Entropy-based artificial viscosity
Parabolic regularization and related topics

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Outline

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The (not so new) idea

- Regularize the PDE from the start.
- Clearly identify the viscous regularization.
- Discretize $\Rightarrow$ artificial viscosity should be independent of discretization (except for a notion of mesh-size). Should work for finite diff, finite elements, DG, spectral method, spectral finite elements, etc.

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- Viscous regularization gives $\mu_{\max}$ (First-order viscosity. Low order method).
- Use the physical principle of entropy production to limit the amount of artificial viscosity: μ_E
- Entropy Viscosity: $\mu = \min(\mu_{\max}, \mu_E)$.

PDE-based vs Entropy-based artificial viscosities

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PDE-residual is less robust than entropy residual

- The residual of the PDE goes to zero in the distribution sense (solve the PDE!)
- The entropy residual converges to a Dirac measure supported in the physical shocks.

Example (Riemann problem for 1D Burgers' equation)

IVP:

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases} \end{cases}$$

Solution:

$$u(x,t) = 1 - H\left(x - \frac{1}{2}t\right)$$

PDE Residual:

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \frac{1}{2}H' - \frac{1}{2}H' = 0$$

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If $E(u) = \frac{u^2}{2}$ and $F(u) = \frac{u^3}{3}$, then the Entropy Residual:

$$\partial_t \left(\frac{u^2}{2} \right) + \partial_x \left(\frac{u^3}{3} \right) = \frac{1}{4} H' - \frac{1}{3} H' = -\frac{1}{12} H' = -\frac{1}{12} \delta\left(x - \frac{1}{2} t \right) < 0$$

Contact and other waves

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Contact and other waves

- The residual of an entropy equation is large in shocks
- But it goes to zero in contacts
- Automatic distinction between shock and other waves

Nonlinear scalar conservation equations

Transport,
mixing

Model problem

$$\begin{cases} \partial_t u + \nabla\cdot \mathbf{f}(u) = 0, \quad (\mathbf{x},t) \in \Omega\times(0,T] \\ u(\mathbf{x},0) = u_0(\mathbf{x}) \\ u(\mathbf{x},t)|_\Gamma = g \end{cases}$$

Entropy inequality

$$\partial_t E(u) + \nabla\cdot \mathbf{F}(u) \le 0$$

$$\mathbf{F}'(u) = E'(u)\mathbf{f}'(u)$$

Regularized model problem

Add viscous dissipation to stabilize the model problem:

$$\begin{cases} \partial_t u + \nabla \cdot \mathbf{f}(u) = -\nabla \cdot \mathbf{q}, & (\mathbf{x}, t) \in \Omega \times (0, T] \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \\ u(\mathbf{x}, t)|_\Gamma = g \end{cases}$$

- $\mathbf{q} = -\mu \nabla u$ is a viscous flux.
- μ will be the entropy viscosity (will depend on u).

Space discretization

- Discretize the domain Ω into $\cup_{K\in\mathbb{T}_h} K = \bar{\Omega}$
- K is assumed to be either a polygon or a polyhedron
- Finite element space $\mathcal{V}_h^p$ consists of continuous polynomials of degree $p \geq 0$
- $h : \Omega \longrightarrow \mathbb{R}_+$ is defined by $\forall K \in \mathbb{T}_h : h\big|_K \equiv h_K = diam(K)/p^2$.

Key idea 1: $\boxed{\textbf{Entropy viscosity should not exceed } \frac{1}{2}|\mathbf{f}'|h}$

- Numerical analysis 101: Up-winding=centered approx + $\frac{1}{2}|\beta|h$ viscosity
- 1D Proof: Assume $f_i' \geq 0$

$$f_i' \frac{u_i - u_{i-1}}{h_i} = f_i' \frac{u_{i+1} - u_{i-1}}{2h_i} - \frac{1}{2} f_i' h_i \frac{u_{i+1} - 2u_i + u_{i-1}}{h_i^2}$$

In 1D

$$\mu_{\max} = \frac{1}{2}|f'|h$$

Key idea 2: **Use entropy residual to construct viscosity**

- Evaluate entropy residual

$$D_h := \partial_t E(u_h) + \mathbf{f}'(u_h) \cdot \nabla E(u_h)$$

  at each time step
- Set

$$\mu_E = h^2 \frac{D_h}{\text{normalization}(E(u_h))}.$$

The algorithm

- Choose one entropy functional (or more).
  EX: $E(u) = |u - \overline{u_0}|$, $E(u) = (u - \overline{u_0})^2$, etc.

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- Construct viscosity associated with entropy residual over each mesh cell K:

$$\mu_{E,K} := c_E h_K^2 \frac{\max(\|D_h\|_{L^\infty(K)}, \|J_h\|_{L^\infty(\partial K)})}{\overline{E(u_h)}}$$

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- Compute maximum upwind viscosity over each mesh cell K:

$$\mu_{\max,K} = c_{\max} h_K \|\mathbf{f}'(u_h)\|_{L^\infty(K)}$$

- Compute viscosity over each mesh cell K by comparing $\mu_{\max,K}$ and $\mu_{E,K}$:

$$\mu_K := \min(\mu_{\max,K}, \mu_{E,K})$$

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- Definition of μ_K can be localized when polynomial degree p is large.
- $c_{\max} = \frac{1}{2}$ in 1D, with $p = 1$.
- $c_{\max}$ can be theoretically estimated (depends on space dimension, p, and type of mesh).
- $c_E \approx 1$ in applications.

The algorithm

- Space approximation: Galerkin + entropy viscosity:

$$\underbrace{\int_\Omega (\partial_t u_h + \nabla\cdot(\mathbf{f}(u_h)))v_h \mathrm{d}\mathbf{x}}_{\text{Galerkin(centered approximation)}} + \underbrace{\sum_K \int_K \mu_K \nabla u_h \nabla v_h \mathrm{d}\mathbf{x}}_{\text{Entropy viscosity}} = 0, \quad \forall v_h \in \mathcal{V}_h^p$$

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- Time approximation: Use an explicit time stepping: BDF2, RK3, RK4, etc.
- Make the viscosity explicit $\Rightarrow$ Stability under CFL condition.

Example (Finite differences + RK2)

- (u^n, μ^n) Given. Advance half time step to get w^n

$$w_i^n = u_i^n - \frac{1}{2}\Delta t \frac{f(u_{i+1}^n) - f(u_{i-1}^n)}{2\overline{h_i}} + \left(\mu_i^n \frac{u_{i+1}^n - u_i^n}{h_i} - \mu_{i-1}^n \frac{u_i^n - u_{i-1}^n}{h_{i-1}}\right)$$

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- Compute entropy residuals (volume and interface)

$$D_i := \frac{E(w_i^n) - E(u_i^n)}{\Delta t/2} + \frac{F(w_{i+1}^n) - F(w_i^n)}{h_i}$$

$$D_{i+1} := \frac{E(w_{i+1}^n) - E(u_{i+1}^n)}{\Delta t/2} + \frac{F(w_{i+1}^n) - F(w_i^n)}{h_i}$$

$$J_i := \frac{F(w_{i+1}^n) - F(w_i^n)}{h_i} - \frac{F(w_i^n) - F(w_{i-1}^n)}{h_{i-1}}$$

Example (Finite differences + RK2)

- Compute entropy viscosity μ^{n+1}

$$\mu_{i,\max} = \frac{1}{2}\|f'\|_{L^\infty(x_{i-1},x_{i+1})}\overline{h_i}$$

$$\mu_{i,E} = \overline{h_i}^2 \frac{\max(|D_i|,|D_{i+1}|,|J_i|)}{\overline{E(w^n)}}$$

$$\mu_i^{n+1} = \min(\mu_{i,\max},\mu_{i,E}).$$

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$$\mu_i^{n+1} = \min(\mu_{i,\max},\mu_{i,E}).$$

- Compute u^{n+1}

$$u_i^{n+1} = u_i^n - \Delta t \frac{f(w_{i+1}^n) - f(w_{i-1}^n)}{2\overline{h_i}} + \left(\mu_i^{n+1}\frac{w_{i+1}^n - w_i^n}{h_i} - \mu_{i-1}^{n+1}\frac{w_i^n - w_{i-1}^n}{h_{i-1}}\right)$$

Theorem (AB,JLG,BP (2012))

The RK2 time approximation with finite element approximation is stable under CFL condition for all polynomial degrees. (Better than usual $\delta < ch^{\frac{4}{3}}$ condition for piecewise linear approximation.)

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Conjecture

Convergence to the entropy solution is under way for convex, Lipschitz flux.

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Conjecture

Convergence to the entropy solution is under way for convex, Lipschitz flux.

Why convergence is so difficult to prove?

- Key a priori estimate
$$\int_0^T \mu(u)|\nabla u|^2 \mathrm{d}\mathbf{x} \leq c$$
- Ok in $\{\mu(u)(\mathbf{x},t) = \frac{1}{2}\|\mathbf{f}'\|_{L^\infty} h\}$ (non-smooth region)
- The estimate is useless in smooth region. 
- Explicit time stepping makes the viscosity depend on the past.

Extensions

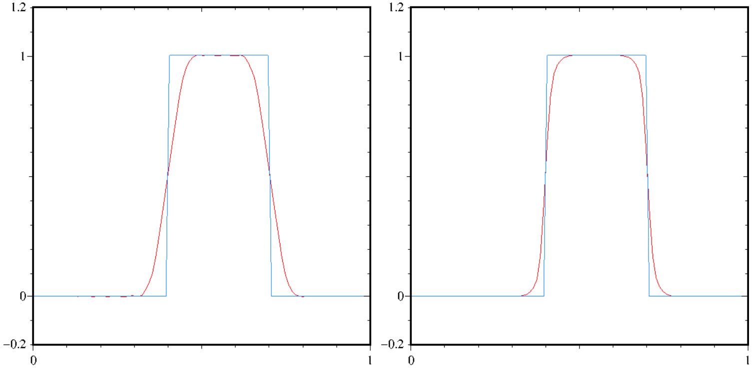
- Algorithm extends naturally to Discontinuous Galerkin setting (PhD thesis Valentin Zingan (2011) Texas A&M).
- Lagrangian formulation under way (PhD thesis Vladimir Tomov, Texas A&M).

Nonlinear scalar conservation equations

Johannes Martinus Burgers

Example (1D scalar transport)

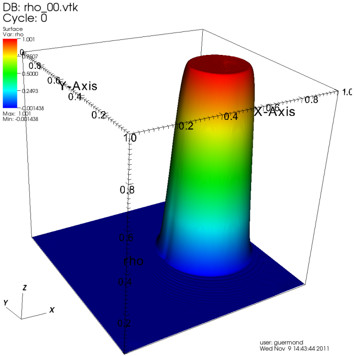
- $\partial_t u + \partial_x u = 0$, periodic BCs.
- $\mathbb{P}_1$ finite elements, RKx ($x \geq 2$).
- Using very nonlinear entropies help to satisfy the maximum principle for scalar conservation and steepen contacts.



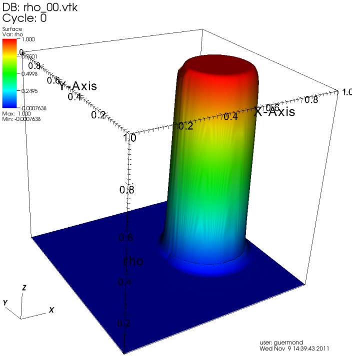
(a) $E(u) = (u - \frac{1}{2})^2$, $N = 100$, $t = 1$ (b) $E(u) = (u - \frac{1}{2})^{30}$, $N = 100$, $t = 1$

Example (2D scalar transport)

- $\partial_t u + \beta \cdot \nabla u = 0$, ($\beta$ solid rotation).
- $\mathbb{Q}_1$ finite elements, RKx ($x \geq 2$).
- Using very nonlinear entropies help to satisfy the maximum principle for scalar conservation and steepen contacts.



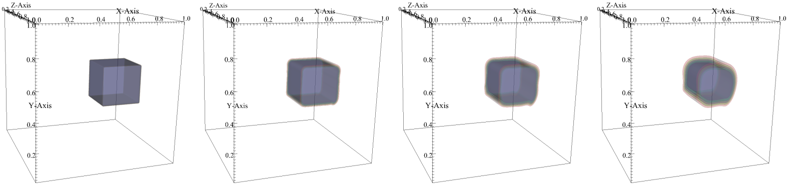
(c) $E(u) = (u - \frac{1}{2})^2$, $N = 100^2$, $t = 1$



(d) $E(u) = (u - \frac{1}{2})^{30}$, $N = 100^2$, $t = 1$

Example (3D scalar transport)

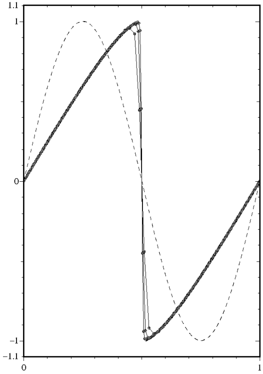
- $\partial_t u + \beta \cdot \nabla u = 0$, ($\beta$ solid rotation about Oz)
- $\mathbb{Q}_1$ finite elements, RKx ($x \geq 2$).
- Level sets of a cube in rotation on a $(100)^3$ grid in the original configuration and after 1, 10, and 100 rotations. $E(u) = (u - \frac{1}{2})^{20}$, $0 \leq u \leq 1$.



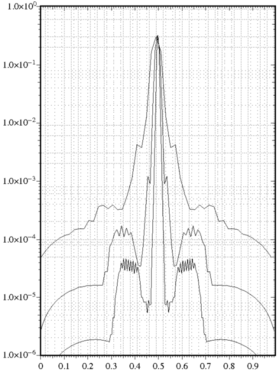
(e) $t = 0$ (f) $t = 1$ (g) $t = 10$ (h) $t = 100$

Example (1D Burgers)

- Second-order Finite Differences + RKx
- Burgers, $t = 0.25$, $N = 50$, 100, and 200 grid points.



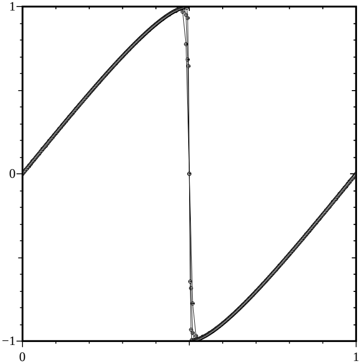
(i) u_h



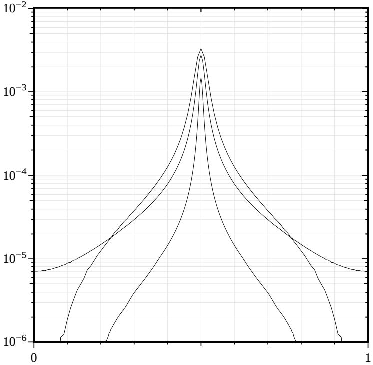
(j) $\nu_h(u_h)|\partial_x u_h|$

Example (1D Burgers)

- Fourier approximation + RKx
- Burgers at $t = 0.25$ with $N = 50$, 100, and 200.



(k) u_h



(l) $\nu_N(u_N)$

Example (1D Burgers)

- DG1 + RKx (V. Zingan)
- Entropy viscosity preserve accuracy outside shocks.
- Compute error in $[0, 0.5 - 0.025] \cup [0.5 + 0.025]$ at $t = 0.25$ with DG1

cells	dofs	h	L_1-error	R_1	L_2-error	R_2
5	10	2e-01	1.677e-01	-	2.450e-01	-
10	20	1e-01	7.866e-02	1.09	1.420e-01	0.79
20	40	5e-02	2.133e-02	1.88	4.891e-02	1.54
40	80	2.5e-02	1.779e-03	3.58	4.918e-03	3.31
80	160	1.25e-02	1.517e-04	3.55	1.894e-04	4.69
160	320	6.25e-03	2.989e-05	2.34	4.075e-05	2.22
320	640	3.125e-03	6.903e-06	2.11	9.832e-06	2.05
640	1280	1.5625e-03	1.720e-06	2.01	2.464e-06	2.00

Example (1D Burgers)

- DG2 + RKx (V. Zingan)
- Entropy viscosity preserve accuracy outside shocks.
- Compute error in $[0, 0.5 - 0.025] \cup [0.5 + 0.025]$ at $t = 0.25$ with DG2.

cells	dofs	h	L_1-error	R_1	L_2-error	R_2
5	15	2e-01	4.039e-02	-	8.362e-02	-
10	30	1e-01	8.040e-03	2.33	1.398e-02	2.58
20	60	5e-02	2.242e-03	1.84	6.584e-03	1.08
40	120	2.5e-02	2.149e-04	3.38	5.229e-04	3.65
80	240	1.25e-02	1.366e-05	3.98	1.621e-05	5.01
160	480	6.25e-03	1.644e-06	3.06	1.949e-06	3.06
320	960	3.125e-03	2.018e-07	3.03	2.410e-07	3.02
640	1920	1.5625e-03	2.505e-08	3.01	3.003e-08	3.01

Example (1D Burgers)

- DG3 + RKx (V. Zingan)
- Entropy viscosity preserve accuracy outside shocks.
- Compute error in $[0, 0.5-0.025] \cup [0.5+0.025]$ at $t = 0.25$ with DG3.

cells	dofs	h	L_1-error	R_1	L_2-error	R_2
5	20	2e-01	1.678e-02	-	2.556e-02	-
10	40	1e-01	9.932e-03	0.76	2.445e-02	0.10
20	80	5e-02	2.019e-03	2.30	6.712e-03	1.86
40	160	2.5e-02	1.761e-04	3.52	6.608e-04	3.35
80	320	1.25e-02	5.716e-06	4.95	7.317e-06	6.50
160	640	6.25e-03	5.791e-07	3.30	7.531e-07	3.28
320	1280	3.125e-03	6.225e-08	3.22	7.843e-08	3.26
640	2560	1.5625e-03	7.485e-09	3.06	9.052e-09	3.12

Example (1D Nonconvex flux)

- Fourier approximation

1D equation

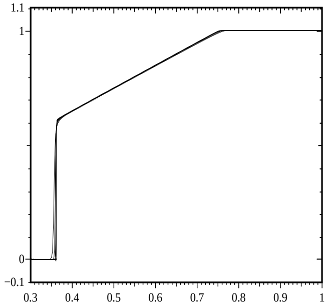
$\partial_t u + \partial_x f(u) = 0$, $u(x,0) = u_0(x)$

Flux

$$f(u) = \begin{cases} \frac{1}{4}u(1-u) & \text{if } u < \frac{1}{2}, \\ \frac{1}{2}u(u-1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2}, \end{cases}$$

Initial data

$$u_0(x) = \begin{cases} 0, & x \in (0, 0.25], \\ 1, & x \in (0.25, 1] \end{cases}$$



$t = 1$ with $N = 200$, 400, 800, and 1600.

Example (2D Burgers)

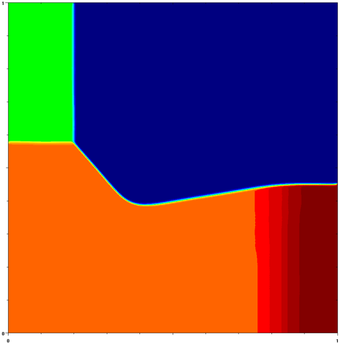
- $\mathbb{P}_1$ finite elements.

2D Burgers

$$\partial_t u + \partial_x(\tfrac{1}{2}u^2) + \partial_y(\tfrac{1}{2}u^2) = 0$$

Initial data

$$u^0(x,y) = \begin{cases} -0.2 & \text{if } x < 0.5,\ y > 0.5 \\ -1 & \text{if } x > 0.5,\ y > 0.5 \\ 0.5 & \text{if } x < 0.5,\ y < 0.5 \\ 0.8 & \text{if } x > 0.5,\ y < 0.5 \end{cases}$$



Solution at $t = \frac{1}{2}$, 3×10^4 nodes.

Example (2D Burgers)

- $\mathbb{P}_1$ and $\mathbb{P}_2$ finite elements.

$\mathbb{P}_1$ approximation

h	$\mathbb{P}_1$			
	L^2	rate	L^1	rate
5.00E-2	2.3651E-1	–	9.3661E-2	–
2.50E-2	1.7653E-1	0.422	4.9934E-2	0.907
1.25E-2	1.2788E-1	0.465	2.5990E-2	0.942
6.25E-3	9.3631E-2	0.449	1.3583E-2	0.936
3.12E-3	6.7498E-2	0.472	6.9797E-3	0.961

$\mathbb{P}_2$ approximation

h	$\mathbb{P}_2$			
	L^2	rate	L^1	rate
5.00E-2	1.8068E-1	–	5.2531E-2	–
2.50E-2	1.2956E-1	0.480	2.7212E-2	0.949
1.25E-2	9.5508E-2	0.440	1.4588E-2	0.899
6.25E-3	6.8806E-2	0.473	7.6435E-3	0.932

Example (Buckley Leverett)

- $\mathbb{P}_2$ finite elements.

The equation

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0.$$

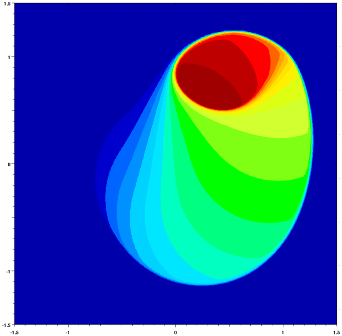
Flux

$$f(u) = \frac{u^2}{u^2+(1-u)^2},$$
$$g(u) = f(u)(1-5(1-u)^2)$$

Non-convex fluxes (composite waves)

Initial data

$$u(x,y,0) = \begin{cases} 1, & \sqrt{x^2+y^2} \le 0.5 \\ 0, & \text{else} \end{cases}$$



Solution at $t = \frac{1}{2}$, 3×10^4 nodes.

Example (KPP)

- $\mathbb{P}_2$ and $\mathbb{Q}_4$ finite elements.

The equation

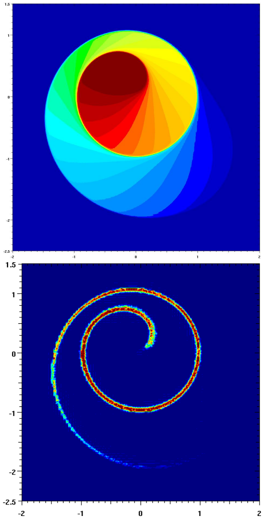
$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0.$

Flux

$f(u) = \sin(u)$, $g(u) = \cos(u)$,
Non-convex fluxes (composite waves)

Initial data

$$u(x,y,0) = \begin{cases} \frac{7}{2}\pi, & \sqrt{x^2+y^2} \le 1 \\ \frac{1}{4}\pi, & \text{else} \end{cases}$$



$\mathbb{P}_2$

Solution u_h

$\mathbb{Q}_4$

Viscosity μ_h

Compressible Euler equations


Leonhard Euler

Compressible Euler equations

$$\partial_t \mathbf{c} + \nabla \cdot \mathbf{F}(\mathbf{c}) = 0, \qquad \mathbf{c} = \begin{pmatrix} \rho \\ \mathbf{m} \\ E \end{pmatrix}, \qquad \mathbf{F}(\mathbf{c}) = \begin{pmatrix} \mathbf{m} \\ \frac{1}{\rho}\mathbf{m} \otimes \mathbf{m} \\ \frac{1}{\rho}\mathbf{m}(E+p) \end{pmatrix}$$

Equation of state

Ideal gas e.g.

$$p = (\gamma - 1)(E - \frac{1}{2\rho}\mathbf{m}^2).$$

Compressible Euler equations

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Equation of state

Ideal gas e.g.

$$p = (\gamma - 1)(E - \frac{1}{2\rho}\mathbf{m}^2).$$

Entropy inequality

$$\partial S + \nabla\cdot(\mathbf{u}S) \geq 0, \qquad \mathbf{u} := \frac{\mathbf{m}}{\rho}$$

$$S = \rho \log(e\rho^{1-\gamma}), \qquad e := \frac{1}{\rho}(E - \frac{1}{2\rho}\mathbf{m}^2)$$

Viscous regularization?

- Entropy viscosity $= \min(\mu_{\max}, \mu_E)$.

Viscous regularization?

- Entropy viscosity $= \min(\mu_{\max}, \mu_E)$.
- What is a good viscous regularization of Euler? $\mu_{\max}$?

Lax-Friedrich regularization (parabolic regularization)

In 1D, LxF is an approximation of

$$\partial_t \mathbf{c} + \nabla\cdot\mathbf{F}(\mathbf{c}) - \frac{1}{2}(|\mathbf{u}| + a)h\nabla^2\mathbf{c} = 0$$

where h is the mesh size, a is the speed of sound (Perthame, CW Shu (1996)).

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- Not Gallilean/rotational invariant.

Navier-Stokes regularization

$$\partial_t \mathbf{c} + \nabla\cdot\mathbf{F}(\mathbf{c}) - \nabla\cdot\mathbf{q} = 0, \qquad \mathbf{q} = \begin{pmatrix} 0 \\ \mu\nabla^s\mathbf{u} \\ \kappa\nabla T \end{pmatrix}$$

- T is the temperature.
- $\mu > 0$, $\kappa > 0$.

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Case $\kappa \neq 0$, ideal gas

$$\rho(\partial_t s + \mathbf{u}\cdot\nabla s) - \nabla\cdot(\kappa e^{-1}\nabla T) = \frac{\mu}{e}|\nabla^s\mathbf{u}|^2 + \frac{\kappa}{e^2}\nabla T\cdot\nabla e$$

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- Sets $\{s(\rho, e) > s_0\}$ are not positively invariant if $\kappa \neq 0$. (See e.g. Serre (1999)
  Discrete positivity of e?

Minimum principle on the specific entropy

- Formally, solutions to Euler equations should satisfy

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- Provided $\rho > 0 \Rightarrow e > 0$ (minimum principle on e).
- Is there a viscous regularization that can reproduce this property?

Minimum entropy preserving regularization

$$\partial_t \mathbf{c} + \nabla{\cdot}\mathbf{F}(\mathbf{c}) - \nabla{\cdot}\mathbf{q} = 0, \qquad \mathbf{q} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} + \mathbf{g}{\cdot}\mathbf{u} \end{pmatrix}$$

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- **f**, **g**, **h** to be determined so that

$$\rho(\partial_t s + \mathbf{u}{\cdot}\nabla s) - \nabla{\cdot}(\kappa(\rho, e)\nabla\varphi(s)) + \text{conservative} \geq 0,$$

and

$$\partial_t S + \nabla{\cdot}(\mathbf{u}S) \geq 0.$$

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Key hypotheses

- $\mathbf{f}\cdot\nabla\rho \geq 0 \Rightarrow \{\rho > 0\}$ positively invariant set.

Minimum entropy preserving regularization

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and

$$\partial_t S + \nabla\cdot(\mathbf{u}S) \geq 0.$$

Key hypotheses

- $\mathbf{f}\cdot\nabla\rho \geq 0 \Rightarrow \{\rho > 0\}$ positively invariant set.
- $\varphi'(s) \geq 0$, $\kappa(\rho, e) \geq 0 \Rightarrow \{s(\rho, e) > s_0\}$ positively invariant sets.

Strategy

- $\rho s_\rho \times$mass balance $+\, s_e \times$internal energy balance
- Recombine the terms so that conservative term is $-\nabla\cdot\kappa\nabla s$, rhs is positive, and hope for the best.

Simple choice

$$\mathbf{f} = \kappa \frac{s_\rho}{\rho s_\rho - e s_e} \nabla \rho.$$

$$\mathbf{g} = \mu \nabla^s \mathbf{u} + \mathbf{u} \otimes \mathbf{f}.$$

$$\mathbf{h} = \kappa \nabla e - \frac{1}{2} \mathbf{u}^2 \mathbf{f}.$$

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Proposition (JLG-BP (2012))

Assume ideal gas, $\gamma > 1$. Assume existence of a smooth solution. The sets $\{s(\rho, e) > s_0\}$ are positively invariant and

$$\rho(\partial_t s + u \nabla s) - \nabla\cdot(\kappa \nabla s) = \frac{\mu}{e} |\nabla^s \mathbf{u}|^2 + \frac{\kappa}{e^2} \nabla T \cdot \nabla e.$$

$$\partial_t S + \nabla\cdot(\mathbf{u} S + \kappa(\nabla s + \frac{\gamma - 1}{\gamma} s \nabla \log(\rho))) \geq 0.$$

Similar properties hold for a stiffened gas (conjecture: holds on a large class of eos)

Example

Ideal gas

$$\mathbf{f} = \frac{\kappa}{c_v}\frac{\gamma-1}{\gamma}\frac{\nabla\rho}{\rho}.$$

Connection with a phenomenological model by H. Brenner (2006)

- Seems a bit controversial in the physics literature
- Seems to give some leeway in the analysis of Navier-Stokes? (Feireisl-Vasseur (2008))

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Brenner's model (ideal gas)

$$\mathbf{u}_m = \mathbf{u} - \rho^{-1}\mathbf{f}$$

$$\mathbf{f} = \frac{\kappa}{c_p}\frac{\nabla\rho}{\rho}$$

$$\partial_t\rho + \nabla\cdot(\mathbf{u}_m\rho) = 0$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\mathbf{u}\otimes\rho\mathbf{u}_m) + \nabla p - \nabla\cdot\tau_v = 0$$

$$\partial_t(\rho e) + \nabla\cdot(\mathbf{u}_m e) + p\nabla\cdot\mathbf{u} - \nabla\cdot(\kappa\nabla T) - \nabla\cdot(\tau_v\cdot v) = 0$$

Our regularization (ideal gas)

$$\mathbf{u}_m = \mathbf{u} - \rho^{-1}\mathbf{f}$$

$$\mathbf{f} = \frac{\kappa}{c_p}\frac{1}{\gamma-1}\frac{\nabla\rho}{\rho}$$

$$\partial_t\rho + \nabla\cdot(\mathbf{u}_m\rho) = 0$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\mathbf{u}\otimes\rho\mathbf{u}_m) + \nabla p - \nabla\cdot\tau_v = 0$$

$$\partial_t(\rho e) + \nabla\cdot(\mathbf{u}e) + p\nabla\cdot\mathbf{u} - \nabla\cdot(\kappa\nabla T) - \nabla\cdot(\tau_v\cdot v) = 0$$

The algorithm, $S = \frac{\rho}{\log}(e\rho^{1-\gamma})$

- Compute cell entropy residual, $D_{h|K} := \partial_t S + \nabla\cdot(\mathbf{u}S)$
- Compute interface entropy residual $J_{h|\partial K} = [\![(\nabla \mathbf{u} S) : (\mathbf{n}\otimes\mathbf{n})]\!]$
- Define

$$\mu_{E|K} = c_E h_K^2 \max(\|D_{h|K}\|_{L^\infty(K)}, \|J_{h|\partial K}\|_{L^\infty(\partial K)})$$

- Compute maximum local viscosity: $\mu_{\max,K} = c_{\max} h_k \rho \| \|\mathbf{u}\| + (\gamma T)^{\frac{1}{2}} \|_{\infty,K}$
- Compute entropy viscosity

$$\mu_K = \min(\mu_{\max,K}, \mu_{E|K}).$$

- Define artificial thermal diffusivity

$$\kappa_K = \mathcal{P}\mu_K, \qquad \mathcal{P} \approx 0.2.$$

The algorithm (continued)

- Use Galerkin for space approximation (use your favorite method: FE, FD, Fourier, Spectral, DG, etc.)
- Use explicit RK to step in time.

1D Euler flows + Fourier

- Solution method: Fourier + RK4 + entropy viscosity

1D Euler flows + Fourier

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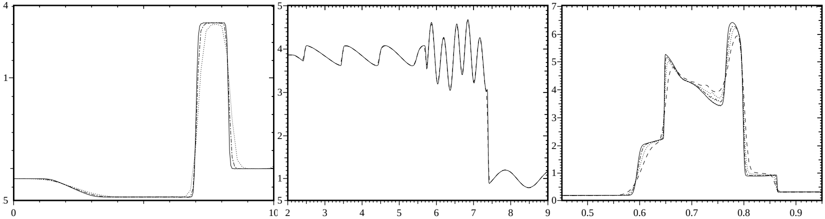
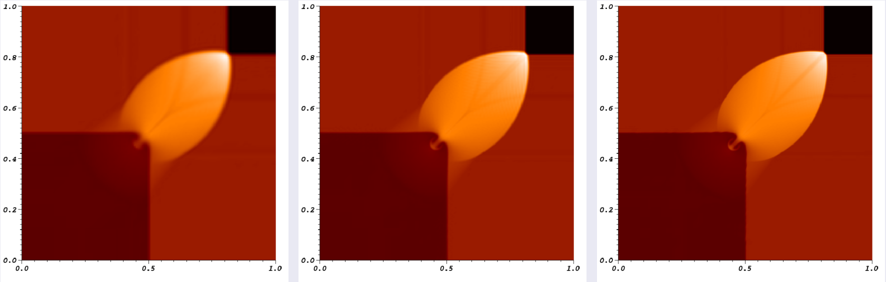


Figure: Lax shock tube, $t = 1.3$, 50, 100, 200 points. Shu-Osher shock tube, $t = 1.8$, 400, 800 points. Right: Woodward-Collela blast wave, $t = 0.038$, 200, 400, 800, 1600 points.

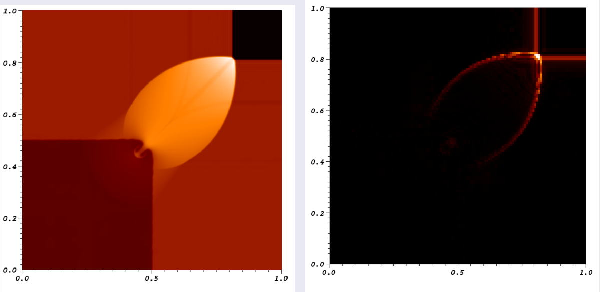
DG, 2D Riemann problem

Density Q_1, Q_2, and Q_3

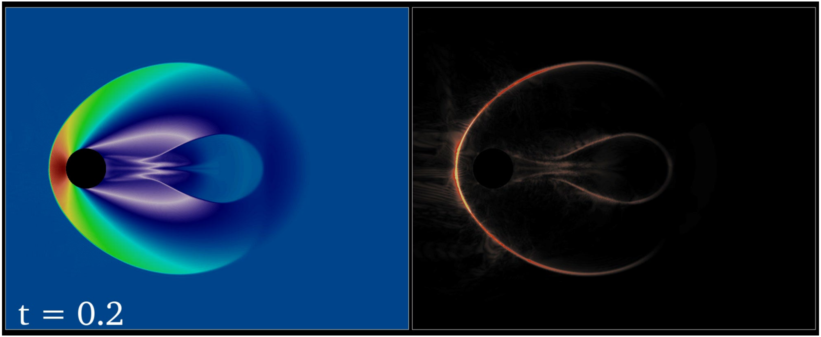


DG, 2D Riemann problem

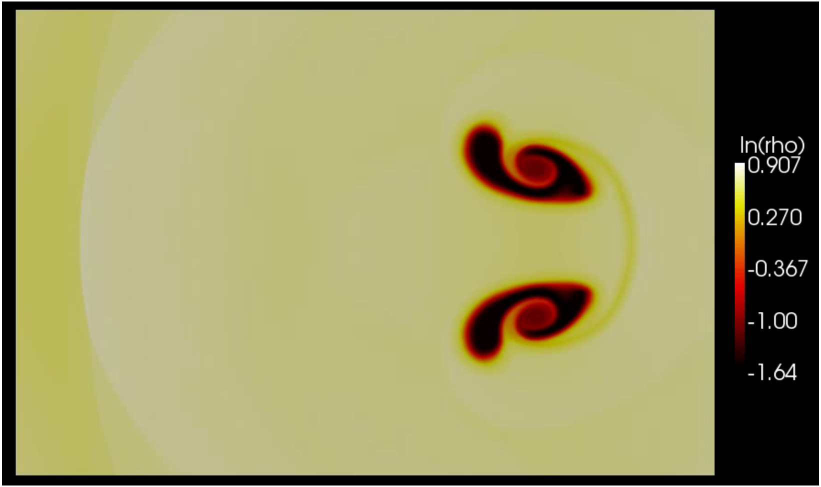
Density Q_3 and associated dynamic viscosity



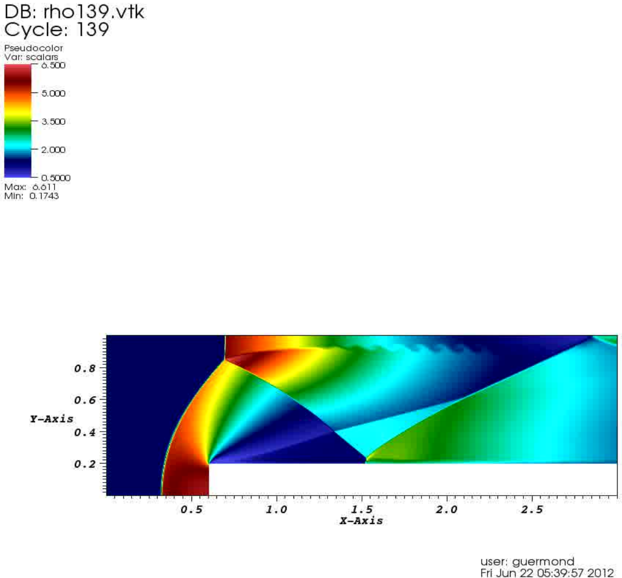
Cylinder in a channel, Mach 2, $\mathbb{P}_1$ FE (By M. Nazarov)



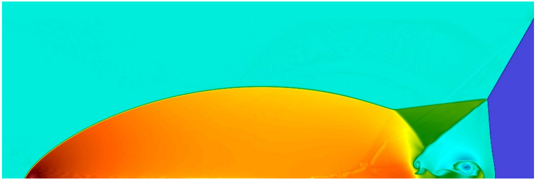
Bubble, density ratio 10^{-1}, Mach 1.65, $\mathbb{P}_1$ FE (by M. Nazarov)



Mach 3 Wind Tunnel with a Step, $\mathbb{P}_1$ finite elements, 1.3 10^5 nodes



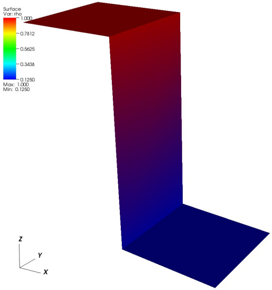
Mach 10 Double Mach reflection, $\mathbb{P}_1$ finite elements

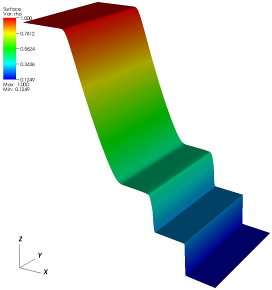


$\mathbb{P}_1$ FE, $4.5\,10^5$ nodes, $t = 0.2$

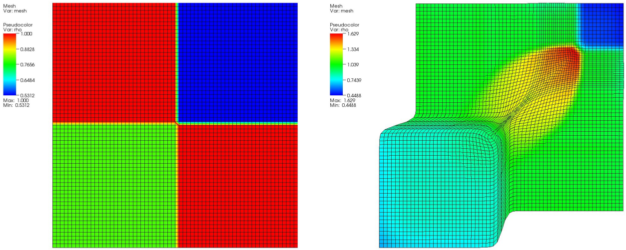
Movie, density field

Sod shocktube. Lagrangian hydro. $\mathbb{Q}_1$ FEM, 1×1024 (V. Tomov)





Riemann pb. Lagrangian hydro. $\mathbb{Q}_2$ FEM, 32×32, (V. Tomov)



Sedov explosion. Lagrangian hydro. $\mathbb{Q}_3$ FEM, 32×32, (V. Tomov)