Tutorial: Discontinous Galerkin

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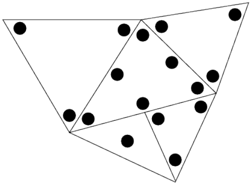
Outline

Introduction

- Discontinuous Galerkin (DG) methods were introduced in the 70's
  - Hyperbolic PDE's (Reed and Hill 73, Lesaint and Raviart 74)
  - Elliptic PDE's (Nitsche 71, Douglas and Dupont 76, Baker 77, Wheeler 78, Arnold 82)
- General principles and motivations
  - Handle and compute accurately discontinuous fields
  - FE-based method using piecewise polynomials discontinuous across mesh elements
  - FV-based high-order method using numerical fluxes flexibility (non-matching grids, variable polynomial degree)
  - Locally conservative, stable, high order accurate methods



Motivation for DG methods

- Spurious pressure modes in mixed FE methods
- Other spurious solutions in mixed FE methods
- Stabilized Finite element methods

1) – Spurious pressure modes in mixed methods

Let $a(.,.)$, $b(.,.)$ and $c(.,.)$, continuous bilinear forms on $V \times V$, $V \times Q$ and $Q \times Q$, resp., where V and Q are some Hilbert spaces. We assume $a(.,.)$ is positive semidefinite.

We introduce the continuous problem: find $u \in V$ and $p \in Q$ such that

$$\begin{cases} a(u,v) + b(v,p) = < f, v >_{V' \times V} & \forall v \in V, \\ b(u,q) = < g, q >_{Q' \times Q} & \forall q \in Q, \end{cases} \quad \begin{cases} < grad\, u, grad\, v > - < p, div\, v > = < f, v > \\ < div\, u, q > = 0, \end{cases}$$

and for the discrete problem we search $u_h \in V_h$ and $p_h \in Q_h$ such that:

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = < f, v_h >_{V_h' \times V_h} & \forall v_h \in V_h \\ b(u_h, q_h) = < g, q_h >_{Q_h' \times Q_h} & \forall q_h \in Q_h. \end{cases}$$

We also let A and B the linear continuous operators defined as:

$< Au, v >_{V' \times V} = a(u,v), \quad < Bv, q >_{Q' \times Q} = < v, B^t q >_{V \times V'} = b(v,q), \quad \forall u, v \in V, \forall q \in Q.$

Finally, let A_0 the restriction of A to $\ker B$.

If the five following conditions are filled:

- $a(.,.)$is invertible on $\ker B$, that is there exists $\alpha_1 > 0$ such that

$$\inf_{u_0 \in \ker B} \sup_{v_0 \in \ker B} \frac{a(u_0, v_0)}{\|u_0\|_V \ \|v_0\|_V} \geq \alpha_1 > 0, \text{ i.e. } \ker A_0 = 0, \text{ that is } A_0 \text{ is injective,}$$

$$\inf_{v_0 \in \ker B} \sup_{u_0 \in \ker B} \frac{a(u_0, v_0)}{\|u_0\|_V \ \|v_0\|_V} \geq \alpha_1 > 0, \text{ i.e. } \ker A_0^t = 0, \text{ that is } A_0 \text{ is surjective,}$$

- There exists a positive constant $\alpha_2 > 0$ such that

$$\inf_{u_h \in \ker B_h} \sup_{v_h \in \ker B_h} \frac{a(u_h, v_h)}{\|u_h\|_V \ \|v_h\|_V} \geq \alpha_2 > 0,$$

- Im B is closed in $Q^{'}$,
- There exists a positive **constant** $\beta > 0$ such that

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \ \|q_h\|_{Q/\ker B^t}} \geq \beta > 0,$$

Then, for every $f \in V^{'}$ and $g \in \text{Im } B$, the continuous and discrete problems have a unique solution and for K a bounded non linear function depending on $\|a\|, \|b\|, \alpha, \beta$:

$$\|u - u_h\|_V + \|p - p_h\|_Q \leq K\Big(\inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q \Big).$$

If $B_h : V_h \longrightarrow Q'_h$ is such that

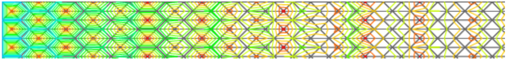
$$< B_h u_h, q_h >_{Q'_h \times Q_h} = < u_h, B_h^t q_h >_{V_h \times V'_h} = b(u_h, q_h), \quad \forall q_h \in Q_h,$$

then β is the smallest eigenvalue of the matrix $\begin{pmatrix} 0 & B_h^t \\ B_h & 0 \end{pmatrix}$.

If a spurious pressure mode exists (i.e. $\dim(\ker B_h^t) > 1$) we have $\beta = 0$, and the discrete problem fails to admit a unique solution.

For example, B_h^t is the discrete gradient operator for Stokes flow.

The spurious mode is a numerical artifact introduced by the discrete scheme. It is a physical eigenmode of the system which appears as a stationary internode oscillation for pressure (see below for the $P_1 - P_1$ FE pair). The velocity part of such a spurious eigenvector is zero.



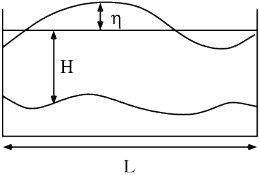
We need to perform Fourier and kernel discrete operators analyses.

2) – Other spurious solutions in mixed FE methods

The Shallow-water equations are derived by vertical integration of the Navier-Stokes system assuming several asumptions ($u_z = v_z = 0$, $H \ll L$, ρ is constant, $p_z = -\rho g$).

Linear shallow-water system in the non conservative form with IC and BC, H is constant:

$$\begin{aligned} \mathbf{u}_t + f\mathbf{k}\times\mathbf{u} + g\nabla\eta &= 0\,, \\ \eta_t + H\nabla\cdot\mathbf{u} &= 0. \end{aligned}$$



The continuous solution is examined by considering the behavior of one Fourier mode: $(\widetilde{u},\widetilde{v},\widetilde{\eta}) = (\widehat{u},\widehat{v},\widehat{\eta})\, e^{i(kx+ly+\omega t)}$. The so-called dispersion relation reads: $\omega\left(\omega^2 - f^2 - gH(k^2+l^2)\right) = 0$.

RT_0 BDM_1 P_0 P_1 P_2 P_1^{NC} P_1^{DG}

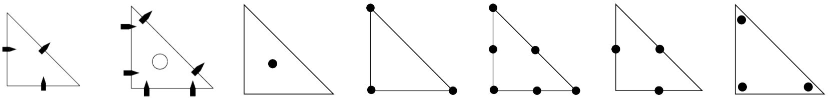


Table: Number of frequencies of type $\omega = 0$, $O(h^2)$, $\pm f$, $O(1)$, $O(\frac{1}{h})$, solutions of the dispersion relation of degree n for inertia-gravity waves.

FE pair	n	$\omega = 0$	$O(h^2)$	f or $-f$	$O(1)$ modes when $h \to 0$	Spurious η modes	$O(\frac{1}{h})$
$RT_0 - P_0$	5	**1**			$\omega_{AN} + O(h^2)$	no	2
$RT_0 - P_1$	4	**2**			$\omega_{AN} + O(h^2)$	yes	
$BDM_1 - P_0$	8	**4**			$\omega_{AN} + O(h^2)$	no	2
$BDM_1 - P_1$	7	**3**	**2**		$\omega_{AN} + O(h^4)$	yes	
$P_1 - P_1$	3	**1**			$\omega_{AN} + O(h^4)$	yes	
$P_2 - P_1$	9	**1**		**6**	$\omega_{AN} + O(h^2)$	no	
$P_1^{NC} - P_1$	7	**1**		**4**	$\omega_{AN} + O(h^4)$	no	
$P_0 - P_1$	5	**1**		**2**	$\omega_{AN} + O(h^2)$	no	
$P_1^{DG} - P_1$	13	**1**		**10**	$\omega_{AN} + O(h^2)$	no	
$P_1^{DG} - P_2$	16	**4**		**4**	$\omega_{AN} + O(h^4)$	no	6

Table: Dimension of the discrete operator kernels on a $m \times n$ grid (made up of biased triangles) for several FE pairs with periodic boundary conditions.

		r	s	C	G	D (1)	(2)	(3)	(4)
A	$RT_0 - P_0$	$\frac{3}{2}mn$	$2mn$	mn	1	$\mathbf{mn}$			$+\ 1$
	$RT_0 - P_1$	$\frac{3}{2}mn$	mn	mn	>1	$\mathbf{2mn}$			$+>1$
	$BDM_1 - P_0$	$3mn$	$2mn$	$2mn$	1	$\mathbf{4mn}$			$+\ 1$
	$BDM_1 - P_1$	$3mn$	mn	$2mn$	>1	$\mathbf{3mn}$	$+\mathbf{2mn}$		$+>1$
B	$P_1 - P_1$	mn	mn	0	>1	$\mathbf{mn}$			$+>1$
	$P_2 - P_1$	$4mn$	mn	0	1	$\mathbf{mn}$		$+\mathbf{6mn}$	$+\ 1$
	$P_1^{NC} - P_1$	$3mn$	mn	0	1	$\mathbf{mn}$		$+\mathbf{4mn}$	$+\ 1$
	$P_0 - P_1$	$2mn$	mn	0	1	$\mathbf{mn}$		$+\mathbf{2mn}$	$+\ 1$
	$P_1^{DG} - P_1$	$6mn$	mn	0	1	$\mathbf{mn}$		$+\mathbf{10mn}$	$+\ 1$
	$P_1^{DG} - P_2$	$6mn$	$4mn$	0	1	$\mathbf{4mn}$		$+\mathbf{4mn}$	$+\ 1$

$\ker D$ contains the stationary modes: physical ($\omega = 0$) and spurious (inertial and pressure). For the inertial modes ($\ker D$ (3)) the reason lies in $\ker D = (\mathrm{Im}\ G)^{\perp}$. Number of inertial spurious modes: $2(r-s)$.

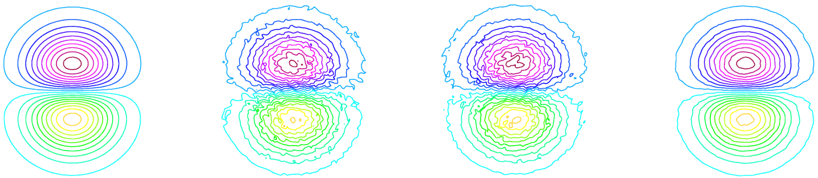
Since many years there are open questions about problems observed iin geophysical fluid dynamics involving Coriolis (for pairs of Group B):

- Numerical noise observed in the velocity field.
- Sub-optimal convergence of velocity, (e.g. $O(h)$ instead of $O(h^2)$).

In fact, the spurious inertial modes are responsible of such behavior.

Example: u component of geostrophic adjustment for the $P_2 - P_1$ pair.

$t = 0$ s | $t = 20$ s | $t = 80$ s | $t = 100\,\mathrm{s} = 2\pi/f$



The $P_1^{DG} - P_1$ pair: the inertial modes take the control of the solution.

3) – Stabilized Finite element methods

Consider the Stokes problem: Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases} (\mathbf{grad}\mathbf{u}_h, \mathbf{grad}\mathbf{v}_h) - (p_h, div\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in V_h \\ div(\mathbf{u}_h, q_h) = 0 & \forall q_h \in Q_h. \end{cases}$$

The term $(p_h, div\mathbf{v}_h)$ may be problematic as pressure is obtained via:

$$(p_h, div\mathbf{v}_h) = (\mathbf{grad}\mathbf{u}_h, \mathbf{grad}\mathbf{v}_h) - (\mathbf{f}, \mathbf{v}_h).$$

When there are not enough functions $\mathbf{v}_h$, as it is the case when $\mathbf{grad}$ is not injective ($\dim(\ker B_h^t) > 1$), there are more p_h unknowns than equations to satisfy them. Hence, the uniqueness for pressure is lost.

This is the case when $\mathbf{v}_h$ is too "small", and a lot of informations are lost about $\mathbf{grad}p_h$, as we only get $\Pi_{V_h}\mathbf{grad}p_h$, i.e. the orthogonal projection of $\mathbf{grad}p_h$ on V_h, w.r.t. the scalar product in L^2. Hence, only the part $\Pi_{V_h}\mathbf{grad}p_h$ of $\mathbf{grad}p_h$ is retained. For example, for the $P_1 - P_1$ and $P_1 - P_0$ pairs, Π_{V_h} "kills" too many components.

The purpose of stabilized methods is to retrieve the information lost by the projection Π_{V_h} for a bad choice of V_h and Q_h (when V_h is too "small") i.e.: $\mathbf{grad} p_h - \Pi_{V_h}\mathbf{grad} p_h$. It is sufficient to put this term in the system and the problem now reads: Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases} (\mathbf{grad}\mathbf{u}_h, \mathbf{grad}\mathbf{v}_h) - (p_h, div\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h \\ -div(\mathbf{u}_h, q_h) - h^2(\mathbf{grad} p_h - \Pi_{V_h}\mathbf{grad} p_h, \mathbf{grad} p_h - \Pi_{V_h}\mathbf{grad} p_h) = 0, \forall q_h \in \end{cases}$$

The sign "-" leads to a symmetric problem, and h^2 is for the dimension. A number of methods follow this idea ...

Example: the $Q_1 - P_0$ pair. In this case the stabilization term reads $h^2 \sum_{\sigma \in \Gamma_h} h_\sigma \int_\sigma [p_h]_\sigma [q_h]_\sigma$, where Γ_h is the set of faces σ of the triangulation, h_σ is the length of h_σ and $[p_h]$ denotes the jump of p_h through σ. Simple calculation shows it is $h^2(4p_1 - p_2 - p_3 - p_4 - p_5)$, i.e. the discrete Laplacian on the dual mesh obtained bu joining the element centers.

The wave equation

The wave equation

For an enclosed domain of length L, consider the 1–D wave equation

$$v_{tt} - gH\,v_{xx} = 0,$$

written (using $u = v_t, \eta = -H\,v_x$) on the form

$$\begin{aligned} u_t + g\eta_x &= 0, \qquad (1)\\ \eta_t + H\,u_x &= 0, \qquad (2) \end{aligned}$$

with apropriate boundary and initial data.
Let $\mathbf{w} = (\eta, u)$, equations (1) and (2) can be conveniently expressed as

$$\mathbf{w}_t + A\mathbf{w}_x = 0, \quad \text{where } A = \begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix}. \qquad (3)$$

Matrix A as two real eigenvalues $\pm\sqrt{gH}$, two eigenvectors $(H, \pm\sqrt{gH})$ with

$$A = M\,D\,M^{-1}, \text{ where } M = \begin{pmatrix} H & H \\ -\sqrt{gH} & \sqrt{gH} \end{pmatrix} \text{ and } D = \begin{pmatrix} -\sqrt{gH} & 0 \\ 0 & \sqrt{gH} \end{pmatrix}. \qquad (4)$$

The characteristic variables $\mathbf{q} = (q_1, q_2)^T$ are defined via

$$\mathbf{q} = M^{-1}\mathbf{w}, \quad q_1 = \sqrt{gH}\eta - H u, \quad q_2 = \sqrt{gH}\eta + H u. \tag{5}$$

The original system (3) then becomes a simple set of decoupled equations. Indeed, we have $\mathbf{w}_t = M\mathbf{q}_t$ and $\mathbf{w}_x = M\mathbf{q}_x$, and from (3) and (4) we obtain

$$\mathbf{q}_t + D\mathbf{q}_x = 0. \tag{6}$$

In the following we let $q = q_1$, and will only consider the first equation in (6), i.e.

$$q_t - \sqrt{gH}\, q_x = 0, \tag{7}$$

and will deduce the solution of the second equation in (6) by symmetry arguments.

For this analysis, we assume time is continuous, and we seek periodic Fourier solutions of the form $q_1 = \widehat{q}(x)\, e^{i\omega t}$, where $\widehat{q}(x)$ is the amplitude. Equation (7) then becomes (by dropping the hats)

$$i\omega\, q - \sqrt{gH}\, q_x = 0. \tag{8}$$

Equation (8) is now spatially discretized using the DG method.

Let ε_h denote a partition of the model domain $\Omega = (0, L)$, i.e. ε_h is a finite collection of m open elements e_j, $j = 1, 2, ..., m$, of the real line,

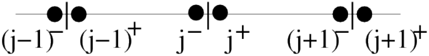
$$\bar{\Omega} = \bigcup_{e_j \in \varepsilon_h} \bar{e}_j \quad \text{and} \quad e_i \cap e_j = \emptyset \ \text{ for } \ i \neq j .$$

Consider a uniform mesh of m intervals on $(0, L)$ and let $h = L/m$ denote the meshlength parameter with elements $e_j = (x_j, x_{j+1})$ for $j = 1, 2, ..., m$ and 'knots' $x_j = (j-1)\,h$ for $j = 1, 2, ..., m+1$.

The so-called (mesh-dependent) *broken space* $H^1(\varepsilon_h)$ is defined as

$$H^1(\varepsilon_h) = \{ v \in L^2(\Omega);\ v|_e \in H^1(e),\ \forall\, e \in \varepsilon_h \},$$

where e simply denotes an element e_j, $j = 1, 2, ..., m$, of ε_h.



The discontinuous variables (q_h, u_h et η_h) are located at the same nodal locations and both are approximated with linear polynomials on e_j. Let say they belong to $Q(e_j)$.

To obtain the DG formulation we seek a discontinuous approximate solution $q_h \in Q(e_j)$, integrate over the domain, decompose the integrals

$$\sum_{j=1}^{m} \int_{e_j} q_h \varphi \, dx - \sqrt{gH} \sum_{j=1}^{m} \left(- \int_{e_j} q_h \varphi_x \, dx + q_h \varphi \,|_{j^+}^{(j+1)^-} \right) = 0. \tag{9}$$

Regrouping the boundary terms leads to

$$\sum_{j=1}^{m} q_h \varphi \,|_{j^+}^{(j+1)^-} = \sum_{j=1}^{m+1} (q_{j^-} \varphi_{j^-} - q_{j^+} \varphi_{j^+}), = \sum_{j=1}^{m+1} \left(< q_j >_\lambda [\varphi_j] + [q_j] < \varphi_j >_{1-\lambda} \right),$$

where $< \chi_j >_\lambda = (1-\lambda) \chi_{j^-} + \lambda \chi_{j^+}$ and $[\chi_j] = \chi_{j^-} - \chi_{j^+}$, and λ is real.

When $q_h \in H^1(\Omega) \subset H^1(\varepsilon_h)$ the jump $[q_j]$ vanishes on each node x_j, $j = 1, 2, ..., m+1$, and we obtain

$$\sum_{j=1}^{m} \int_{e_j} q_h \varphi \, dx + \sqrt{gH} \sum_{j=1}^{m} \int_{e_j} q_h \varphi_x \, dx - \sqrt{gH} \sum_{j=1}^{m+1} < q_j >_\lambda [\varphi_j] = 0. \tag{10}$$

In the present study we use $\lambda = 1$ (i.e. the upwind case), due to the choice made in (8), as the wave is progressing from the right part of the domain to the left one.

Equation (10) at node j^+ for $\varphi = \varphi_{j^+}$ becomes

$$\frac{h}{3} q_{j^+} + \frac{h}{6} q_{(j+1)^-} + \frac{1}{2}\sqrt{gH}\,\frac{1}{i\omega}\left(q_{j^+} - q_{(j+1)^-}\right) = 0, \tag{11}$$

and equation (10) at node $(j+1)^-$ for $\varphi = \varphi_{(j+1)^-}$, leads to

$$\frac{h}{3} q_{(j+1)^-} + \frac{h}{6} q_{(j)^+} + \frac{1}{2}\sqrt{gH}\,\frac{1}{i\omega}\left(q_{j^+} + q_{(j+1)^-} - 2q_{(j+1)^+}\right) = 0. \tag{12}$$

Periodic solutions of system (11) - (12) corresponding to $\widetilde{q}_{j^\pm} = \widehat{q}_\pm\, e^{ikx_{j^\pm}}$ are sought. Substituting in (11) - (12) leads to a matrix system for the amplitudes $\widehat{q}_\pm$. For a nontrivial solution $(\widehat{q}_+, \widehat{q}_-)^t$ to exist, the determinant of the matrix must vanish which leads to a polynomial in ω, the so-called dispersion relation. We obtain the solutions

$$\omega_{1,2} = i\frac{\sqrt{gH}}{h}\left(e^{ikh} + 2 \pm \sqrt{e^{2ikh} + 10e^{ikh} - 2}\right). \tag{13}$$

In the limit as mesh spacing $h \to 0$, it follows

$$\omega_1 = \sqrt{gH}\left(k + \frac{i}{72}k^4 h^3 + O(h^4)\right), \tag{14}$$

$$\omega_2 = \sqrt{gH}\left(\frac{6i}{h} - 3k - i k^2 h + \frac{1}{3}k^3 h^2 + \frac{5i}{72}k^4 h^3 + O(h^4)\right). \tag{15}$$

Note that ω_1 coincide with the continuous solution obtained in the limit as mesh spacing $h \to 0$, while ω_2 correspond to spurious modes from the DG scheme.

A few remarks

- The DG method is appealing for its ability to exactly represent discontinuities.
- However, one has to carefully choose the variable of which continuity is weakly enforced.
- DG schemes where upwinding weighting is naively applied to the primitive variables (velocity, pressure) appear to poorly perform for all values of λ.
- It is mandatory to impose the continuity of suitable combinations of primitive variables.
- Enforcing the weak continuity of the so-called Riemann variables would perform quite better. In fact, the numerical flux function (numerical trace) at the element interfaces is based on the solution of Riemann problems.
- Such an approach is known as a DG method with a Riemann solver and its numerical performances have been well documented in the literature (Roe 81, Schwanenberger and Kongeter 00, Cockburn and Shu 01, Flaherty et al. 02 ....).
- In higher dimensions the definition of Riemann variables is not obvious. The approach consists in considering a simplified 1–D Riemann problem along the normal direction of each segment.
- The higher is the accuracy order of the numerical method, the less crucial is the choice of Riemann solver.

The linear transport equation

The linear transport equation

We consider the space discretization of

$$u_t + div(\mathbf{a}u) = 0, \quad \text{in } \mathbb{R}^2 \times (0,T)$$
$$u(t=0) = u_0 \quad \text{on } \mathbb{R}^2.$$

The objective here is to examine three properties of the DG methods:

1. Strong link with FV methods (e.g. up-winding, Lax-Friedrichs).
2. High-order accuracy when high order polynomials are used.
3. The artificial viscosity is given by the size of the jumps associated with the residual.

To discretize the transport equation in space using DG, we first triangulate the domain (τ_h), seek a discontinuous approximation u_h belonging of $V(K)$ (usually $P^k(K)$) in each element K of τ_h, and determine u_h on K by weakly enforcing the transport equation as:

$$\int_K (u_h)_t\, v - \int_K \mathbf{a} u_h \cdot \nabla v + \int_{\partial K} \widehat{\mathbf{a} u_h} \cdot \mathbf{n}\, v\, ds = 0, \quad \forall v \in V(K).$$

To complete the definition of the DG method it only remains to define the numerical trace $\widehat{\mathbf{a} u_h}$. The choice of the numerical trace is perhaps the most delicate and crucial aspect of the definition of the DG methods as it can affect their consistency, stability and even accuracy.

First: a stability result. Taking $v = u_h$ in the weak formulation, integrate over space and time and adding on the elements K, we get

$$\frac{1}{2}\int_{\mathbb{R}^2} u_h^2(\mathbf{x}, T)\, d\mathbf{x} + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2} \nabla \cdot \mathbf{a}(\mathbf{x})\, u_h^2(\mathbf{x}, T)\, d\mathbf{x} dt + \int_0^T \Theta_h(t)\, dt = \frac{1}{2}\int_{\mathbb{R}^2} u_{h,0}^2(\mathbf{x})\, d\mathbf{x},$$

where $\Theta_h(t) = \sum_{K \in \tau_h} \left(-\frac{1}{2}\int_K \nabla \cdot (\mathbf{a}\, u_h)(\mathbf{x}, t)\, d\mathbf{x} + \int_{\partial K} \widehat{\mathbf{a} u_h}(\mathbf{x}, t) \cdot \mathbf{n}\, u_h(\mathbf{x}, t)\, ds \right).$

Next we investigate if it is possible to define $\widehat{\mathbf{a} u_h}$ in such a way as to render Θ_h non-negative (link with the continuous stability result).

Notation: Let $\mathbf{x}$ be a point on the set $e = \overline{\partial K^+} \bigcap \overline{\partial K^-}$ and let $n_\pm$ denote the unit outward normal to $\overline{\partial K^\pm}$ at the point $\mathbf{x}$.
Let $u_h^\pm(\mathbf{x})$ denote the value $\lim_{\epsilon \to 0}(\mathbf{x} - \epsilon \mathbf{n}^\pm)$ and set

$$\{u_h\} = \frac{1}{2}(u_h^+ + u_h^-), \quad [u_h] = u_h^- \mathbf{n}^- + u_h^+ \mathbf{n}^+.$$

Finally, let ε_h denote the set of sets e for all K^+ and $K^- \in \tau_h$.
By considering a sum over ε_h and using $\frac{1}{2}[u_h^2] = \{u_h\}[u_h]$, we obtain

$$\begin{aligned} \Theta_h &= \sum_{K \in \tau_h} \int_{\partial K} \left(\widehat{\mathbf{a} u_h} \cdot \mathbf{n}\, u_h - \frac{1}{2} \mathbf{a}\, u_h^2 \cdot \mathbf{n} \right) ds, \\ &= \sum_{e \in \varepsilon_h} \int_e \left(\widehat{\mathbf{a} u_h} - \mathbf{a}\{u_h\} \right) \cdot [u_h]\, ds. \end{aligned}$$

Thus if we take: $\widehat{\mathbf{a} u_h} = \mathbf{a}\{u_h\} + C\,[u_h]$, we get

$$\Theta_h = \sum_{e \in \varepsilon_h} \int_e C\,[u_h] \cdot [u_h]\, ds \geq 0,$$

if C is a non-negative definite matrix and the method is stable.

Examples of the DG methods:

1. $C = \frac{1}{2}|\mathbf{a}\cdot\mathbf{n}|\,\mathrm{Id}$. This implies that the numerical trace is
$$\widehat{\mathbf{a}u_h} = \lim_{\epsilon\to 0} u_h(\mathbf{x}-\epsilon\mathbf{a}),$$
which is nothing but the classical up-winding numerical flux.
2. $C = \frac{1}{2}|\mathbf{a}|\,\mathrm{Id}$. For this choice we have
$$\widehat{\mathbf{a}u_h} = \mathbf{a}\{u_h\} + \frac{1}{2}|\mathbf{a}|\,[u_h],$$
which is the so-called local Lax-Friedrichs numerical flux.

Properties of the DG methods:

- From the two examples above, we see that the DG methods are strongly related to finite volume methods. Indeed, the discretization in space for up-winding scheme and the local Lax-Friedrichs scheme coincide with the corresponding DG method under consideration when the local space $V(K)$ is taken to be the space of constant functions.

- The DG methods, like finite volume methods, can easily handle complex computational domains. Also like finite volume methods, they have the property of being locally conservative, that is,

$$\int_K (u_h)_t \, d\mathbf{x} + \int_{\partial K} \widehat{\mathbf{a}u_h} \cdot \mathbf{n} \, ds = 0,$$

  provided $V(K)$ contains the constant function. This property is obtained by simply taking the test function $\mathbf{v}$ to be a constant.
- Unlike finite volume methods, DG achieve with ease high-order accuracy. Moreover, this is achieved while keeping a high degree of locality since to evolve the degrees of freedom of the approximate solution u_h in an element, only the degrees of freedom of u_h in the immediate neighbors are involved (block diagonal mass matrices).
- The method is highly parallelizable when time discretized explicit methods are employed (e.g. RK schemes).

- The dissipation of the DG methods is given by the jumps of their approximate solution. This is because DG has a higher rate of dissipation of the energy, which here is nothing but the square of the L^2-norm, than the exact solution of the transport equation.

  The extra rate of dissipation for the DG method is given by

$$\Theta_h = \sum_{e \in \varepsilon_h} \int_e C\,[u_h] \cdot [u_h]\, ds.$$

  For monotone finite difference schemes for hyperbolic problems, the above term, when $C = \nu\,\mathrm{Id}$, is introduced by what could be considered to be a term modeling a viscosity effect with ν being the viscosity coefficient, artificially inserted to render the scheme stable. That is why it is also called artificial viscosity.

  We thus see that the artificial viscosity of the DG method solely depends on the jumps of their approximate solution.

Moreover, the jumps and the local residual $(u_h)_t + div(\mathbf{a}u_h)$, which we denote by R are strongly related. Indeed, a simple integration by parts in the definition of the approximate solution leads to

$$\int_K R\,v = \int_{\partial K} \left(\mathbf{a}u_h \cdot \mathbf{n}\,v - \widehat{\mathbf{a}u_h} \cdot \mathbf{n}\,v\right) ds.$$

For up-winding fluxes, with $\partial K^- = \{x \in \partial K : \mathbf{a}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \leq 0\}$ we get

$$\int_K R\,v = \int_{\partial K^-} \mathbf{a} \cdot [u_h]\,ds.$$

In other words, the residual of u_h in K is linearly related to the jump of u_h on its inflow boundary ∂K^-. A similar, but more complicated relation holds for general DG methods.

The artificial viscosity generated by the method will generally depends on the polynomial degree of the approximate solution and the way of computing the fluxes.

Second-order elliptic problems

Second-order elliptic problems

We consider the space discretization of

$$\Delta u = f, \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain of $\mathbb{R}^d$. The elliptic problem is rewritten as

$$\mathbf{q} = \nabla u, \quad -\nabla \cdot \mathbf{q} = f \quad \text{in } \Omega, u = 0 \quad \text{on } \partial\Omega.$$

The objective here is to examine a few properties of the DG methods:

1. We emphasized the DG methods are a generalization of VF methods for hyperbolic problems.

   Here we show that DG methods are in fact mixed FE methods.
2. Finally, the dissipation mechanism of the DG methods is associated to the idea of penalization of the discontinuity jumps.

Following the same approach than previously the DG numerical method reads

$$\int_K \mathbf{q}_h \cdot \mathbf{v}\, dx = -\int_K u_h \nabla \cdot \mathbf{v}\, dx + \int_{\partial K} \widehat{u_h}\, \mathbf{v} \cdot \mathbf{n}\, ds = 0, \quad \forall v \in Q(K).$$

$$\int_K \mathbf{q}_h \cdot \nabla w\, dx - \int_{\partial K} w\, \widehat{\mathbf{q}_h} \cdot \mathbf{n}\, ds = \int_K f\, w\, dx, \quad \forall w \in U(K),$$

where the approximate solution $(\mathbf{q}_h, u_h)$ is taken in the space $Q(K) \times U(K)$.

The numerical traces $\widehat{\mathbf{q}_h}$ and $\widehat{u_h}$ remain to be defined in order to make the method stable. It is enough to take , inside Ω

$$\widehat{\mathbf{q}_h} = \{\mathbf{q}_h\} + C_{11}\, [u_h] + \mathbf{C}_{12}\, [\mathbf{q}_h], \quad \widehat{u_h} = \{u_h\} - \mathbf{C}_{12}\, [u_h] + C_{22}\, [\mathbf{q}_h],$$

and on its boundary

$$\widehat{\mathbf{q}_h} = \mathbf{q}_h - C_{11}\, u_h\, \mathbf{n}, \quad \widehat{u_h} = 0,$$

to obtain

$$\Theta_h = \sum_{e \in \varepsilon_h} \int_e \left(C_{22}\, [\mathbf{q}_h]^2 + C_{11}\, [u_h]^2 \right) ds + \int_{\partial\Omega} C_{11}\, u_h^2\, ds \geq 0,$$

provided C_{11} and C_{22} are non-negative. Note that the boundary conditions are imposed weakly through the definition of the numerical traces.

Some properties

- To guarantee the existence and uniqueness of the approximate DG method
  - The parameter C_{11} has to be greater than zero
  - The spaces $Q(K)$ and $U(K)$ must satisfy the compatibility condition

$$u_h \in U(K): \quad \int_K \nabla u_h \mathbf{v}\, dx = 0, \quad \forall \mathbf{v} \in Q(K) \quad \text{then} \quad \nabla u_h = \mathbf{0}.$$

- DG methods are in fact mixed FE methods. Indeed, the DG approximate solution $(\mathbf{q}_h, u_h)$ can be also characterized as the solution of

$$\begin{cases} a(\mathbf{q}_h, \mathbf{v}) + b(u_h, \mathbf{v})0 & \forall \mathbf{v} \in Q_h \\ -b(\mathbf{q}_h, w) + c(u_h, w) = F(w) & \forall w \in U_h. \end{cases}$$

  where $c(u,w) = \int_{\varepsilon_h} C_{11}\, [u] \cdot [v]\, ds + \int_{\partial\Omega} C_{11}\, u\, v\, ds \geq 0$, which is typical of stabilized mixed FE method. For DG methods the stabilizing form $c(.,.)$ solely depends on the parameter C_{11} and the jumps across elements of function in U_h. This shows that DG methods may be interpreted as stabilized FE methods, (penalization methods by the jumps). The jumps act as dampers that stabilized the DG method.

- For DG methods penalizing the jumps is also a way of introducing stabilization by using residuals, as for the hyperbolic case.

Non linear hyperbolic conservation laws

Non linear hyperbolic conservation laws

We solve a non linear hyperbolic conservation law on the form

$$u_t + \nabla \mathbf{f}(u) = 0.$$

- DG space discretization
- RK time discretization
- The general slope limiter

Conclusions

- The discretization of the shallow-water equations usually leads to spurious (non physical) modes, dispersion and dissipation effects. In particular the spurious pressure modes, inertial oscillations (when the Coriolis term is considered) pose significant problems. Further modes of type $0(1/h)$ have not been explored.
- Modes of type $0(i/h)$ have been found for the 1–D DG scheme. Such modes are dissipated quite instantaneously. A 2–D DG approach merits to be studied.
- Hence, DG methods have the potential can be an improved alternative for modelling geophysical flows, compared to FE methods.
- The DG method is appealing for its ability to exactly represent discontinuities.
- Strong link with FV methods (e.g. up-winding, Lax-Friedrichs).
- High-order accuracy when high order polynomials are used.
- The artificial viscosity is given by the size of the jumps associated with the residual for hyperbolic problems.
- For elliptic problems, DG methods may be interpreted as stabilized FE methods, (penalization methods by the jumps). The jumps act as dampers that stabilized the DG method. Penalizing the jumps is also a way of introducing stabilization by using residuals, as for the hyperbolic case.