GENEO: une méthode de décomposition de domaine à deux niveaux pour des systèmes d'équations très hétérogènes

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Outline

Motivation

Large discretized system of PDEs
strongly heterogeneous coefficients
(high contrast, nonlinear, multiscale)

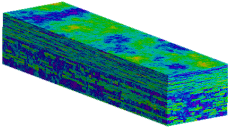
E.g. Darcy pressure equation,
P^1-finite elements:

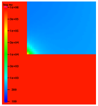
$$A\mathbf{U} = \mathbf{F}$$

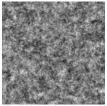
$$\mathrm{cond}(A) \sim \frac{\alpha_{\max}}{\alpha_{\min}} h^{-2}$$

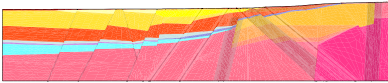
Goal:
iterative solvers
robust in size and heterogeneities

Applications:
flow in heterogeneous / stochastic / layered media
structural mechanics
electromagnetics
etc.









Why iterative solvers?

Limitations of direct solvers

In practice all direct solvers work well until a certain barrier:

- **two-dimensional problems** (10^6 unknowns)
- **three-dimensional problems** (10^5 unknowns).

Beyond, the factorization cannot be stored in memory any more.

To summarize:

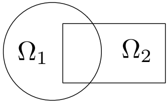
- below a certain size, direct solvers are chosen.
- beyond the critical size, iterative solvers are needed.

Direct	DDM	Iterative
Cons: Memory Difficult to || Pros: Robustness	Pro: Flexible Naurally ||	Pros: Memory Easy to || Cons: Robustness
solve(MAT,RHS,SOL)	Few black box routines Few implementations of efficient DDM	solve(MAT,RHS,SOL)

Multigrid methods: very efficient but may lack robustness, not always applicable (Helmholtz type problems, complex systems) and difficult to parallelize.

The First Domain Decomposition Method

The original Schwarz Method (H.A. Schwarz, 1870)



$$
\begin{aligned}
-\Delta(u) &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$
\begin{aligned}
-\Delta(u_1^{n+1}) &= f \quad \text{in } \Omega_1 \\
u_1^{n+1} &= 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \\
u_1^{n+1} &= u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}.
\end{aligned}
\qquad
\begin{aligned}
-\Delta(u_2^{n+1}) &= f \quad \text{in } \Omega_2 \\
u_2^{n+1} &= 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\
u_2^{n+1} &= u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}.
\end{aligned}
$$

Parallel algorithm, converges but very slowly, overlapping subdomains only.
The parallel version is called **Jacobi Schwarz method (JSM)**.

Continuous ASM and RAS - I

The algorithm acts on the local functions $(u_i)_{i=1,2}$.
To make things global, we need:

- extension operators, E_i, s.t. for a function $w_i : \Omega_i \mapsto \mathbb{R}$, $E_i(w_i) : \Omega \mapsto \mathbb{R}$ is the extension of w_i by zero outside Ω_i.
- partition of unity functions $\chi_i : \Omega_i \mapsto \mathbb{R}$, $\chi_i \geq 0$ and $\chi_i(x) = 0$ for $x \in \partial\Omega_i$ and s.t.

$$w = \sum_{i=1}^{2} E_i(\chi_i \, w_{|\Omega_i}) \, .$$

Let u^n be an approximation to the solution to the global Poisson problem and u^{n+1} is computed by solving first local subproblems and then gluing them together.

Local problems to solve

$$\begin{array}{rll} -\Delta(u_i^{n+1}) = f & \text{in} & \Omega_i \\ u_i^{n+1} = 0 & \text{on} & \partial\Omega_i \cap \partial\Omega \\ u_i^{n+1} = u^n & \text{on} & \partial\Omega_i \cap \overline{\Omega}_{3-i}. \end{array}$$

Two ways to ”glue” solutions

- Using the partition of unity functions **Restricted Additive Schwarz (RAS)**

$$u^{n+1} := \sum_{i=1}^{2} E_i(\chi_i \, u_i^{n+1})\,.$$

- Not based on the partition of unity **Additive Schwarz (ASM)**

$$u^{n+1} := \sum_{i=1}^{2} E_i(u_i^{n+1})\,.$$

Schwarz setting - algebraic level

Denote $\mathcal{N} = dof(\Omega)$ and $\mathcal{N}_j = dof(\Omega_j)$. We have the restriction operators

$$R_i : \mathbb{R}^{\#\mathcal{N}} \longmapsto \mathbb{R}^{\#\mathcal{N}_i}$$

and the transpose is a prolongation operator

$$R_i^T : \mathbb{R}^{\#\mathcal{N}_i} \longmapsto \mathbb{R}^{\#\mathcal{N}} .$$

The local Dirichlet matrices are given by

$$A_i := R_i A R_i^T .$$

The partition of unity defined by matrices D_i

$$D_i : \mathbb{R}^{\#\mathcal{N}_i} \longmapsto \mathbb{R}^{\#\mathcal{N}_i}$$

so that we have:

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id$$

Schwarz methods variants - RAS and ASM

Definition: RAS (Restricted Additive Schwarz)

$$M_{RAS}^{-1} := \sum_{i=1}^{N} R_i^T \, D_i \, (R_i \, A \, R_i^T)^{-1} \, R_i \tag{1}$$

so that the iterative RAS algorithm reads:

$$U^{n+1} = U^n + M_{RAS}^{-1} r^n \,, r^n := F - A \, U^n.$$

Definition: ASM (Additive Schwarz Method)

$$M_{ASM}^{-1} := \sum_{i=1}^{N} R_i^T \, (R_i \, A \, R_i^T)^{-1} \, R_i \tag{2}$$

so that the iterative ASM algorithm reads:

$$U^{n+1} = U^n + M_{ASM}^{-1} r^n \,.$$

Schwarz methods as preconditioners

ASM and RAS in iterative version are preconditioned fixed point iterations $\Rightarrow$ use Krylov methods instead.

- RAS (in conjunction with BiCGStab or GMRES) to solve

$$M_{RAS}^{-1} A U = M_{RAS}^{-1} F.$$

- ASM (in a CG methods)

$$M_{ASM}^{-1} A U = M_{ASM}^{-1} F.$$

Weak scalability

How to evaluate the efficiency of a domain decomposition?

Weak scalability – definition

"How the solution time varies with the number of processors for a fixed problem size per processor."

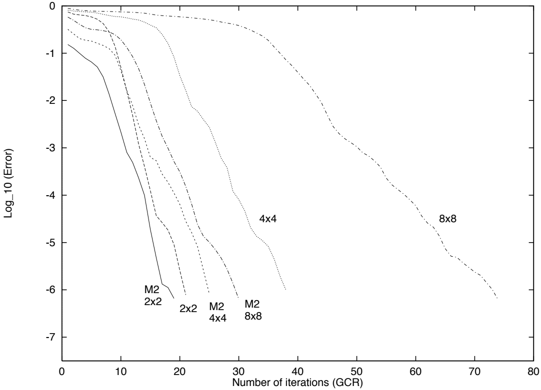
It is not achieved with the one level method

Number of subdomains	8	16	32	64
ASM	18	35	66	128

The iteration number increases linearly with the number of subdomains in one direction.

Convergence curves- more subdomains

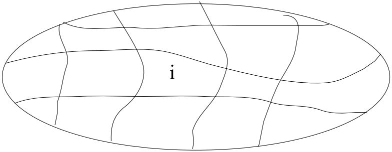
Plateaus appear in the convergence of the Krylov methods.



How to achieve scalability

Stagnation corresponds to a few very low eigenvalues in the spectrum of the preconditioned problem. They are due to the lack of a global exchange of information in the preconditioner.

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$



The mean value of the solution in domain i depends on the value of f on all subdomains.
A classical remedy consists in the introduction of a coarse problem that couples all subdomains. This is closely related to deflation technique classical in linear algebra (see Nabben and Vuik's papers in SIAM J. Sci. Comp, 200X).

Adding a coarse space

We add a coarse space correction (*aka* second level)
Let V_H be the coarse space and Z be a basis, $V_H = \mathrm{span}\, Z$, writing $R_0 = Z^T$ we define the two level preconditioner as:

$$M^{-1}_{ASM,2} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i .$$

The Nicolaides approach is to use the kernel of the operator as a coarse space, this is the constant vectors, in local form this writes:

$$Z := (R_i^T\, D_i R_i \mathbf{1})_{1 \le i \le N}$$

where D_i are chosen so that we have a partition of unity:

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id .$$

Theoretical convergence result

Theorem (Widlund, Dryija)

Let $M_{ASM,2}^{-1}$ be the two-level additive Schwarz method:

$$\kappa(M_{ASM,2}^{-1} A) \leq C \left(1 + \frac{H}{\delta}\right)$$

where δ is the size of the overlap between the subdomains and H the subdomain size.

This does indeed work very well

Number of subdomains	8	16	32	64
ASM	18	35	66	128
ASM + Nicolaides	20	27	28	27

Outline

Darcy equation with heterogeneities

$$
\begin{aligned}
-\nabla \cdot (\alpha(x,y)\nabla u) &= 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \\
u &= 0 \quad \text{on} \quad \partial\Omega_D, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega_N.
\end{aligned}
$$

Decomposition

$\alpha(x,y)$

Jump	1	10	10^2	10^3	10^4
ASM	39	45	60	72	73
ASM + Nicolaides	30	36	50	61	65

Objectives

Strategy

Define an appropriate coarse space $V_{H2} = \mathrm{span}(Z_2)$ and use the framework previously introduced, writing $R_0 = Z_2^T$ the two level preconditioner is:

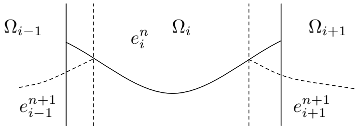
$$P_{ASM2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i.$$

The coarse space must be

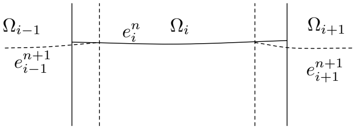
- Local (calculated on each subdomain) $\to$ parallel
- Adaptive (calculated automatically)
- Easy and cheap to compute
- Robust (must lead to an algorithm whose convergence does not depend on the partition nor the jumps in coefficients)

Heuristic approach: what functions should be in Z_2?

The error satisfies the Schwarz algorithm, it is harmonic, so it satisfies a maximum principle.



Fast convergence



Slow convergence

Idea

Ensure that the error decreases quickly on the subdomain boundaries which translates to making $\left.\frac{\partial e}{\partial n_i}\right|_{\Gamma_i}$ big.

Using the DtN operator

The Dirichlet to Neumann operator is defined as follows: Let $g : \Gamma_i \mapsto \mathbb{R}$,

$$\mathrm{DtN}_{\Omega_i}(g) = \alpha \left.\frac{\partial v}{\partial n_i}\right|_{\Gamma_i},$$

where v satisfies

$$\begin{cases} (-\mathrm{div}(\alpha\nabla))v = 0, & \text{in } \ \Omega_i, \\ v = g, & \text{on } \partial\Omega_i. \end{cases}$$

To construct the coarse space, we use the low frequency modes associated with the DtN operator:

$$\mathrm{DtN}_{\Omega_i}(v_i^\lambda) = \lambda\,\alpha\,v_i^\lambda$$

with λ small. The functions v_i^λ are extended harmonically to the subdomains.

Theoretical convergence result

Suppose we have $(v_i^{\lambda_k}, \lambda_i^k)_{1\le k\le n_{\Gamma_i}}$ the eigenpairs of the local DtN maps ($\lambda_i^1 \le \lambda_i^2 \le \ldots$) and that we have selected m_i in each subdomain. Then let Z be the coarse space built via the local DtN maps:

$$Z := (R_i^T\, D_i\, \tilde{V}_i^{\lambda_i^k})_{1\le i\le N;\, 1\le k\le m_i}$$

Theorem (Dolean, N., Scheichl and Spillane 2010)

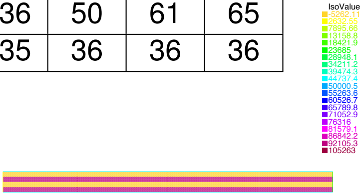
Under the monotonicity of α in the overlapping regions:

$$\kappa(M_{ASM,2}^{-1} A) \le C\Big(1 + \max_{1\le i\le N} \frac{1}{\delta_i\, \lambda_i^{m_i+1}}\Big)$$

where δ_i is the size of the overlap of domain Ω_i and C is independent of the jumps of α.

If m_i is chosen so that, $\lambda_i^{m_i+1} \ge 1/H_i$ the convergence rate will be analogous to the constant coefficient case.

Jump	1	10	10^2	10^3	10^4
ASM	39	45	60	72	73
ASM + Nicolaides	30	36	50	61	65
ASM + DtN	31	35	36	36	36



Decomposition

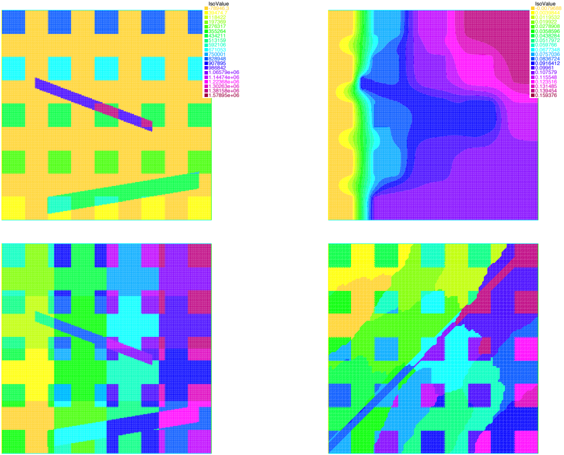
$\alpha(x, y)$

With DtN the jumps do not affect convergence

We put at most two modes per subdomain in the coarse space (using the automatic selection process)

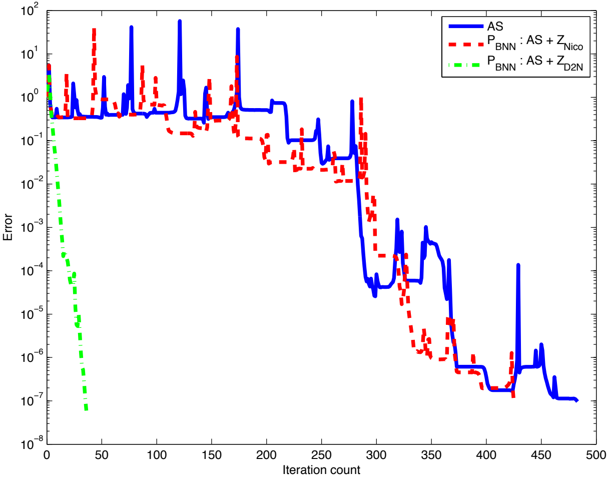
Numerical results

Using FreeFEM++ `http://www.freefem.org/ff++`



Channels and inclusions: $1 \leq \alpha \leq 1.5 \times 10^6$, the solution and partitionings (Metis or not)

Numerical results



ASM convergence for channels and inclusions – 4×4 Metis partitioning

Numerical results – Optimality

$\#Z$ per subd.	ASM	ASM+Z_{Nico}	ASM+Z_{D2N}
$\max(m_i - 1, 1)$			273
m_i	614	543	36
$m_i + 1$			32

m_i is given automatically by the strategy.

- Taking one fewer eigenvalue has a huge influence on the iteration count
- Taking one more has only a small influence

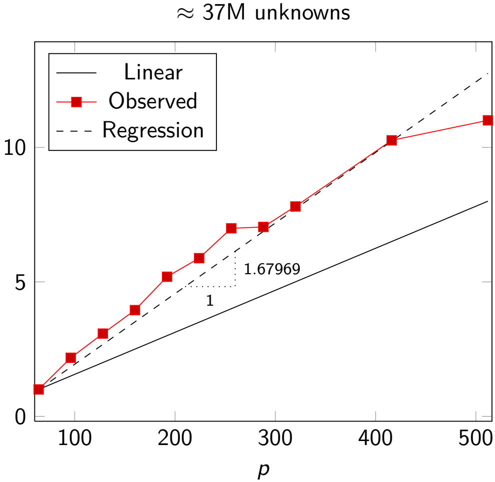
PhD of Pierre Jolivet.
Since version 1.16, bundled with the `Message Parsing Interface`. `FreeFem++` is working on the following parallel architectures (among others):

	N° of cores	Memory	Peak perf	Compilers
hpc1@LJLL	64@2.00 Ghz	252 Go	$<$ 1 TFLOP/s	`Intel`
titane@CEA	12192[1]@2.93 Ghz	37 To	140 TFLOP/s	`Intel`
babel@IDRIS	40960@850 Mhz	20 To	139 TFLOP/s	`IBM+GNU`

* + 46080 CUDA cores

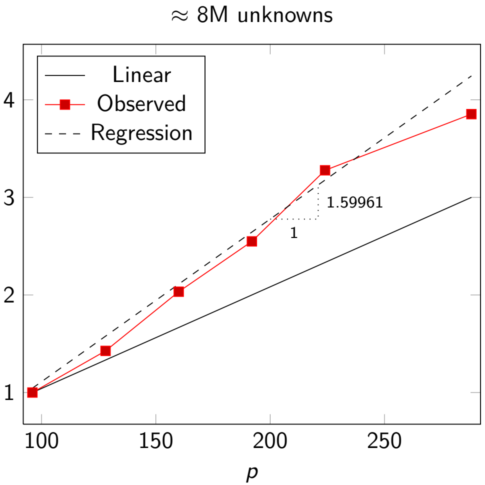
`http://www-ccrt.cea.fr`, Bruyères-le-Châtel, France.
`http://www.idris.fr`, Orsay, France.

p	T	$\sum_{i=1}^{N} \nu_i$
64	65.7 s	1,890
96	30.2 s	2,850
128	21.4 s	3,810
160	16.6 s	4,770
192	12.7 s	5,730
224	11.2 s	6,690
256	9.4 s	7,650
288	9.3 s	8,610
320	8.4 s	9,570
416	6.4 s	12,450
512	6.0 s	15,330



Speed-up for a 2D problem

p	T	$\sum_{i=1}^{N} \nu_i$
96	26.5 s	1,920
128	18.6 s	2,560
160	13.0 s	3,200
192	10.4 s	3,840
224	8.1 s	4,480
288	6.9 s	5,760



Speed-up for a 3D problem

Some "even bigger" problem

On `babel`, allowable memory space per core: 512MB !

4096-way decomposition:

- in $\mathbb{R}^2$, 168M unkowns,
- in $\mathbb{R}^3$, 86M unkowns.

All systems are solved with:

- coarse spaces of size $[100; 20000]$,
- less than 25 iterations.

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Outline

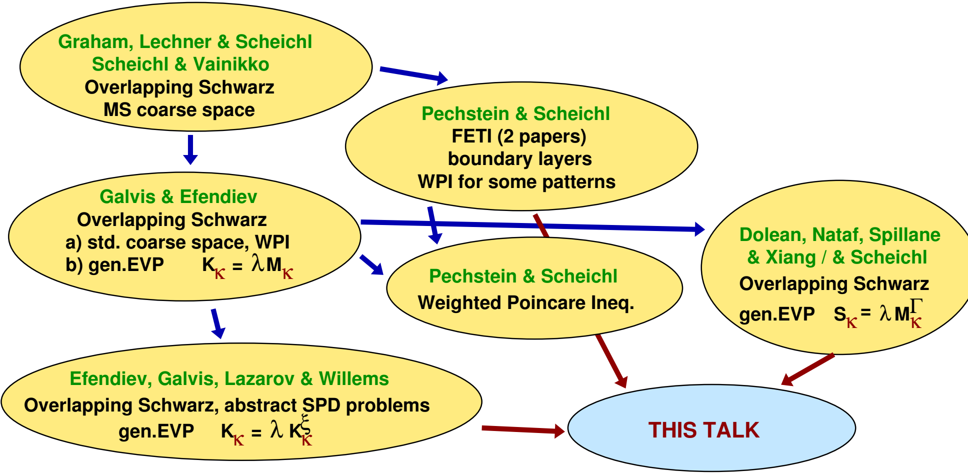
(Pros and) cons of the DtN algorithm

Main disadvantages: extension to other equations/systems such as elasticity

- Not a very natural definition of *DtN* type operators
- Theoretical convergence proof based on Poincaré type inequalities which are not clear for other systems.

$\Rightarrow$ need to re-think the strategy of building the coarse space.

Relation of GENEO to other methods



Problem setting – I

Given $f \in (V^h)^*$ find $u \in V^h$

$$
\begin{aligned}
a(u, v) &= \langle f, v \rangle \qquad \forall v \in V^h \\
\iff \qquad \mathbf{A}\,\mathbf{u} &= \mathbf{f}
\end{aligned}
$$

Assumption throughout: $\mathbf{A}$ *symmetric positive definite (SPD)*

Examples:

- Darcy $\quad a(u, v) = \int_\Omega \boldsymbol{\kappa} \nabla u \cdot \nabla v \, dx$
- Elasticity $\quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{C}\, \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, dx$
- Eddy current $\quad a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\nu}\, \mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} + \boldsymbol{\sigma}\, \boldsymbol{u} \cdot \boldsymbol{v}\, dx$

Heterogeneities / high contrast in parameters

Problem setting – II

1. $V^h \ldots$ FE space of functions in Ω based on mesh $\mathcal{T}^h = \{\tau\}$
2. $\{\phi_k\}_{k=1}^n$ (FE) basis of V^h
3. Technical assumptions fulfilled by standard FE and bilinear forms $a(\cdot, \cdot)$

Overlapping decomposition: $\Omega = \bigcup_{j=1}^{N} \Omega_j$ (Ω_j union of elements)

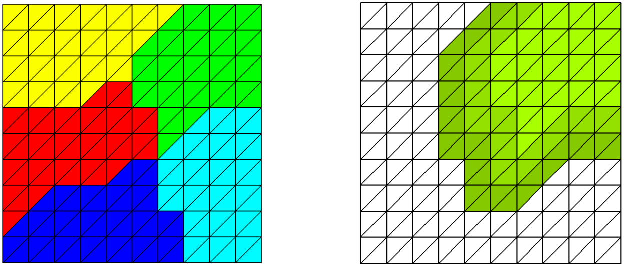
$$V_j := \mathrm{span}\{\phi_k : \mathrm{supp}(\phi_k) \subset \overline{\Omega}_j\}$$

such that every ϕ_k is contained in one of those spaces, i.e.

$$V^h = \sum_{j=1}^{N} V_j$$

Example: adding “layers” to non-overlapping partition

(partition and adding layers based on matrix information only!)



Local subspaces:

$$V_j \subset V^h \qquad j = 1, \ldots, N$$

Coarse space (defined later):

$$V_0 \ \subset V^h$$

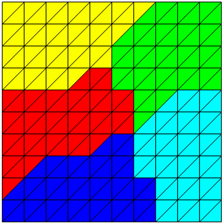
Additive Schwarz preconditioner:

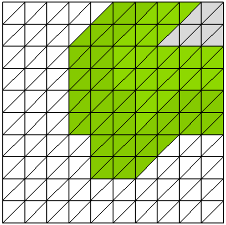
$$\mathbf{M}_{AS,2}^{-1} \ = \ \sum_{j=0}^{N} \mathbf{R}_j^\top \mathbf{A}_j^{-1} \mathbf{R}_j$$

where $\mathbf{A}_j = \mathbf{R}_j^\top \mathbf{A} \mathbf{R}_j$
and $\mathbf{R}_j^\top \leftrightarrow R_j^\top : V_j \to V^h$ natural embedding

Overlapping zone / Choice of coarse space

Overlapping zone: $\Omega_j^\circ = \{x \in \Omega_j : \exists i \neq j : x \in \Omega_i\}$





Observation: $\Xi_{j|\Omega_j \setminus \Omega_j^\circ} = \mathrm{id}$

Coarse space should be a sum of **local** contributions:

$$V_0 = \sum_{j=1}^{N} V_{0,j} \qquad \text{where } V_{0,j} \subset V_j$$

E.g. $V_{0,j} = \mathrm{span}\{\Xi_j p_{j,k}\}_{k=1}^{m_j}$

ASM theory needs **stable splitting**:

$$v = v_0 + \sum_{j=1}^{N} v_j$$

Suppose $v_0 = \sum_{j=1}^{N} \Xi_j \Pi_j v_{|\Omega_j}$ where $\Pi_j \ldots$ local projector

$$|\underbrace{\Xi_j(v - \Pi_j v)}_{v_j}|^2_{a,\Omega_j} = |\Xi_j(v - \Pi_j v)|^2_{a,\Omega_j^\circ} + |\Xi_j(v - \Pi_j v))|^2_{a,\Omega_j \setminus \Omega_j^\circ}$$

$$\overset{\text{HOW?}}{\leq} C\,|v|^2_{a,\Omega_j}$$

(a,D denotes the restriction of a to D)

"Minimal" requirements:

- Π_j be a-orthogonal
- Stability estimate: $|\Xi_j(v - \Pi_j v)|^2_{a,\Omega_j^\circ} \leq c\,|v|^2_{a,\Omega_j}$

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Abstract eigenvalue problem

"Minimal" requirements:

1. Π_j be a-orthogonal
2. Stability estimate: $|\Xi_j(v - \Pi_j v)|^2_{a,\Omega_j^\circ} \;\leq\; c\,|v|^2_{a,\Omega_j}$

Fulfillment of 2

If there exist a non zero function w such that $|w|_{a,\Omega_j} = 0$, it is necessary to project on $Span(w)$.
The kernel of a Darcy equation is the constant function and that of elasticity is spanned by rigid body motions.
The corresponding coarse space will be refered to as ZEM (zero energy modes).

For highly heterogeneous problems, we take a larger coarse space deduced from the stability estimate.

Abstract eigenvalue problem

Gen.EVP per subdomain:

Find $p_{j,k} \in V_{h|\Omega_j}$ and $\lambda_{j,k} \geq 0$:

$$a_{\Omega_j}(p_{j,k},\, v) \;=\; \lambda_{j,k}\, a_{\Omega_j^\circ}(\Xi_j p_{j,k},\, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

$$A_j \mathbf{p}_{j,k} \;=\; \lambda_{j,k}\, \mathbf{X}_j A_j^\circ \mathbf{X}_j\, \mathbf{p}_{j,k} \qquad (\mathbf{X}_j \ldots \text{diagonal})$$

$a_D \ldots$ restriction of a to D

In the two-level ASM:

Choose first m_j eigenvectors per subdomain:

$$V_0 \;=\; \text{span}\{\Xi_j p_{j,k}\}_{k=1,\ldots,m_j}^{j=1,\ldots,N}$$

This automatically includes Zero Energy Modes.

Abstract eigenvalue problem

Gen.EVP per subdomain:

Find $p_{j,k} \in V_{h|\Omega_j}$ and $\lambda_{j,k} \geq 0$:

$$a_{\Omega_j}(p_{j,k},\ v) \ = \ \lambda_{j,k}\, a_{\Omega_j^\circ}(\Xi_j p_{j,k},\ \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

$$A_j \mathbf{p}_{j,k} \ = \ \lambda_{j,k}\, \mathbf{X}_j A_j^\circ \mathbf{X}_j\, \mathbf{p}_{j,k} \qquad (\mathbf{X}_j \ldots \text{diagonal})$$

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In the two-level ASM:

Choose first m_j eigenvectors per subdomain:

$$V_0 \ = \ \mathrm{span}\{\Xi_j p_{j,k}\}_{k=1,\ldots,m_j}^{j=1,\ldots,N}$$

This automatically includes Zero Energy Modes.

Comparison with existing works

Galvis & Efendiev (SIAM 2010):

$$\int_{\Omega_j} \boldsymbol{\kappa}\, \nabla p_{j,k} \cdot \nabla v \, dx \;=\; \lambda_{j,k} \int_{\Omega_j} \boldsymbol{\kappa}\, p_{j,k}\, v \, dx \qquad \forall v \in V_{h|\Omega_j}$$

Efendiev, Galvis, Lazarov & Willems (submitted):

$$a_{\Omega_j}(p_{j,k},\, v) \;=\; \lambda_{j,k} \sum_{i \in \text{neighb}(j)} a_{\Omega_j}(\xi_j\, \xi_i\, p_{j,k},\, \xi_j\, \xi_i\, v) \qquad \forall v \in V_{|\Omega_j}$$

$\xi_j \ldots$ partition of unity, calculated adaptively (MS)

Our gen.EVP:

$$a_{\Omega_j}(p_{j,k},\, v) \;=\; \lambda_{j,k}\, a_{\Omega_j^\circ}(\Xi_j p_{j,k},\, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

both matrices typically singular $\implies \lambda_{j,k} \in [0,\, \infty]$

Two technical assumptions.

Theorem (Spillane, Dolean, Hauret, N., Pechstein, Scheichl)

If for all j: $\quad 0 < \lambda_{j,m_{j+1}} < \infty$*:*

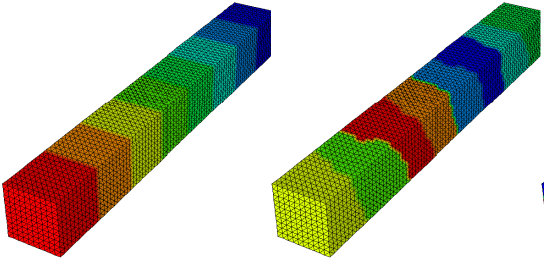
$$\kappa(M_{ASM,2}^{-1}A) \;\leq\; (1+k_0)\Big[2 + k_0\,(2k_0+1)\,\max_{j=1}^{N}\Big(1+\frac{1}{\lambda_{j,m_j+1}}\Big)\Big]$$

Possible criterion for picking m_j: (used in our Numerics)

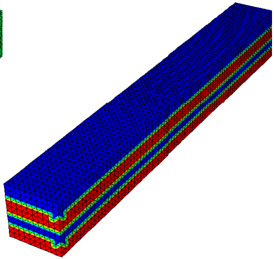
$$\lambda_{j,m_j+1} \;<\; \frac{\delta_j}{H_j}$$

$H_j \ldots$ subdomain diameter, $\delta_j \ldots$ overlap

Domain & Partitions



Coefficient



Iterations (CG) vs. jumps

Code: Matlab & FreeFem++

κ_2	AS-1	AS-ZEM	$dim(V_H)$	GENEO	$dim(V_H)$
1	22	16	*(8)*	16	*(8)*
10^2	31	24	*(8)*	17	*(15)*
10^4	37	30	*(8)*	21	*(15)*
10^6	36	29	*(8)*	18	*(15)*

AS-1: 1-level ASM

AS-ZEM: $m_j = 1$

GENEO: $\lambda_{j,m_j+1} < \delta_j / H_j$

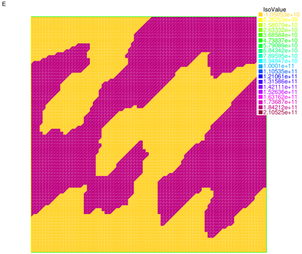
Iterations (CG) vs. number of subdomains

regular partition

subd.	dofs	AS-1	AS-ZEM	$dim(V_H)$	GENEO	$dim(V_H)$
4	4840	14	15	*(4)*	10	*(6)*
8	9680	26	22	*(8)*	11	*(14)*
16	19360	51	36	*(16)*	13	*(30)*
32	38720	$>$100	61	*(32)*	13	*(62)*

METIS partition

subd.	dofs	AS-1	AS-ZEM	$dim(V_H)$	GENEO	$dim(V_H)$
4	4840	21	18	*(4)*	15	*(7)*
8	9680	36	29	*(8)*	18	*(15)*
16	19360	65	45	*(16)*	22	*(31)*
32	38720	$>$100	79	*(32)*	34	*(63)*



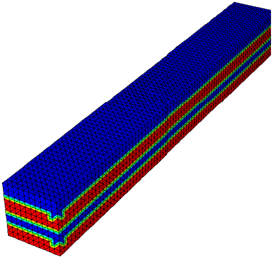
$$E_1 = 2 \cdot 10^{11}$$
$$\nu_1 = 0.3$$

$$E_2 = 2 \cdot 10^{7}$$
$$\nu_2 = 0.45$$

METIS partitions with 2 layers added

subd.	dofs	AS-1	AS-ZEM	(V_H)	GENEO	(V_H)
4	13122	93	134	*(12)*	42	*(42)*
16	13122	164	165	*(48)*	45	*(159)*
25	13122	211	229	*(75)*	47	*(238)*
64	13122	279	167	*(192)*	45	*(519)*

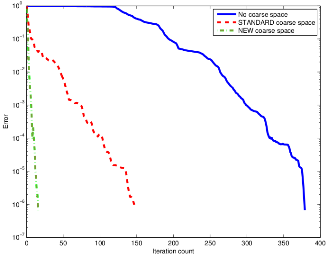
Iterations (CG) vs. number of subdomains



$$E_1 = 2 \cdot 10^{11}$$
$$\nu_1 = 0.3$$

$$E_2 = 2 \cdot 10^{7}$$
$$\nu_2 = 0.45$$

Relative error vs. iterations
16 regular subdomains



subd.	dofs	AS-1	AS-ZEM	(V_H)	GENEO	(V_H)
4	1452	79	54	*(24)*	16	*(46)*
8	29040	177	87	*(48)*	16	*(102)*
16	58080	378	145	*(96)*	16	*(214)*

AS-ZEM (Rigid body motions): $m_j = 6$

Remarks:

- Implementation requires only element stiffness matrices + connectivity
- Proof works for any partition of unity
  (changes the eigenproblem and coarse space)

Outlook:

- More testing & comparison to other methods
- Solution of the Eigenproblems (LAPACK $\mapsto$ LOBPCG)
- Coarse space dimension reduction?
- Coarse problem satisfies assembling property
  $\mapsto$ multilevel method — link to σAMGe ?
- More applications within a FreeFem++ MPI implementation
- Other discretizations: finite volume, finite difference, . . .

Conclusion & Outlook

Remarks:

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Outline

Preprints available on HAL:
http://hal.archives-ouvertes.fr/

- N. Spillane, V. Dolean, P. Hauret,**F. Nataf**, C. Pechstein, R. Scheichl, "An Algebraic Local Generalized Eigenvalue in the Overlapping Zone Based Coarse Space : A first introduction", *C. R. Mathématique*, Vol. 349, No. 23-24, pp. 1255-1259, 2011.
- **F. Nataf**, H. Xiang, V. Dolean, N. Spillane, "A coarse space construction based on local Dirichlet to Neumann maps", *SIAM J. Sci Comput.*, Vol. 33, No.4, pp. 1623-1642, 2011.
- **F. Nataf**, H. Xiang, V. Dolean, "A two level domain decomposition preconditioner based on local Dirichlet-to-Neumann maps", *C. R. Mathématique*, Vol. 348, No. 21-22, pp. 1163-1167, 2010.
- V. Dolean,**F. Nataf**, R. Scheichl, N. Spillane, "Analysis of a two-level Schwarz method with coarse spaces based on local Dirichlet–to–Neumann maps", submitted to *SIAM J. Num. An.*, http://hal.archives-ouvertes.fr/hal-00586246/fr/, 2011.
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THANK YOU FOR YOUR ATTENTION!

Bibliography Schwarz: DtN and GenEO

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