Discontinuous Galerkin method on hybrid meshes for time domain electromagnetics

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Outline

Ω, bounded polyhedral domain of $\mathbb{R}^3$, boundary $\Gamma = \Gamma^a \cup \Gamma^m$;
the system of Maxwell's equation in three space dimensions is given by :

$$\boxed{\begin{cases} \epsilon \dfrac{\partial \mathbf{E}}{\partial t} - \mathrm{curl}(\mathbf{H}) & = & 0, \\ \mu \dfrac{\partial \mathbf{H}}{\partial t} + \mathrm{curl}(\mathbf{E}) & = & 0, \end{cases}}$$

where :

- $\mathbf{E} \equiv {}^t(E_1(\mathbf{x},t), E_2(\mathbf{x},t), E_3(\mathbf{x},t))$ & $\mathbf{H} \equiv {}^t(H_1(\mathbf{x},t), H_2(\mathbf{x},t), H_3(\mathbf{x},t))$ are the electric field and the magnetic field
- $\epsilon \equiv \epsilon(\mathbf{x})$, $\mu \equiv \mu(\mathbf{x})$, are the electric permittivity and the magnetic permeability, respectively
- Metallic boundary condition on Γ^m : $\mathbf{n} \times \mathbf{E} = 0$ ($\mathbf{n}$ outwards normal to Γ)
- Silver-Müller boundary condition on Γ^a : $\mathbf{n} \times \mathbf{E} - \sqrt{\dfrac{\mu}{\epsilon}}\mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = 0$
- Pseudo-conservative form : $Q(\partial_t \mathbf{W}) + \nabla \cdot F(\mathbf{W}) = 0 \quad (\mathbf{W} = {}^t(\mathbf{E},\mathbf{H}) \in \mathbb{R}^6)$

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Outline

Objective : Formulate, study and validate a **DGTD**$-\mathbb{P}_p/\mathbb{Q}_k$ method to solve Maxwell's equations :

- mesh objects with complex geometry by tetrahedra (triangles in 2D) for high precision
- mesh the surrounding space by square elements (large size) for simplicity and speed

- Ω is discretized by $\mathscr{C}_h = \bigcup_{i=1}^{N} c_i = \mathscr{T}_h \bigcup \mathscr{Q}_h$, where c_i are tetrahedra ($\in \mathscr{T}_h$) or hexahedra ($\in \mathscr{Q}_h$) in 3D (triangles or quadrangles in 2D)

- We multiply the system by ψ, a **test function** (scalar) and we integrate on c_i (integration by parts)

- $\mathbb{P}_p[c_i]$ the space of polynomial functions with degree at most p in $c_i \in \mathscr{T}_h$, $\mathbb{Q}_k[c_i]$ the space of polynomial functions with degree at most k with respect to each variable separately on $c_i \in \mathscr{Q}_h$ (ex : form of polynomials $\mathbb{Q}_1$ in 2D : $\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1 x_2$)

- $\phi_i = (\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{id_i})$ local basis of $\mathbb{P}_p[c_i]$
  $\theta_i = (\vartheta_{i1}, \vartheta_{i2}, \ldots, \vartheta_{ib_i})$ local basis of $\mathbb{Q}_k[c_i]$

- The discrete solution vector $\mathbf{W}_h$ is searched for in the approximation space V_h^6 defined by :

$$V_h = \left\{ v_h \in L^2(\Omega) \,\middle|\, \begin{array}{l} \forall c_i \in \mathscr{T}_h,\ v_{h|c_i} \in \mathbb{P}_p[c_i] \\ \forall c_i \in \mathscr{Q}_h,\ v_{h|c_i} \in \mathbb{Q}_k[c_i] \end{array} \right\}$$

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- Local degrees of freedom denoted by $\mathbf{W}_{il} \in \mathbb{R}^6$
- $\mathbf{W}_i$ defines the restriction of the approximate solution to the cell c_i ($\mathbf{W}_h|_{c_i}$)
- $$c_i \in \mathscr{T}_h \Longrightarrow \mathbf{W}_i \in \mathbb{P}_p[c_i] : \mathbf{W}_i(\mathbf{x}) = \sum_{l=1}^{d_i} \mathbf{W}_{il}\varphi_{il}(\mathbf{x}) \in \mathbb{R}^6$$

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- The local representation of $\mathbf{W}$ does not provide any form of continuity from one element to another. We use a centered numerical flux on $a_{ij} = c_i \cap c_j$

  $$\mathbf{W}_h|_{a_{ij}} = \frac{\mathbf{W}_i|_{a_{ij}} + \mathbf{W}_j|_{a_{ij}}}{2}$$

  If a_{ij} on the metallic boundary : ${}^t(\mathbf{E}_j, \mathbf{H}_j) = {}^t(-\mathbf{E}_i, \mathbf{H}_i)$
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- Two cases for weak formulation

Case (A) :

c_i is a tetrahedron. a_{ij} face of c_i, is **either** on boundary, **or** common to another tetrahedron, **or** to a hexahedron (**hybrid**)

6d$_\mathbf{i}$ semi-discretized equations system :

$$\boxed{\left\{\begin{array}{l} 2\mathcal{X}_{\epsilon,i}\dfrac{d\overline{\mathbf{E}}_i}{dt} + \displaystyle\sum_{k=1}^{3}\mathcal{X}_i^{x_k}\overline{\mathbf{H}}_i + \sum_{a_{ij}\in\mathcal{T}_d^i}\mathcal{X}_{ij}\overline{\mathbf{H}}_j + \sum_{a_{ij}\in\mathcal{T}_m^i}\mathcal{X}_{im}\overline{\mathbf{H}}_i + \sum_{a_{ij}\in\mathcal{H}_d^i}\mathcal{A}_{ij}\widetilde{\mathbf{H}}_j = 0, \\ 2\mathcal{X}_{\mu,i}\dfrac{d\overline{\mathbf{H}}_i}{dt} - \displaystyle\sum_{k=1}^{3}\mathcal{X}_i^{x_k}\overline{\mathbf{E}}_i - \sum_{a_{ij}\in\mathcal{T}_d^i}\mathcal{X}_{ij}\overline{\mathbf{E}}_j + \sum_{a_{ij}\in\mathcal{T}_m^i}\mathcal{X}_{im}\overline{\mathbf{E}}_i - \sum_{a_{ij}\in\mathcal{H}_d^i}\mathcal{A}_{ij}\widetilde{\mathbf{E}}_j = 0, \end{array}\right.}$$

with :

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- $\mathcal{X}_{\epsilon,i}$ and $\mathcal{X}_{\mu,i}$ are mass matrices, $\mathcal{X}_i^{x_k}$ gradient matrix, $\mathcal{X}_{ij}$ surface matrix
  $\Longrightarrow$ All have a $3d_i \times 3d_i$ size, **except** $\mathcal{A}_{ij}$, whose size is $3d_i \times 3b_j$

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c_i is an hexahedron. a_{ij} face of c_i, is **either** on boundary, **or** common to another hexahedron, **or** to a tetrahedron (**hybrid**)

$6b_i$ semi-discretized equations system :

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Second order Leap-Frog scheme :

- **Case (A)** : $\left\{\begin{array}{lcl} \overline{\mathbf{H}}_i^{n+\frac{1}{2}} & = & \overline{\mathbf{H}}_i^{n-\frac{1}{2}} + \dfrac{\Delta t}{2} \left[\mathcal{X}_{\mu,i}\right]^{-1} \mathbf{A}_{\mathbf{E},i}^{n}, \\ \overline{\mathbf{E}}_i^{n+1} & = & \overline{\mathbf{E}}_i^{n} + \dfrac{\Delta t}{2} \left[\mathcal{X}_{\epsilon,i}\right]^{-1} \mathbf{A}_{\mathbf{H},i}^{n+\frac{1}{2}} \end{array}\right.$

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Outline

- We define a discrete energy $\mathfrak{E}^n$. We consider only metallic boundary. We assume that this is an energy and we check that it is exactly conserved, i.e. $\Delta\mathfrak{E} = \mathfrak{E}^{n+1} - \mathfrak{E}^n = 0$
- We prove that $\mathfrak{E}^n$ is a positive definite quadratic form under a CFL condition
- For this, we make the hypothesis :

$$\forall \mathbf{X} \in \left(\mathbb{P}_p[c_i]\right)^3, \quad \|\mathrm{rot}(\mathbf{X})\|_{c_i} \leq \left(\alpha_i^T p_i \|\mathbf{X}\|_{c_i}\right)/|c_i|,$$

$$\forall \mathbf{X} \in \left(\mathbb{P}_p[c_i]\right)^3, \quad \|\mathbf{X}\|^2_{a_{ij}} \leq \left(\beta_{ij}^T \|\mathbf{n}_{ij}\| \|\mathbf{X}\|^2_{c_i}\right)/|c_i|$$

- α_i^T and β_{ij}^T ($j \in \{j | c_i \cap c_j \neq \emptyset\}$) defining the constant parameters
- We also admit similar hypothesis $\forall \mathbf{X} \in \left(\mathbb{Q}_k[c_i]\right)^3$ with constants α_i^q and β_{ij}^q
- $\|.\|_{c_i}$ and $\|.\|_{a_{ij}}$ are L^2-norm. $\|\mathbf{n}_{ij}\| = \int_{a_{ij}} 1 d\sigma$ with $\mathbf{n}_{ij}$ non-unitary normal to a_{ij} oriented from c_i towards c_j. $|c_i| = \int_{c_i} 1 d\mathbf{x}$ and $p_i = \sum_{j \in \mathcal{V}_i} \|\mathbf{n}_{ij}\|$

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- For the **DGTD**$-\mathbb{P}_p$ method, the sufficient condition on Δt_τ is :

$$\forall i, \forall j \in \mathcal{V}_i : \quad \Delta t_\tau \left[2\alpha_i^\tau + \beta_{ij}^\tau \max\left(\sqrt{\frac{\epsilon_i}{\epsilon_j}}, \sqrt{\frac{\mu_i}{\mu_j}} \right) \right] < \frac{4|c_i|\sqrt{\epsilon_i \mu_i}}{p_i}$$

- For **DGTD**$-\mathbb{Q}_k$ method, the sufficient condition on Δt_q is :

$$\forall i, \forall j \in \mathcal{V}_i : \quad \Delta t_q \left[2\alpha_i^q + \beta_{ij}^q \max\left(\sqrt{\frac{\epsilon_i}{\epsilon_j}}, \sqrt{\frac{\mu_i}{\mu_j}} \right) \right] < \frac{4|c_i|\sqrt{\epsilon_i \mu_i}}{p_i}$$

Finally, noting Δt the global time step for the hybrid method, we have shown that the sufficient stability condition is defined by :

$$\boxed{\Delta t = \min(\Delta t_\tau, \Delta t_q)}$$

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- $$\begin{cases} m(\mathbf{T},\mathbf{T}') = 2\displaystyle\int_\Omega \langle Q\mathbf{T}\ ,\ \mathbf{T}'\rangle d\mathbf{x} \\ a(\mathbf{T},\mathbf{T}') = \displaystyle\int_\Omega \left(\left\langle \sum_{k=1}^{3} \partial^h_{x_k} \mathcal{O}^k \mathbf{T}\ ,\ \mathbf{T}' \right\rangle - \sum_{k=1}^{3} \langle \partial^h_{x_k} \mathbf{T}'\ ,\ \mathcal{O}^k \mathbf{T} \rangle \right) d\mathbf{x} \\ b(\mathbf{T},\mathbf{T}') = \displaystyle\int_{\mathscr{F}_d} \Big(\langle \{\mathbf{V}\}\ ,\ [\![\mathbf{U}']\!] \rangle - \langle \{\mathbf{U}\}\ ,\ [\![\mathbf{V}']\!] \rangle - \\ \qquad \langle \{\mathbf{V}'\}\ ,\ [\![\mathbf{U}]\!] \rangle + \langle \{\mathbf{U}'\}\ ,\ [\![\mathbf{V}]\!] \rangle \Big)\ d\sigma + \\ \qquad \displaystyle\int_{\mathscr{F}_m} \Big(\langle \mathbf{U}\ ,\ \check{\mathbf{n}} \times \mathbf{V}' \rangle + \langle \mathbf{V}\ ,\ \check{\mathbf{n}} \times \mathbf{U}' \rangle \Big)\ d\sigma \end{cases}$$
- Summing up weak formulations on each c_i, the discrete solution $\mathbf{W}_h$ satisfies :

$$m(\partial_t \mathbf{W}_h, \mathbf{T}') + a(\mathbf{W}_h, \mathbf{T}') + b(\mathbf{W}_h, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

- We assume that the exact solution $\mathbf{W}(t) \in (H(\mathrm{curl}, \Omega))^6$, $\forall t \in [0, t_f]$, then we prove :

$$m(\partial_t \mathbf{W}, \mathbf{T}') + a(\mathbf{W}, \mathbf{T}') + b(\mathbf{W}, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

- $$\begin{cases} m(\mathbf{T},\mathbf{T}') &= 2\displaystyle\int_{\Omega} \langle Q\mathbf{T} \ , \ \mathbf{T}' \rangle d\mathbf{x} \\ a(\mathbf{T},\mathbf{T}') &= \displaystyle\int_{\Omega} \left(\left\langle \sum_{k=1}^{3} \partial_{x_k}^h \mathcal{O}^k \mathbf{T} \ , \ \mathbf{T}' \right\rangle - \sum_{k=1}^{3} \langle \partial_{x_k}^h \mathbf{T}' \ , \ \mathcal{O}^k \mathbf{T} \rangle \right) d\mathbf{x} \\ b(\mathbf{T},\mathbf{T}') &= \displaystyle\int_{\mathscr{F}_d} \Big(\langle \{\mathbf{V}\} \ , \ [\![\mathbf{U}']\!] \rangle - \langle \{\mathbf{U}\} \ , \ [\![\mathbf{V}']\!] \rangle - \\ & \qquad \langle \{\mathbf{V}'\} \ , \ [\![\mathbf{U}]\!] \rangle + \langle \{\mathbf{U}'\} \ , \ [\![\mathbf{V}]\!] \rangle \Big) \ d\sigma \ + \\ & \quad \displaystyle\int_{\mathscr{F}_m} \Big(\langle \mathbf{U} \ , \ \check{\mathbf{n}} \times \mathbf{V}' \rangle + \langle \mathbf{V} \ , \ \check{\mathbf{n}} \times \mathbf{U}' \rangle \Big) \ d\sigma \end{cases}$$
- Summing up weak formulations on each c_i, the discrete solution $\mathbf{W}_h$ satisfies :
$$m(\partial_t \mathbf{W}_h, \mathbf{T}') + a(\mathbf{W}_h, \mathbf{T}') + b(\mathbf{W}_h, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$
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$$m(\partial_t \mathbf{W}, \mathbf{T}') + a(\mathbf{W}, \mathbf{T}') + b(\mathbf{W}, \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

- We also prove :

$$a(\mathbf{T}', \mathbf{T}') + b(\mathbf{T}', \mathbf{T}') = 0, \quad \forall \mathbf{T}' \in V_h^6$$

Let $h_\tau = \max_{\tau_i \in \mathscr{T}_h}(h_{\tau_i})$, $h_q = \max_{q_i \in \mathscr{Q}_h}(h_{q_i})$ and $\eta_h = \max\left\{h_\tau^{\min\{s,p\}}, h_q^{\min\{s,k\}}\right\}$.

Let $\mathbf{W} \in \mathcal{C}^0([0,t_f];(PH^{s+1}(\Omega))^6)$ for $s \leq 0$ with $PH^{s+1}(\Omega) = \{v \mid \forall j, v_{|\Omega_j} \in H^{s+1}(\Omega_j)\}$.

And $\mathbf{W}_h \in \mathcal{C}^1([0,t_f]; V_h^6)$. Then there is a constant $C > 0$ independent of h such that :

$$\max_{t \in [0,t_f]} \left(\|P_h(\mathbf{W}(t)) - \mathbf{W}_h(t)\|_{0,\Omega}\right) \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}$$

Finally, the error $\mathbf{w} = \mathbf{W} - \mathbf{W}_h$ satisfies the estimate :

$$\boxed{\|\mathbf{w}\|_{\mathcal{C}^0([0,t_f],L^2(\Omega))} \leq C\ \eta_h\ t_f\ \|\mathbf{W}\|_{\mathcal{C}^0([0,t_f],PH^{s+1}(\Omega))}}$$

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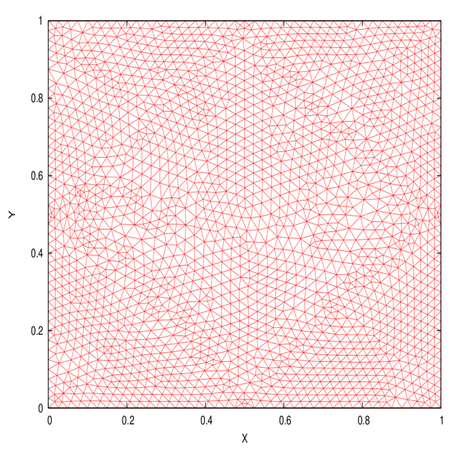
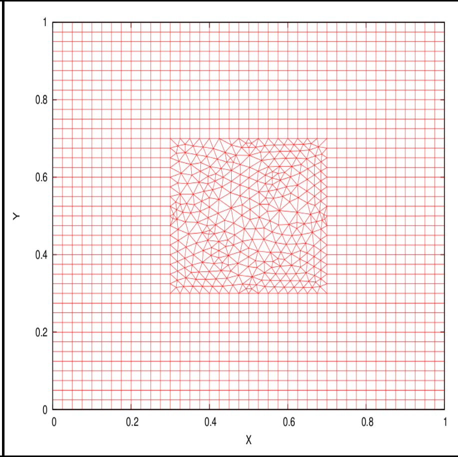
Outline

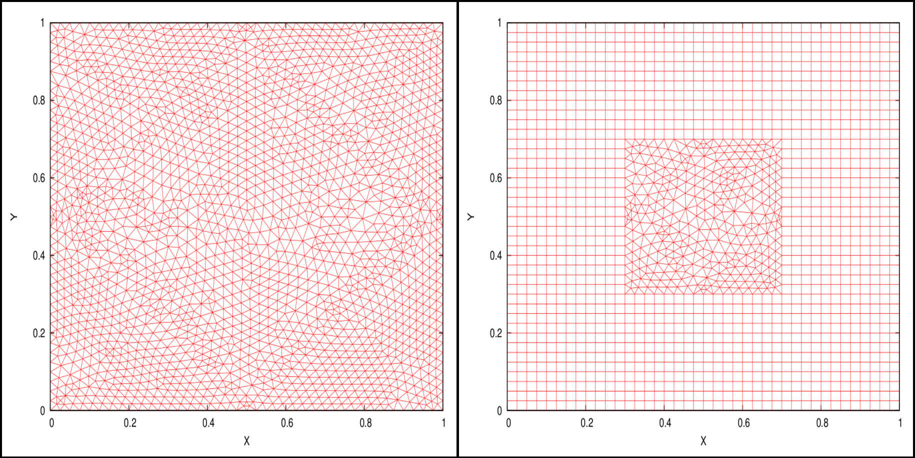
Test problem 1 : Eigenmode in PEC square cavity





Exact solution of the evolution of the (1,1) mode in a PEC square cavity :

$$\begin{cases} H_x(x_1,x_2,t) & = & -(\pi/\omega)\sin(\pi x_1)\cos(\pi x_2)\sin(\omega t), \\ H_y(x_1,x_2,t) & = & (\pi/\omega)\cos(\pi x_1)\sin(\pi x_2)\sin(\omega t), \\ E_z(x_1,x_2,t) & = & \sin(\pi x_1)\sin(\pi x_2)\cos(\omega t). \end{cases} \qquad \omega = 2\pi f,\ f \text{ the frequency}$$



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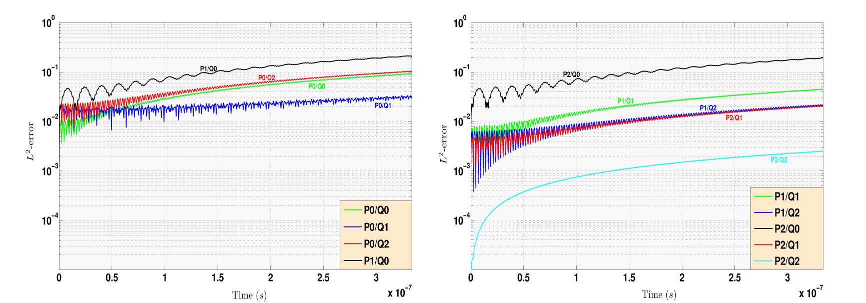

	CPU time	# *dof*	Final L^2-error
DGTD$-\mathbb{P}_0/\mathbb{Q}_0$	9.7 s	1980	9.17×10^{-2}
DGTD$-\mathbb{P}_0/\mathbb{Q}_1$	64.0 s	6012	3.23×10^{-2}
DGTD$-\mathbb{P}_0/\mathbb{Q}_2$	395.0 s	12732	1.05×10^{-1}
DGTD$-\mathbb{P}_1/\mathbb{Q}_0$	38.2 s	3252	2.10×10^{-1}
DGTD$-\mathbb{P}_1/\mathbb{Q}_1$	95.0 s	7284	4.53×10^{-2}
DGTD$-\mathbb{P}_1/\mathbb{Q}_2$	414.0 s	14004	2.20×10^{-2}
DGTD$-\mathbb{P}_2/\mathbb{Q}_0$	129.0 s	5160	1.95×10^{-1}
DGTD$-\mathbb{P}_2/\mathbb{Q}_1$	238.0 s	9192	2.09×10^{-2}
DGTD$-\mathbb{P}_2/\mathbb{Q}_2$	531.0 s	15912	2.70×10^{-3}

- Method stable (energy conserved). The error decreases by refining the mesh
- Most accurate results for $\mathbb{P}_2/\mathbb{Q}_2$ (but long CPU time)
- Best compromise between accuracy and CPU time : $\mathbb{P}_1/\mathbb{Q}_2$ and $\mathbb{P}_2/\mathbb{Q}_1$

	CPU time	# *dof*	Final L^2-error
DGTD$-\mathbb{P}_0$	15.5 s	3778	2.37×10^{-2}
DGTD$-\mathbb{P}_1$	127.0 s	11334	4.75×10^{-2}
DGTD$-\mathbb{P}_2$	601.0 s	22668	2.70×10^{-3}

- Same accuracy for $\mathbb{P}_2/\mathbb{Q}_2$ and $\mathbb{P}_2$ (with slightly lower CPU time for $\mathbb{P}_2/\mathbb{Q}_2$)
- For $\mathbb{P}_1/\mathbb{Q}_2$ and $\mathbb{P}_2/\mathbb{Q}_1$, more important error than $\mathbb{P}_2$ (but very good and smaller than $\mathbb{P}_1$), CPU time reduced by about half compared to $\mathbb{P}_2$

	Time step		Time step		Time step
DGTD$-\mathbb{P}_0/\mathbb{Q}_0$	58.9 ps	**DGTD**$-\mathbb{P}_1/\mathbb{Q}_2$	14.1 ps	**DGTD**$-\mathbb{Q}_1$	29.5 ps
DGTD$-\mathbb{P}_0/\mathbb{Q}_1$	29.5 ps	**DGTD**$-\mathbb{P}_2/\mathbb{Q}_0$	12.4 ps	**DGTD**$-\mathbb{Q}_2$	14.1 ps
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DGTD$-\mathbb{P}_1/\mathbb{Q}_0$	23.0 ps	**DGTD**$-\mathbb{P}_2/\mathbb{Q}_2$	12.4 ps	**DGTD**$-\mathbb{P}_1$	23.0 ps
DGTD$-\mathbb{P}_1/\mathbb{Q}_1$	23.0 ps	**DGTD**$-\mathbb{Q}_0$	117 ps	**DGTD**$-\mathbb{P}_2$	12.4 ps

We note that each time step used in the **DGTD**$-\mathbb{P}_p/\mathbb{Q}_k$ method exactly corresponds to the minimum between the time step for **DGTD**$-\mathbb{P}_p$ and the time step for **DGTD**$-\mathbb{Q}_k$ $\Longrightarrow$ first numerical validation of the stability analysis

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DGTD$-\mathbb{P}_0/\mathbb{Q}_2$	14.1 ps	**DGTD**$-\mathbb{P}_2/\mathbb{Q}_1$	12.4 ps	**DGTD**$-\mathbb{P}_0$	58.9 ps
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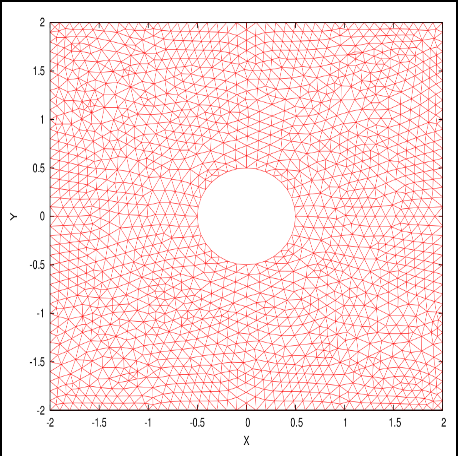
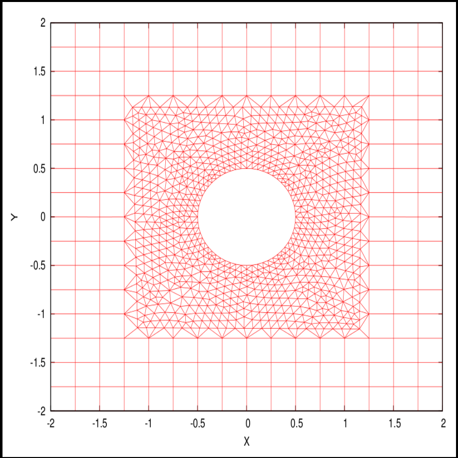
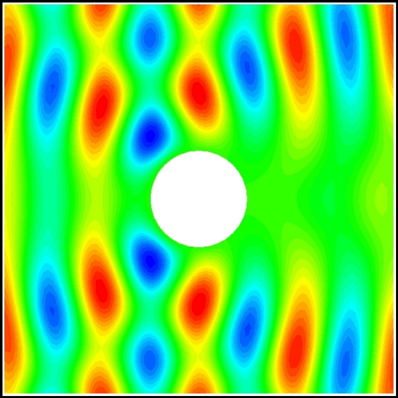
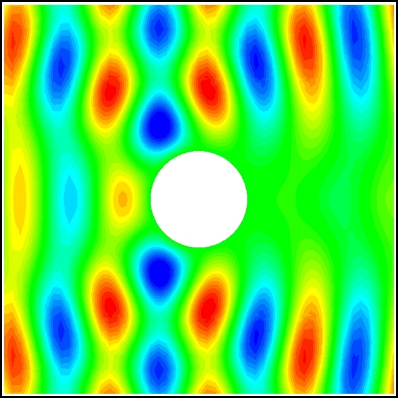
DGTD$-\mathbb{P}_2$ method and **DGTD**$-\mathbb{P}_2/\mathbb{Q}_3$ method :



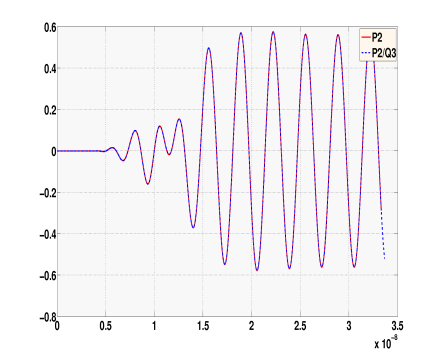


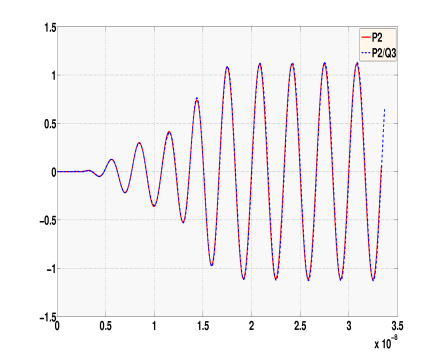
Contour lines of component E_z :





Time evolution of E_z at points $(0.75, 0.75)$ and $(1.3, -1.3)$:





- The curves coincide
- CPU time for $\mathbb{P}_2/\mathbb{Q}_3$ (3.1 s) reduced by about half compared to $\mathbb{P}_2$ (6.3 s)

Outline

1. 3D Maxwell's equations
2. DGTD method on hybrid meshes
   - Objective
   - Spatial discretization
   - Time discretization
3. 3D Convergence and stability
   - Stability analysis
   - A priori convergence analysis
4. 2D Numerical results (TM_z)
   - Test problem 1 : Eigenmode in PEC square cavity
   - Test problem 2 : Scattering of a plane wave by PEC cylinder
5. Conclusion

- First validation of this method
- Interesting results with the first test case for $\mathbb{P}_1/\mathbb{Q}_2$ and $\mathbb{P}_2/\mathbb{Q}_1$ and with the second test case $(\mathbb{P}_2/\mathbb{Q}_3)$
- Recent work :
  - Fourth order Leap-Frog scheme
  - Hybridizations $\mathbb{P}_p/\mathbb{Q}_k$ for $p = 0, \ldots, 4$ and $k = 0, \ldots, 4$
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- Work in progress :
  - Non-conforming meshes (with a large number of new test cases)
  - Transition to 3D

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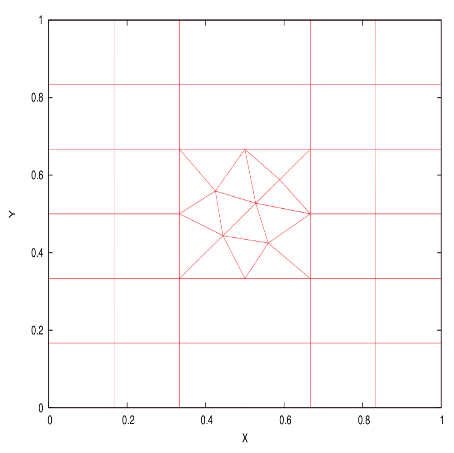
Y
X
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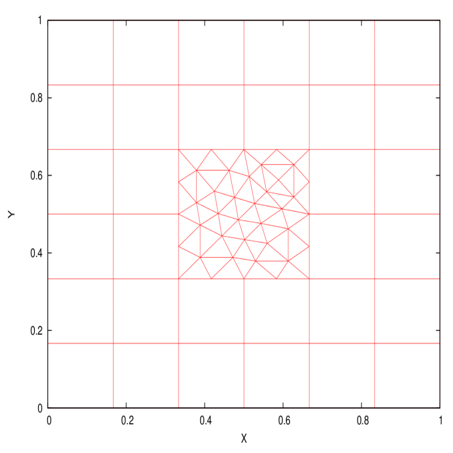
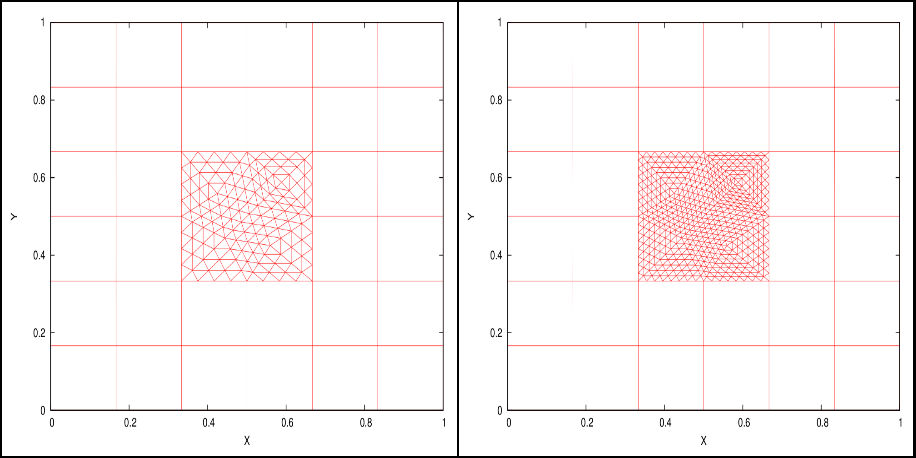
0.2
0.4
0.6
0.8
1


Y
X
0
0.2
0.4
0.6
0.8
1



1
0.8
0.6
0.4
0.2
0
Y
0
0.2
0.4
0.6
0.8
1
X
1
0.8
0.6
0.4
0.2
0
Y
0
0.2
0.4
0.6
0.8
1
X

Thank you for your attention