Overview of fractional step techniques for the incompressible Navier-Stokes equations

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CEMRACS'12
CIRM, Marseille
July 16-20, 2012

Acknowledgments

Collaborators: Peter Minev (Edmonton), Luigi Quartapelle (Milano), Jie Shen (Purdue)

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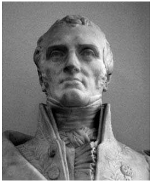
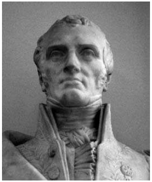
Navier-Stokes equations



Claude L. M. H. Navier

George G. Stokes

Navier-Stokes equations



Claude L. M. H. Navier

$$\begin{cases} \partial_t \mathrm{u} - \nu \nabla^2 \mathrm{u} + \mathrm{u}\cdot\nabla \mathrm{u} + \nabla \mathrm{p} = f \quad \text{in } \Omega \times [0, T], \\ \nabla\cdot\mathrm{u} = 0 \quad \text{in } \Omega \times [0, T], \\ \mathrm{u}|_\Gamma = 0 \quad \text{in } [0,\mathrm{T}], \qquad \text{and } \mathrm{u}|_{t=0} = u_0 \quad \text{in } \Omega, \end{cases}$$

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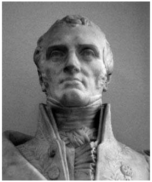

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George G. Stokes

- Ω fluid domain
- T some time
- f smooth source term
- u_0 smooth solenoidal data

Objectives

Hyp: $\Omega \subset \mathbb{R}^{2 \text{ or } 3}$ is a bounded and smooth domain, and all compatibility conditions are satisfied for a smooth solution to exist.

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Strategy: Chorin–Temam idea (1968-1969): fractional-step technique

$$[L^2(\Omega)]^d = H \oplus \nabla H^1(\Omega),$$

where $H = \{v \in [L^2(\Omega)]^d;\ \nabla\cdot v = 0;\ v\cdot n|_\Gamma = 0\}$,

non-incremental pressure-correction schemes

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$$p^{k+1} = \phi^{k+1}$$

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- Step 2 amounts to

$$\tilde{u}^{k+1} = u^{k+1} + \nabla(\Delta t \phi^{k+1}), \qquad u^{k+1} \in H, \; \phi^{k+1} \in H^1(\Omega)$$

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(i) $$\nabla^2 \phi^{k+1} = \frac{1}{dt} \nabla \cdot \tilde{u}^{k+1}; \quad \partial_n \phi^{k+1}|_\Gamma = 0$$

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- Very simple algorithm $\Rightarrow$ Very popular

non-incremental pressure-correction schemes

Theorem (Rannacher (1991), Shen (1992))

$$\|\mathrm{u}_{\Delta t} - u_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} + \|\mathrm{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} \leq c(\mathrm{u}, \mathrm{p}, T)\,\Delta t,$$

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- Observe that $\nabla p^{k+1}\cdot n|_\Gamma = 0$ is enforced on the pressure. Artificial Neumann bc $\Rightarrow$ scheme not fully first-order.
- Irreducible splitting error of order $\mathcal{O}(\Delta t)$ $\Rightarrow$ using higher-order time stepping does not improve the overall accuracy.

Incremental pressure-correction schemes

- Simple idea: use the old pressure p^k in the viscous step and correct the pressure appropriately afterwards (Goda (1979) Van Kan (1986)).

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- Time stepping can be replaced by any 2nd order A-stable stepping.

Rotational incremental pressure-correction schemes

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  Sum viscous prediction + projection + use pressure correction:

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- This implies consistent equations for the pressure:

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- Where is the catch?
  The tangent component of u^{k+1} is still not correct! $\Rightarrow$ sub-optimality

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Theorem (Guermond-Shen (2006))

With appropriate initialization,

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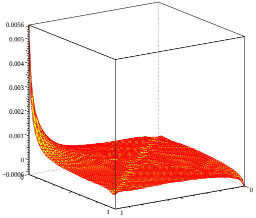
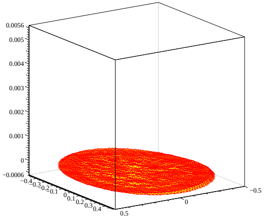
- OPEN QUESTION: can we regain the missing $\Delta t^{\frac{1}{2}}$?

Numerical illustration





Error field on pressure in a rectangular domain (top) and on a circular domain (bottom)

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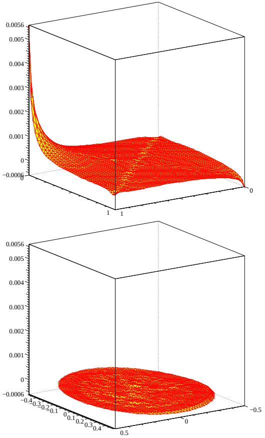

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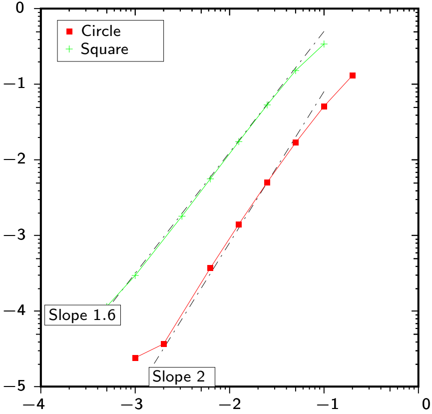

Convergence rates on pressure in L^∞-norm at $T = 2$; ■ for the circular domain; + for the square.

Generalization

- Set

$$p^{\star,k+1} = \begin{cases} 0 & \text{if } r = 0, \\ p^k & \text{if } r = 1, \\ 2p^k - p^{k-1} & \text{if } r = 2. \end{cases}$$

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- Step 1: Viscous prediction (arbitrary time stepping)

$$\frac{1}{\Delta t}\left(\beta_q \tilde{u}^{k+1} - \sum_{j=0}^{q-1} \beta_j u^{k-j}\right) - \nu \nabla^2 \tilde{u}^{k+1} + \nabla p^{\star,k+1} = f(t^{k+1}), \quad \tilde{u}^{k+1}|_{\Gamma}$$

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- Step 2: Projection

$$\begin{cases} \frac{\beta_q}{\Delta t}(u^{k+1} - \tilde{u}^{k+1}) + \nabla \phi^{k+1} = 0, \\ \nabla \cdot u^{k+1} = 0, \quad u^{k+1} \cdot n|_\Gamma = 0, \end{cases}$$

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- Step 3: Pressure correction

$$p^{k+1} = \phi^{k+1} + p^{\star,k+1} - \nu \nabla \cdot \tilde{u}^{k+1}.$$

Implementation

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- Extremely simple to implement

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$$\boxed{p^{\star,k+1} = 2p^k - p^{k-1} \text{ is not recommended}}$$

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- This scheme is written is a clumsy way $\Leftrightarrow$ No convergence analysis has been done.
- Actually, up to change of variables
  (+ 20 years later, Guermond-Shen (2006))

  Rotational incremental pressure-correction = Kim-Moin

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$\Rightarrow$ Switch viscous diffusion and projection.

$\Rightarrow$ Correct the velocity instead of correcting the pressure.

Non-incremental Velocity-correction

- Step 1: Projection

$$\begin{cases} \frac{1}{\Delta t}(u^{k+1} - \tilde{u}^k) + \nabla p^{k+1} = f(t^{k+1}), \\ \nabla\cdot u^{k+1} = 0, \quad u^{k+1}\cdot n|_\Gamma = 0, \end{cases}$$

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- Step 2: Velocity correction by diffusion

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Theorem (Guermond-Shen(2003))

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Incremental Velocity-correction

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- Choosing velocity-correction vs. pressure-correction is a matter of personal taste.
- Same pitfalls as for pressure-correction: second-order extrapolation of velocity not recommended (although advertised in literature)
- Up to changes of variable Velocity-correction schemes are identical to (clumsy) schemes proposed by Orszag, Israeli, Deville (1986) and Karniadakis, Israeli, Orszag (1991)

Consistent splitting: The idea

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- Observe that $(\mathrm{u}_t, \nabla q) = -(\nabla\cdot \mathrm{u}_t, q) = 0$, $\forall q \in H^1(\Omega)$
- That is to say:

$$\int_\Omega \nabla p\cdot\nabla q = \int_\Omega (f + \nu\nabla^2 \mathrm{u})\cdot\nabla q, \quad \forall q \in H^1(\Omega).$$

Consistent splitting in standard form

- Step 0: r-th order extrapolation of pressure,

$$p^{\star,k+1} = \begin{cases} p^k & \text{if } r = 1, \\ 2p^k - p^{k-1} & \text{if } r = 2, \\ 3p^k - 3p^{k-1} + p^{k-2} & \text{if } r = 3. \end{cases}$$

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Consistent splitting in rotational form

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Theorem (Guermond-Shen (2003))

For $q = 2,\ r = 1$

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- Numerical evidences that the rotational consistent scheme is fully second-order (and stable $\forall \Delta t$).

Implementation

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$$p^{k+1} = \psi^{k+1} + p^{\star,k+1} - \nu\nabla\cdot u^{k+1}.$$

Anything new under the sun?

- Up to change of variables consistent splitting is identical to the gauge method proposed by E and Liu(2003)

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- Answer: Both are acceptable and yield the same convergence results if properly done.

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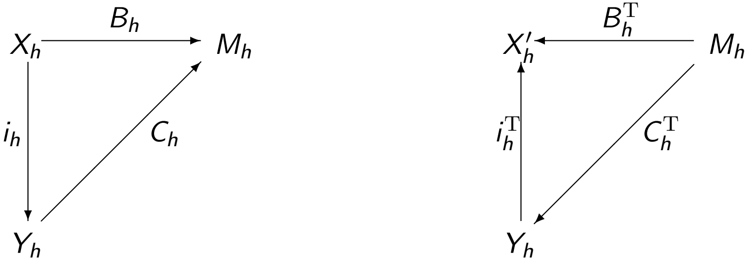

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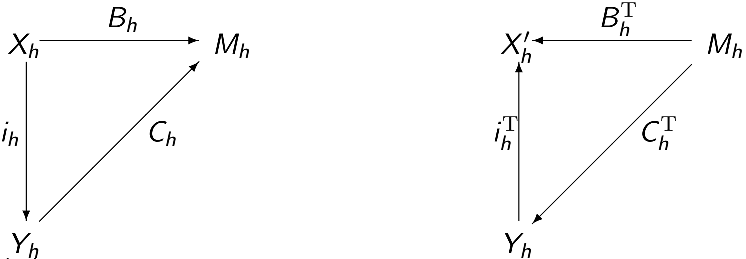

(iii) There exists $c > 0$ s.t., for all q_h in M_h,

$$\|C_h^{\mathrm{T}} q_h\|_{L^2(\Omega)} \leq c\|q_h\|_{H^1(\Omega)}.$$

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- Example 3: Y_h Raviart-Thomas or Brezzi-Douglas-Marini like space
  $\Rightarrow$ Projection is approximated like a Darcy problem, (Azaiez, Bernardi (1993), . . .)

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Theorem (Guermond-Shen-Xiaofeng Yang (2007))

Using X_h, Y_h, and M_h as above yields optimal convergence in space for all velocity-correction schemes in standard and rotational form (+ usual estimates in time).

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  (iv) Does all that work for open BCs? Not so well

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