rih[i]'*zitemp;
}
real[int,int] Ef(npart,npart);    // E = Z^T*A*Z
for(int i=0;i<npart;++i)
{  real[int] vaux = A(Z[i][]);
   for(int j=0;j<npart;++j)
  Ef(j,i) = Z[j][]'*vaux;
}
matrix E;
E = Ef;
set(E,solver=UMFPACK);
```

Coarse grid implementation - II

Then the coarse space correction $Q = ZE^{-1}Z^T$:

```
func real[int] Q(real[int] &l)    // Q = Z*E^-1*Z^T
{
   real[int] res(l.n);
   res=0.;
   real[int] vaux(npart);
   for(int i=0;i<npart;++i)
   {
      vaux[i]=Z[i][]'*l;
   }
   real[int] zaux=E^-1*vaux;      // zaux=E^-1*Z^T*l
   for(int i=0;i<npart;++i)       // Z*zaux
   {
      res +=zaux[i]*Z[i][];
   }
   return res;
}
```

Coarse grid implementation - III

The projector out of the coarse space $P = I - QA$ and its transpose P^T:

```
func real[int] P(real[int] &l)    // P = I - A*Q
{
   real[int] res=Q(l);
   real[int] res2=A(res);
   res2 -= l;
   res2 *= -1.;
   return res2;
}
func real[int] PT(real[int] &l)   // P^T = I-Q*A
{
   real[int] res=A(l);
   real[int] res2=Q(res);
   res2 -= l;
   res2 *= -1.;
   return res2;
}
```

Coarse grid implementation - IV

And finally the preconditioner $P_{BNN} = P^T M^{-1} P + Q$:

```
int j;
func real[int] BNN(real[int] &u)    // precond BNN
{
    real[int] aux1 = Q(u);
    real[int] aux2 = P(u);
    real[int] aux3 = Mm1(aux2);
    aux2 = PT(aux3);
    aux2 += aux1;
    ++j;
return aux2;
}
```

Outline

Motivation

Large discretized system of PDEs
strongly heterogeneous coefficients
(high contrast, nonlinear, multiscale)

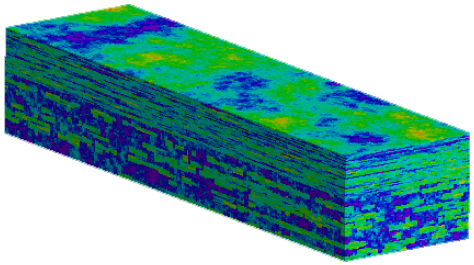
E.g. Darcy pressure equation,
P^1-finite elements:

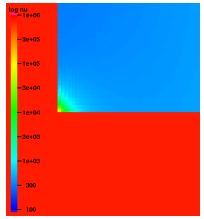
$$\mathbf{Au} = \mathbf{f}$$

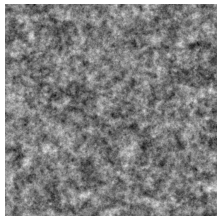
$$\text{cond}(\mathbf{A}) \sim \frac{\alpha_{\max}}{\alpha_{\min}} h^{-2}$$

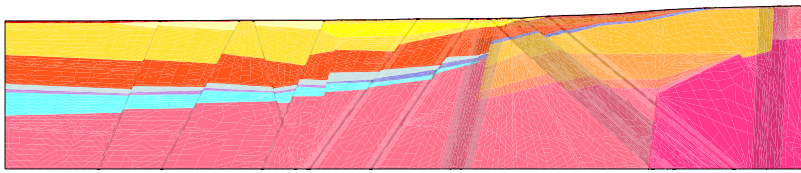
Goal:
iterative solvers
robust in size and heterogeneities

Applications:
flow in heterogeneous /
stochastic / layered media
structural mechanics
electromagnetics
etc.









Darcy equation with heterogeneities

$$
\begin{aligned}
-\nabla \cdot (\alpha(x,y)\nabla u) &= 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \\
u &= 0 \quad \text{on} \quad \partial\Omega_D, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega_N.
\end{aligned}
$$

Decomposition

$\alpha(x,y)$

Jump	1	10	10^2	10^3	10^4
ASM	39	45	60	72	73
ASM + Nicolaides	30	36	50	61	65

Objectives

Strategy

Define an appropriate coarse space $V_{H2} = \text{span}(z_2)$ and use the framework previously introduced, writing $R_0 = Z_2^T$ the two level preconditioner is:

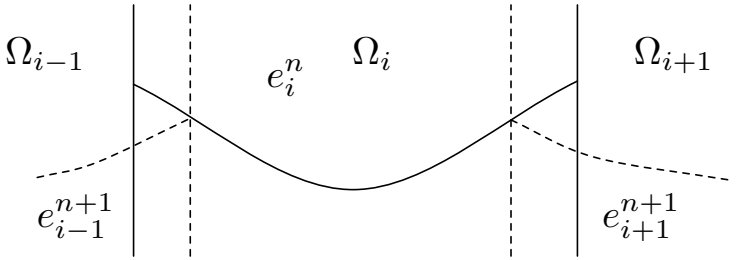
$$P_{ASM2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i.$$

The coarse grid must be

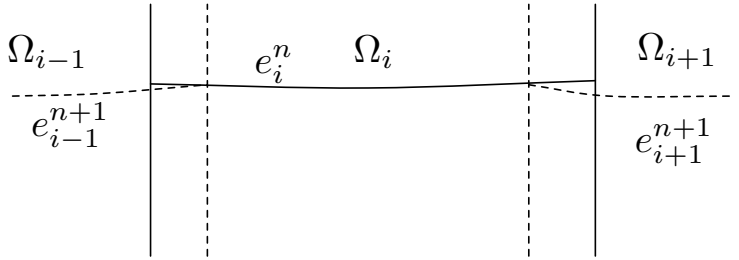
- Local (calculated on each subdomain) $\rightarrow$ parallel
- Adaptive (calculated automatically)
- Easy and cheap to compute (on the boundary for instance)
- Robust (must lead to an algorithm whose convergence does not depend on the partition or the jumps in coefficients)

Heuristic approach: what functions should be in Z_2?

The error satisfies the Schwarz algorithm, it is harmonic, so it satisfies a maximum principle.



Fast convergence



Slow convergence

Idea

Ensure that the error decreases quickly on the subdomain boundaries which translates to making $\left.\frac{\partial e}{\partial n_i}\right|_{\Gamma_i}$ big.

Ensuring that the error decreases quickly on the subdomain boundaries

The Dirichlet to Neumann operator is defined as follows: Let $g : \Gamma_i \mapsto \mathbb{R}$,

$$\mathrm{DtN}_{\Omega_i}(g) = \alpha \left. \frac{\partial v}{\partial n_i} \right|_{\Gamma_i},$$

where v satisfies

$$\begin{cases} (-\mathrm{div}(\alpha \nabla))v = 0, & \text{in } \Omega_i, \\ v = g, & \text{on } \partial\Omega_i. \end{cases}$$

To construct the coarse space, we use the low frequency modes associated with the DtN operator:

$$\mathrm{DtN}_{\Omega_i}(v_i^\lambda) = \lambda \, \alpha \, v_i^\lambda$$

with λ small. The functions v_i^λ are extended harmonically to the subdomains.

Theoretical convergence result

Suppose we have $(v_i^{\lambda_k}, \lambda_i^k)_{1 \leq k \leq n_{\Gamma_i}}$ the eigenpairs of the local DtN maps ($\lambda_i^1 \leq \lambda_i^2 \leq \ldots$) and that we have selected m_i in each subdomain. Then let Z be the coarse space built via the local DtN maps:

$$Z := (R_i^T D_i \tilde{V}_i^{\lambda_i^k})_{1 \leq i \leq N;\, 1 \leq k \leq m_i}$$

Theorem (D., Nataf, Scheichl and Spillane 2010)

Under the monotonicity of α in the overlapping regions:

$$\kappa(M_{ASM,2}^{-1} A) \leq C(1 + \max_{1 \leq i \leq N} \frac{1}{\delta_i \lambda_i^{m_i+1}})$$

where δ_i is the size of the overlap of domain Ω_i and C is independent of the jumps of α.

If m_i is chosen so that, $\lambda_i^{m_i+1} \geq 1/H_i$ the convergence rate will be analogous to the constant coefficient case.

Results with the new DtN method

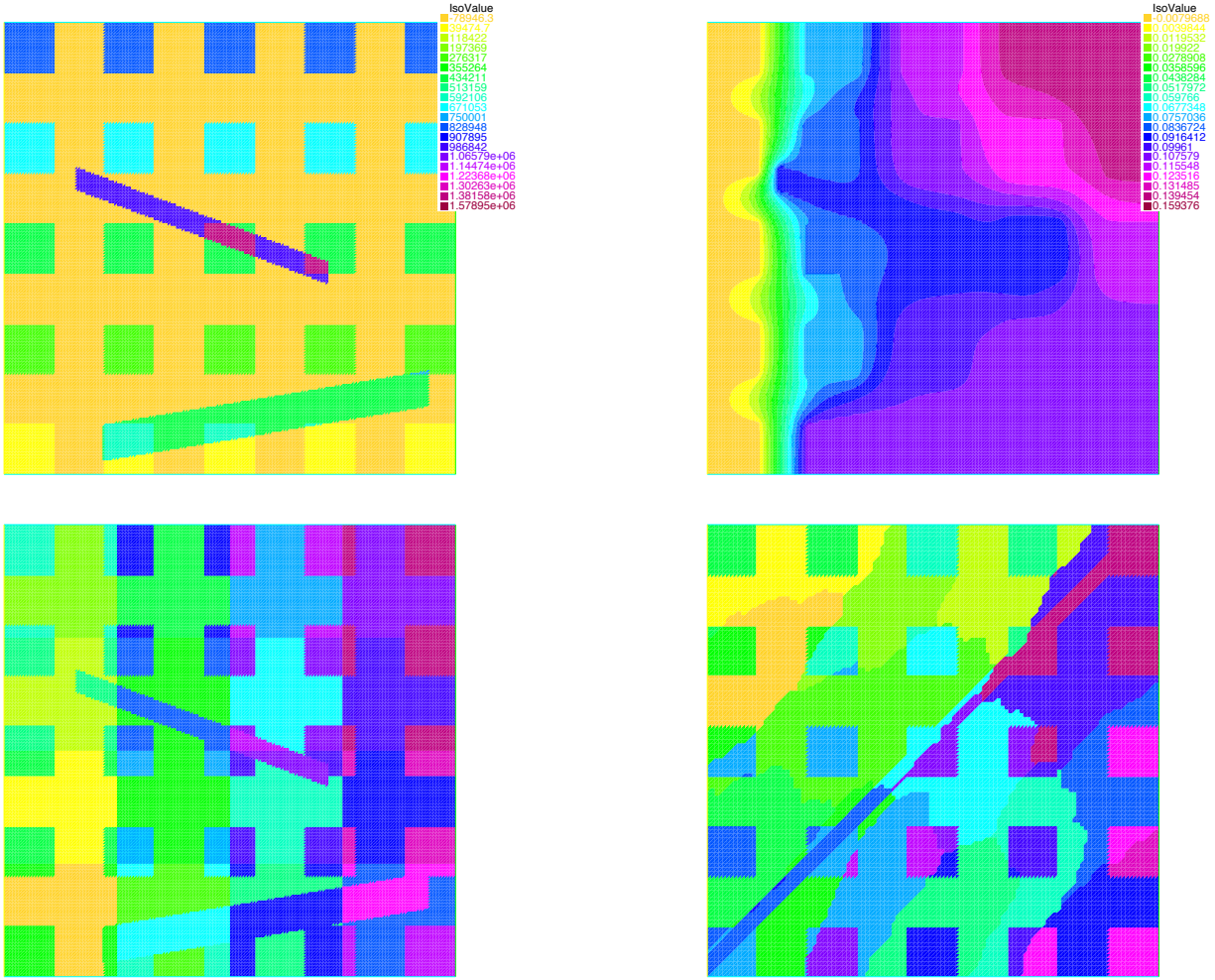
Jump	1	10	10^2	10^3	10^4
ASM	39	45	60	72	73
ASM + Nicolaides	30	36	50	61	65
ASM + DtN	31	35	36	36	36

Decomposition

$\alpha(x, y)$

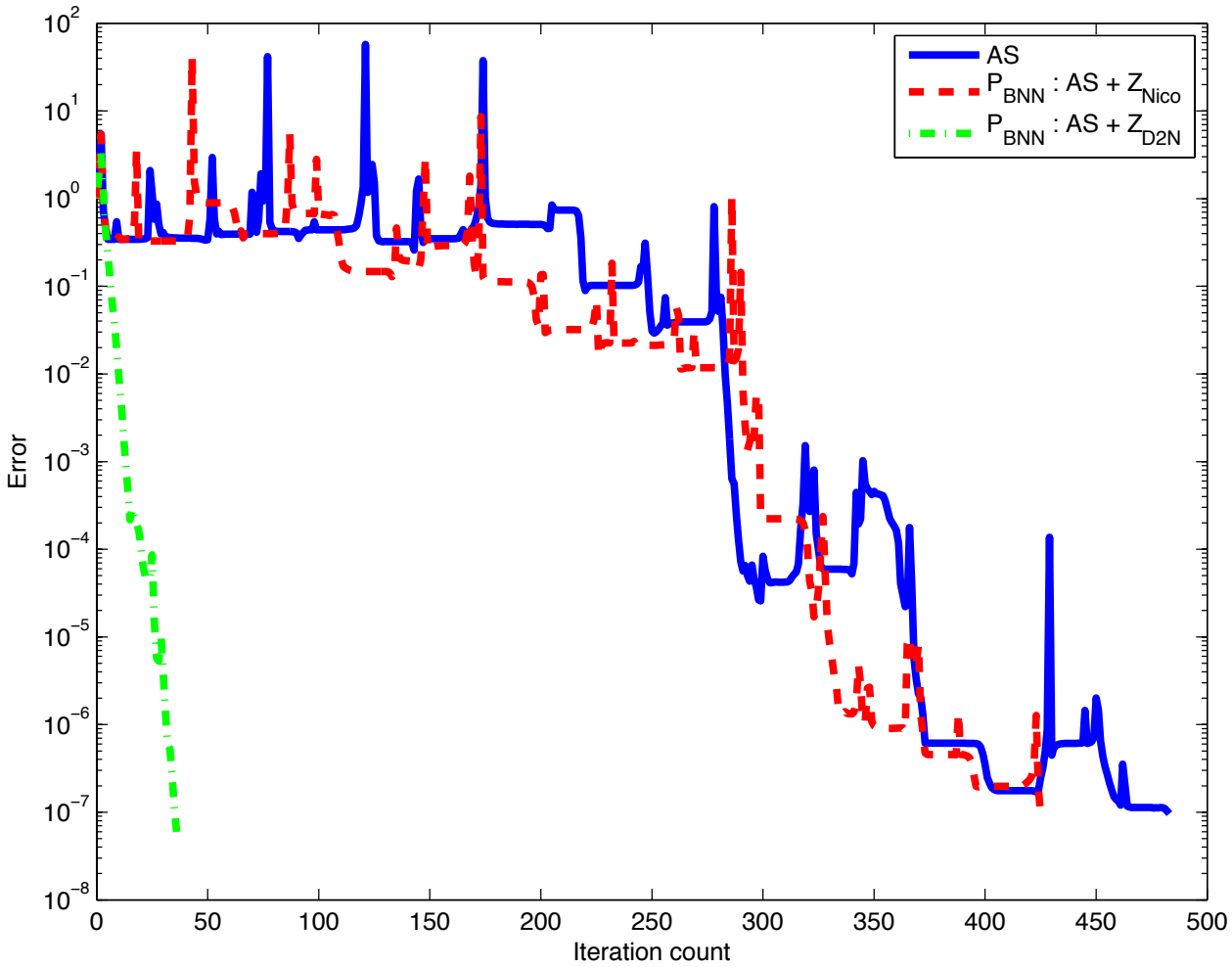
With DtN the jumps do not affect convergence
We put at most two modes per subdomain in the coarse grid (using the automatic selection process)

Numerical results



Channels and inclusions: $1 \leq \alpha \leq 1.5 \times 10^6$, the solution and partitionings (Metis or not)

Numerical results



ASM convergence for channels and inclusions – 4×4 Metis partitioning

Numerical results

subdomain i	m_i	total number of eigenvalues
1	3	155
2	1	109
3	5	175
10	4	174
11	2	71
12	2	128
13	3	166
14	3	127
15	3	188
16	3	106

Metis 4 by 4 decomposition

Numerical results

	ASM	ASM+Nico	ASM+DtN
2×2	103	110	22
2×2 Metis	76	76	22
4×4	603	722	26
4×4 Metis	483	425	36
8×8	461	141	34
8×8 Metis	600	542	31

Convergence results for the ”hard” test case

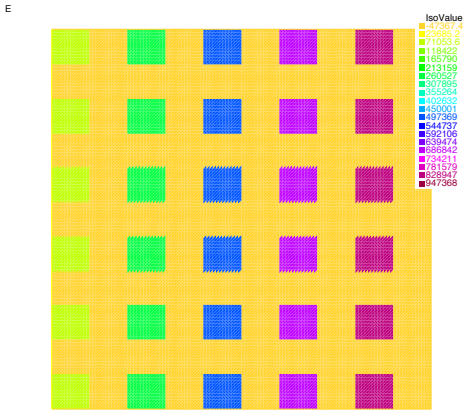
Numerical results – Optimality

$\#Z$ per subd.	ASM	ASM+Z_{Nico}	ASM+Z_{D2N}
$\max(m_i - 1, 1)$			273
m_i	614	543	36
$m_i + 1$			32

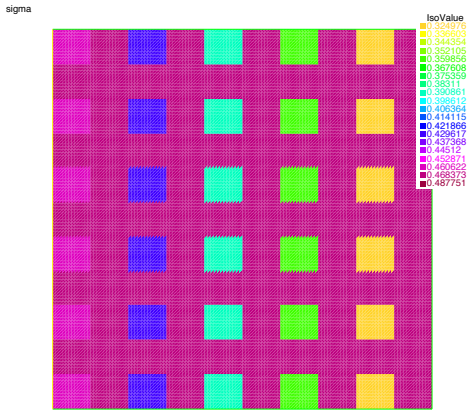
m_i is given automatically by the strategy.

- Taking one fewer eigenvalue has a huge influence on the iteration count
- Taking one more has only a small influence

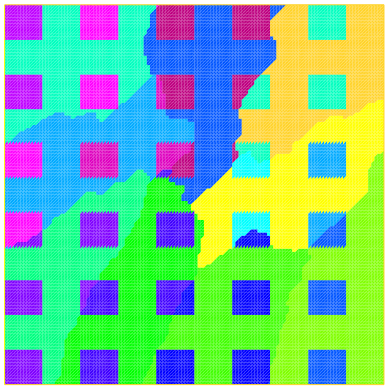
Results for elasticity (Problem)



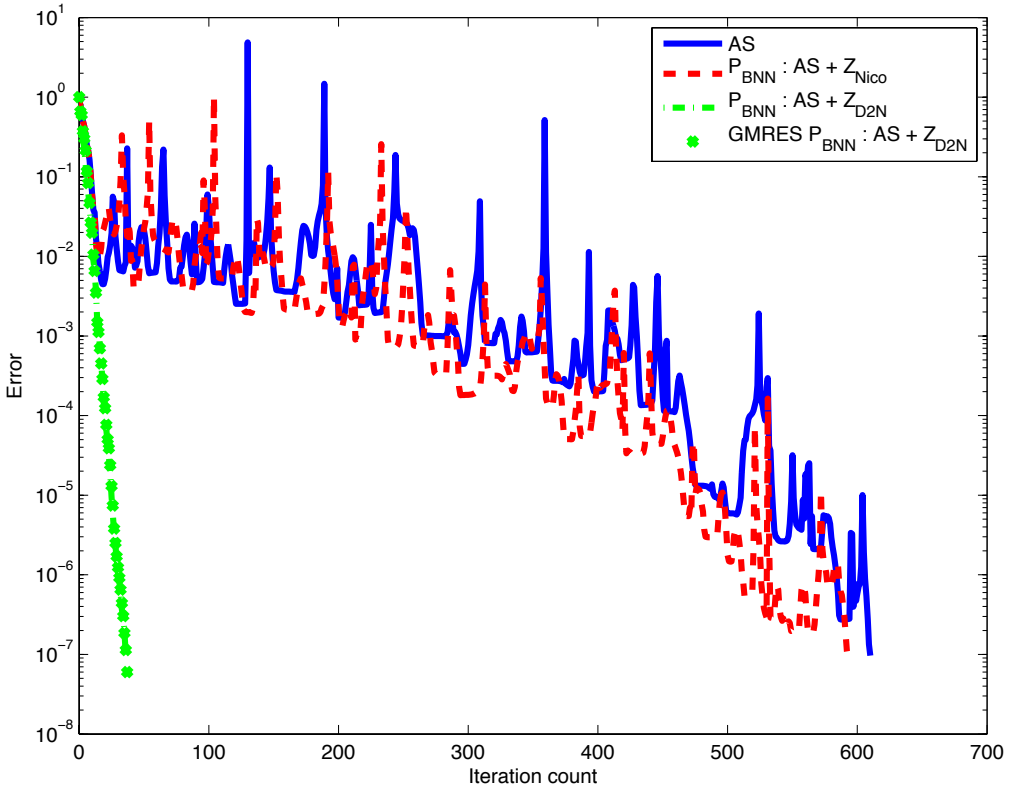
Young's modulus ($1 \leq E \leq 10^6$)



Poisson's ratio ($0.35 \leq \nu \leq 0.48$)



Results for 2d elasticity (Solution)



Overlap is two grid cells

Outline

Problem setting – I

Given $f \in (V^h)^*$ find $u \in V^h$

$$a(u, v) = \langle f, v \rangle \qquad \forall v \in V^h$$
$$\Longleftrightarrow \qquad \mathbf{A}\,\mathbf{u} = \mathbf{f}$$

Assumption throughout: $\mathbf{A}$ *symmetric positive definite (SPD)*

Examples:

- Darcy $a(u, v) = \int_\Omega \boldsymbol{\kappa} \nabla u \cdot \nabla v \, dx$
- Elasticity $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{C}\, \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \, dx$
- Eddy current $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\nu}\, \mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} + \boldsymbol{\sigma}\, \boldsymbol{u} \cdot \boldsymbol{v} \, dx$

Heterogeneities / high contrast / nonlinearities in parameters

Problem setting – II

1. V^h ... FE space of functions in Ω based on mesh $\mathcal{T}^h = \{\tau\}$
2. **A** given as set of **element stiffness matrices**
   + connectivity (list of DOF per element)

   Assembling property:

$$a(v,\, w) \;=\; \sum_{\tau} a_\tau(v_{|\tau},\, w_{|\tau})$$

   where $a_\tau(\cdot,\cdot)$ *symm. pos. semi-definite*
3. $\{\phi_k\}_{k=1}^n$ (FE) basis of V^h
   on each element: *unisolvence*

   set of non-vanishing basis functions linearly independent

   fulfilled by standard FE

   continuous, Nédélec, Raviart-Thomas of low/high order
4. Two more assumptions on $a(\cdot,\cdot)$ later!

Schwarz setting – I

Overlapping partition: $\Omega = \bigcup_{j=1}^{N} \Omega_j$ (Ω_j union of elements)

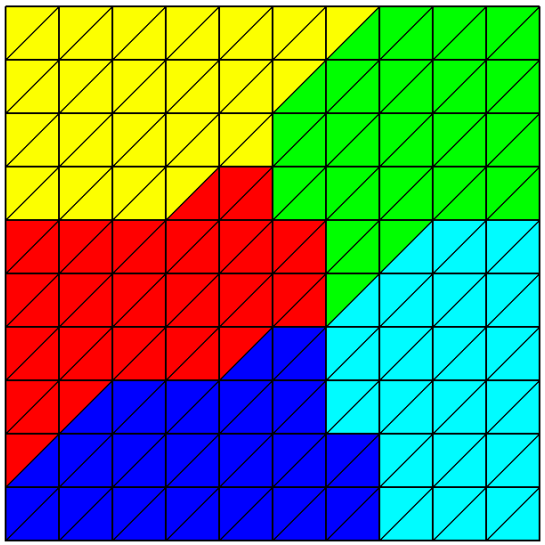
$$V_j := \text{span}\{\phi_k : \text{supp}(\phi_k) \subset \overline{\Omega}_j\}$$

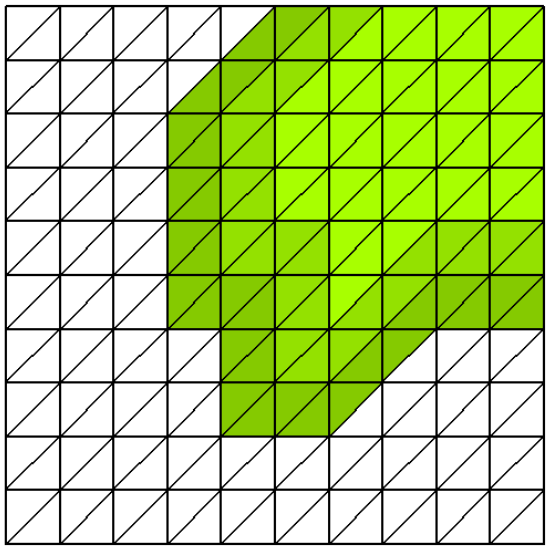
such that every ϕ_k contained in one of those spaces, i.e.

$$V^h = \sum_{j=1}^{N} V_j$$

Example: adding "layers" to non-overlapping partition

(partition and adding layers based on matrix information only!)





Schwarz setting – II

Local subspaces:

$$V_j \subset V^h \qquad j = 1, \ldots, N$$

Coarse space (defined later):

$$V_0 \subset V^h$$

Additive Schwarz preconditioner:

$$\mathbf{M}_{ASM,2}^{-1} = \sum_{j=0}^{N} \mathbf{R}_j^\top \mathbf{A}_j^{-1} \mathbf{R}_j$$

where $\mathbf{A}_j = \mathbf{R}_j^\top \mathbf{A} \mathbf{R}_j$
and $\mathbf{R}_j^\top \leftrightarrow R_j^\top : V_j \to V^h$ natural embedding

Partition of unity

Definitions:

$$dof(\Omega_j) := \{k : \text{supp}(\phi_k) \cap \Omega_j \neq \emptyset\}$$
$$idof(\Omega_j) := \{k : \text{supp}(\phi_k) \subset \overline{\Omega}_j\} \qquad V_j = \text{span}\{\phi_k\}_{k \in idof(\Omega_j)}$$
$$imult(k) := \#\{j : k \in idof(\Omega_j)\}$$

Partition of unity:

(used for design of coarse space and for stable splitting)

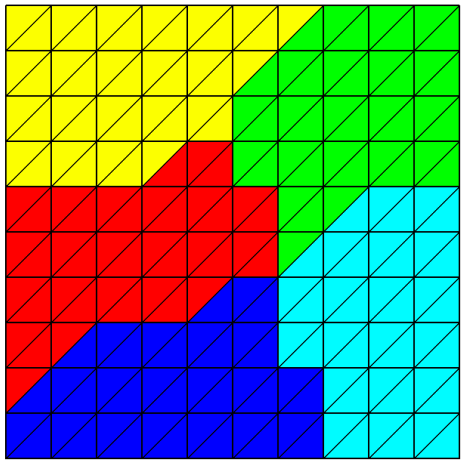
$$\Xi_j v = \sum_{k \in idof(\Omega_j)} \frac{1}{imult(k)} v_k \phi_k \qquad \text{for } v = \sum_{k=1}^{n} v_k \phi_k$$

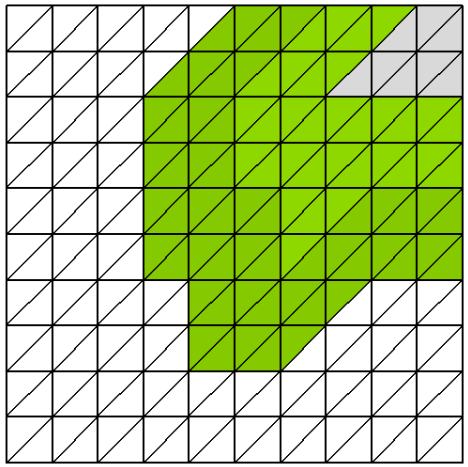
Properties:

$$\sum_{j=1}^{N} \Xi_j v = v \qquad \Xi_j v \in V_j$$

Overlapping zone / Choice of coarse space

Overlapping zone: $\Omega_j^\circ = \{x \in \Omega_j : \exists i \neq j : x \in \Omega_i\}$





Observation: $\Xi_{j|\Omega_j \setminus \Omega_j^\circ} = \mathrm{id}$

Coarse space should be **local**:

$$V_0 = \sum_{j=1}^{N} V_{0,j} \qquad \text{where } V_{0,j} \subset V_j$$

E.g. $V_{0,j} = \mathrm{span}\{\Xi_j p_{j,k}\}_{k=1}^{m_j}$

Abstract eigenvalue problem

Gen.EVP per subdomain:

Find $p_{j,k} \in V_{h|\Omega_j}$ and $\lambda_{j,k} \geq 0$:

$$a_{\Omega_j}(p_{j,k},\, v) \;=\; \lambda_{j,k}\, a_{\Omega_j^\circ}(\Xi_j p_{j,k},\, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

$$\mathbf{A}_j \mathbf{p}_{j,k} \;=\; \lambda_{j,k}\, \mathbf{X}_j \mathbf{A}_j^\circ \mathbf{X}_j\, \mathbf{p}_{j,k} \qquad (\mathbf{X}_j \ldots \text{diagonal})$$

(properties of eigenfunctions discussed soon) $\quad a_D \ldots$ restriction of a to D

In the two-level ASM:

Choose first m_j eigenvectors per subdomain:

$$V_0 \;=\; \operatorname{span}\left\{\Xi_j p_{j,k}\right\}_{k=1,\ldots,m_j}^{j=1,\ldots,N}$$

Two technical assumptions.

Theorem (D., Hauret, Nataf, Pechstein, Scheichl, Spillane)

If for all j: $0 < \lambda_{j,m_{j+1}} < \infty$*:*

$$\kappa(\mathbf{M}_{ASM,2}^{-1}\mathbf{A}) \leq (1+k_0)\left[2 + k_0(2k_0+1)\max_{j=1}^{N}\left(1+\frac{1}{\lambda_{j,m_j+1}}\right)\right]$$

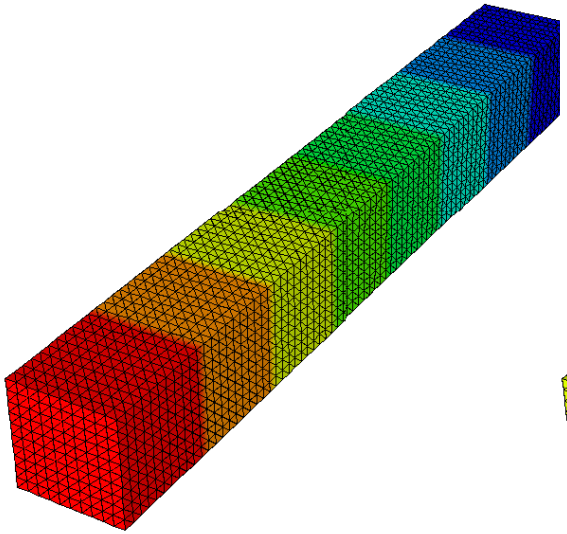
Possible criterion for picking m_j: (used in our Numerics)

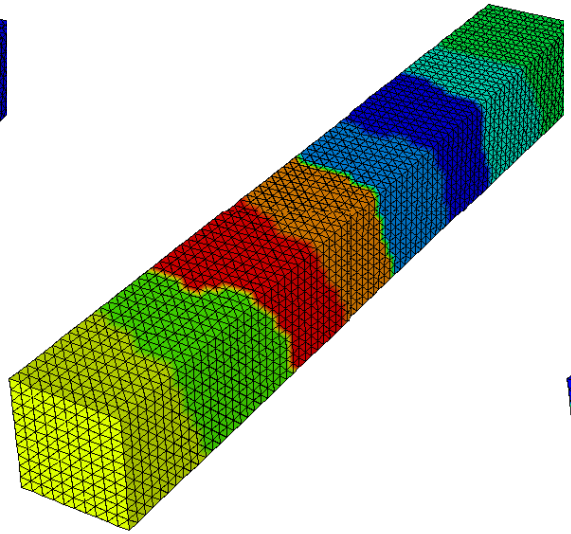
$$\lambda_{j,m_j+1} < \frac{\delta_j}{H_j}$$

H_j . . . subdomain diameter, δ_j . . . overlap

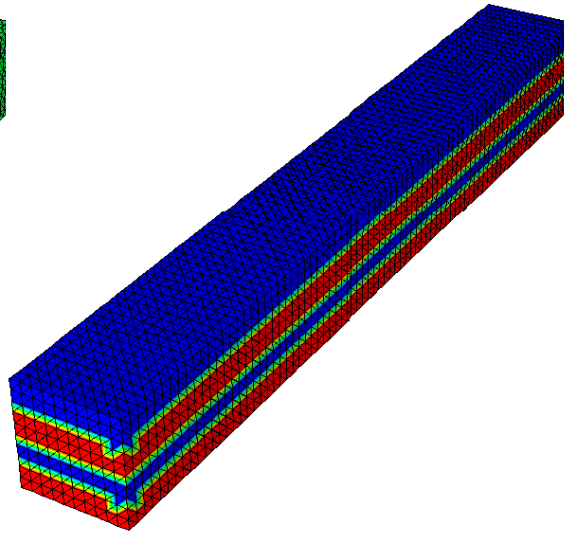
Numerics – Darcy – I

Domain & Partitions





Coefficient



Iterations (CG) vs. jumps

Code: Matlab & FreeFem++

κ_2	ASM-1	ASM-2-low	$dim(V_H)$	GenEO	$dim(V_H)$
1	22	16	*(8)*	16	*(8)*
10^2	31	24	*(8)*	17	*(15)*
10^4	37	30	*(8)*	21	*(15)*
10^6	36	29	*(8)*	18	*(15)*

ASM-1: 1-level ASM ASM-2-low: $m_j = 1$ NEW: $\lambda_{j,m_j+1} < \delta_j / H_j$

Iterations (CG) vs. number of subdomains

regular partition

subd.	dofs	ASM-1	ASM-2-low	$dim(V_H)$	GenEO	$dim(V_H$
4	4840	14	15	*(4)*	10	*(6)*
8	9680	26	22	*(8)*	11	*(14)*
16	19360	51	36	*(16)*	13	*(30)*
32	38720	$>$ 100	61	*(32)*	13	*(62)*

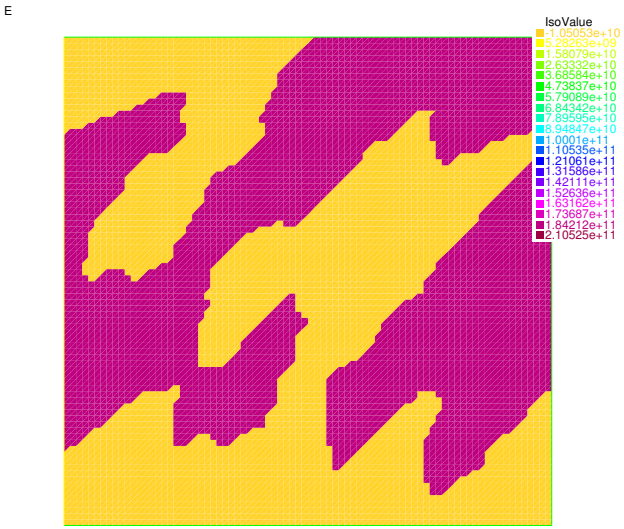
METIS partition

subd.	dofs	ASM-1	ASM-2-low	$dim(V_H)$	GenEO	$dim(V_H$
4	4840	21	18	*(4)*	15	*(7)*
8	9680	36	29	*(8)*	18	*(15)*
16	19360	65	45	*(16)*	22	*(31)*
32	38720	$>$100	79	*(32)*	34	*(63)*

Iterations (CG) vs. overlap

(added) layers	ASM-1	ASM-2-low	(V_H)	GenEO	(V_H)
1	26	22	*(8)*	11	*(14)*
2	22	18	*(8)*	9	*(14)*
3	16	15	*(8)*	9	*(14)*

Numerics – 2D Elasticity



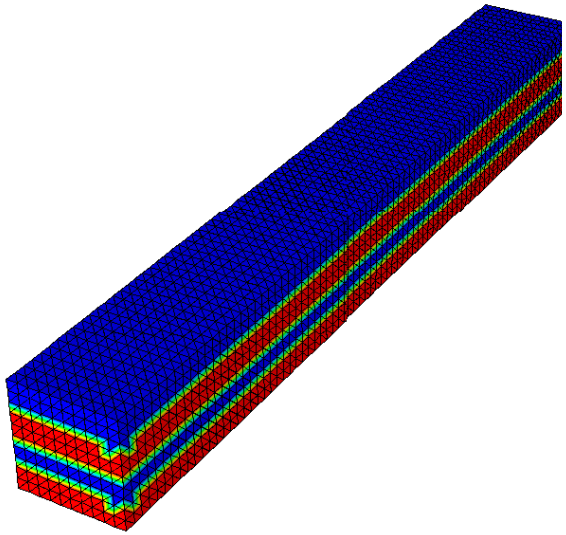
$$E_1 = 2 \cdot 10^{11}$$
$$\nu_1 = 0.3$$

$$E_2 = 2 \cdot 10^{7}$$
$$\nu_2 = 0.45$$

METIS partitions with 2 layers added

subd.	dofs	ASM-1	ASM-2-low	(V_H)	GenEO	(V_H)
4	13122	93	134	*(12)*	42	*(42)*
16	13122	164	165	*(48)*	45	*(159)*
25	13122	211	229	*(75)*	47	*(238)*
64	13122	279	167	*(192)*	45	*(519)*

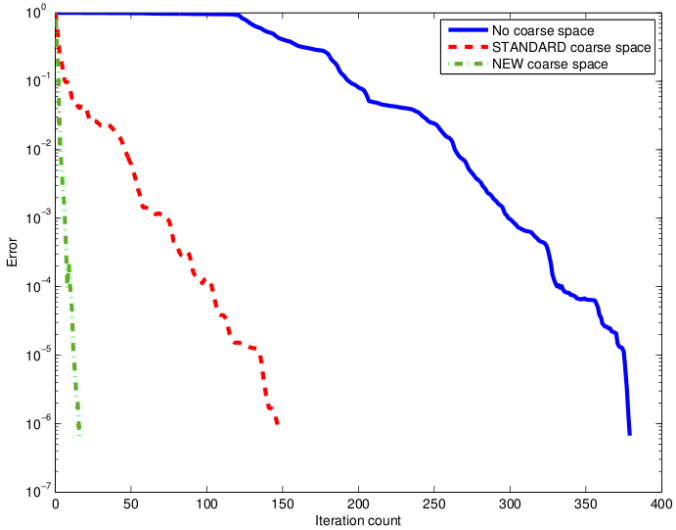
Iterations (CG) vs. number of subdomains



$$E_1 = 2 \cdot 10^{11}$$
$$\nu_1 = 0.3$$

$$E_2 = 2 \cdot 10^{7}$$
$$\nu_2 = 0.45$$

Relative error vs. iterations
16 regular subdomains



subd.	dofs	ASM-1	ASM-2-low	(V_H)	GenEO	(V_H)
4	1452	79	54	*(24)*	16	*(46)*
8	29040	177	87	*(48)*	16	*(102)*
16	58080	378	145	*(96)*	16	*(214)*

Outline