DDFV schemes for elliptic problems

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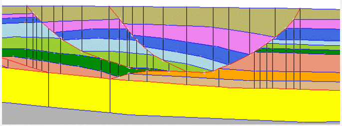
The problem

$$-\operatorname{div}(\varphi(x, \nabla u)) = f$$



where $u \mapsto -\operatorname{div}(\varphi(\cdot, \nabla u))$ is a Leray-Lions operator on $W^{1,p}(\Omega)$.

Heterogeneities, Discontinuities, Anisotropy

- $\varphi(x, \cdot)$ caracteristic of the flow. (Darcy's law $\varphi = -\mathbf{K}(x)\nabla u$, non linear flow $\varphi = -|\mathbf{K}(x)\nabla u|^{p-2}\mathbf{K}(x)\nabla u$).
- One function φ per media (heterogeneity).
- Strong anisotropies due to the main direction of the structure.

Transmission condition

- The pressure u is continuous through the interfaces.
- The mass fluxes $\varphi(x, \nabla u) \cdot n$ are continuous through the interfaces.

The problem

Approximation of the problem

$$-\operatorname{div}(\varphi(x, \nabla u)) = f$$

The finite volume strategy :

- Consider $\mathcal{T} = \cup \mathbb{C}$ a partition of Ω.
  Associate a point $x_{\mathbb{C}}$ to each control volume $\mathbb{C} \in \mathcal{T}$.
- Integrate on any control volume $\mathbb{C}$ the equation :

$$\int_{\mathbb{C}} \operatorname{div} \varphi(x, \nabla u)\, dx = \sum_{\mathrm{F} \in \partial\mathbb{C}} \int_{\mathrm{F}} \varphi(x, \nabla u) \cdot n_{\mathbb{C}} = \int_{\mathbb{C}} f(x)\, dx$$

- Approximate the normal fluxes $\int_{\mathrm{F}} \varphi(x, \nabla u) \cdot n$ in a consistant and conservative way.

The TPFA or VF4 scheme

Approximate the solution (for Dirichlet BC for instance) of

$$-\Delta u = f \qquad (*)$$

in an open bounded set Ω discretized by control volumes K (ex : triangles).

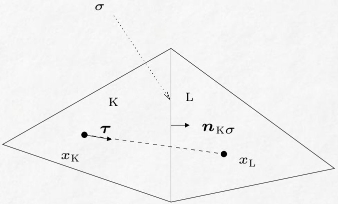
The finite volume scheme principle

- Integrate (*) overall control volumes :

$$\int_{\mathrm{K}} f = \int_{\mathrm{K}} -\Delta u = -\sum_{\sigma \subset \partial \mathrm{K}} \int_{\sigma} \nabla u \cdot \boldsymbol{n}_{\mathrm{K}\sigma}.$$

- Approximate normal fluxes

$$\int_{\sigma} \nabla u \cdot \boldsymbol{n}_{\mathrm{K}\sigma}$$



- Taylor expansion for $\sigma = \mathrm{K}|\mathrm{L}$

$$|\sigma| \frac{u(x_{\mathrm{L}}) - u(x_{\mathrm{K}})}{d_{\mathrm{KL}}} \sim \int_{\sigma} \nabla u \cdot \boldsymbol{\tau}_{\mathrm{KL}} \text{ where } \boldsymbol{\tau}_{\mathrm{KL}} = \frac{\vec{x_{\mathrm{L}} x_{\mathrm{K}}}}{\|\vec{x_{\mathrm{L}} x_{\mathrm{K}}}\|}$$

The FV4 scheme

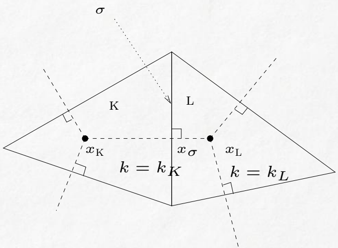
Approximate the solution (for Dirichlet BC for instance) of

$$-\Delta u = f \qquad (*)$$

The classical FV4 scheme

$$\int_K f = \int_K -\Delta u = -\sum_{\sigma\subset\partial K}\int_\sigma \nabla u\cdot \boldsymbol{n}_{K\sigma} \approx -\sum_{\sigma\subset\partial K}|\sigma|\frac{u_L-u_K}{d_{KL}}.$$

Consistency : YES if $[x_K, x_L]\perp\sigma$.



⇒ Such meshes are called admissible.

The FV4 scheme

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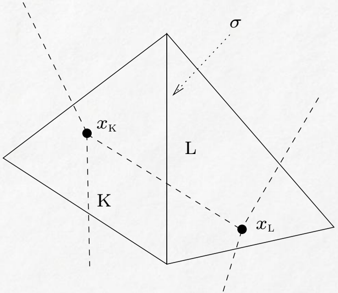
$$\int_K f = \int_K -\Delta u = -\sum_{\sigma\subset\partial K}\int_\sigma \nabla u\cdot \boldsymbol{n}_{K\sigma} \approx -\sum_{\sigma\subset\partial K} |\sigma|\frac{u_L - u_K}{d_{KL}}.$$

Consistency : NO if $[x_K x_L] \perp \sigma$.



⇒ Such control volumes are said to be non admissible.

Error estimates for the FV4 scheme

Admissible Meshes

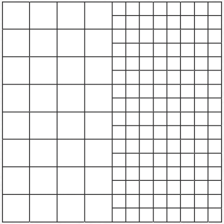
Theorem

The error of the FV4 scheme in case of admissible meshes, is bounded by Ch.

Non admissible meshes

Theorem

If non admissible control volumes are located along a curve Γ, the error of the FV4 scheme is bounded by $Ch^{\frac{1}{2}}$.



Example of non admissible meshes.

Implementation of the TPFA scheme

The mesh

- Starting point : (p,t,f) structure as given by matlab/emc2
  - p : coordinates of the vertices
  - t : reference to the vertices of the volume (here triangle)
  - f : reference to the vertices of the boundary edges
- Creation of an edge structure
  For each edge $\sigma_i = \text{K}_i | \text{L}_i$, we define
  - e(i,_PT1) e(i,_PT2) refer to the vertices's references
  - e(i,_K), e(i,, _L) refer to the volume K_i and L_i with $e(i, \text{L}_i) \leq 0$ if $\sigma_i \in \partial\Omega$.
  - e(i,_MES) and e(i,_DEDGE) stand for the measure of σ_i and the distance between K_i and L_i (or K_i and the boundary)
- Transcription in Matlab

▸ Program

Implementation of the TPFA scheme

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The linear system $AU = b$

- Creation of the matrix A going through the edge structure.
  An edge $\sigma = \text{K}|\text{L}$ in the equation relative to the volumes K and L :
  - Equation for K : ... $+ |\sigma| \frac{u_\text{K} - u_\text{L}}{d_\sigma} +$ ...
  - Equation for L : ... $+ |\sigma| \frac{u_\text{L} - u_\text{K}}{d_\sigma} +$ ...
- Transcription in Matlab
  - m0=find(e(:,_L)> 0 ... we select interior edges
  - l=e(:,_MES)./e(:,_DEDGE)
  - A=A+sparse([e(:,_K) e(m,_K) e(m,_L) e(m,_L)]
    [e(:,_K) e(m,_L) e(m,_L) e(m,_K)]

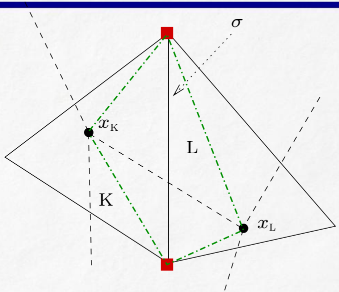
    [l(:) l(m) l(m) l(m)],Nbvol,Nbvol)

What to do in case of non admissible mesh ?

- For "non admissible mesh"

$$\text{For } \mathrm{F} = \mathrm{K}|\mathrm{L}, \qquad \nabla u \cdot n \not\sim \frac{u_{\mathrm{L}} - u_{\mathrm{L}}}{d(x_{\mathrm{K}}, x_{\mathrm{L}})}$$

⇒ New unknowns have to be added to reconstruct a whole discrete gradient.



Approximation of $\int_F \varphi(x, \nabla u) \cdot n$

- Design a "good" discrete gradient $\nabla^{\mathcal{T}} : \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathfrak{D}}$
- We can approximate fluxes on $\mathfrak{D}$ by

$$\varphi(x, \nabla u)_{|_{\mathcal{D}}} \sim \varphi_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}})$$

- A discrete divergence $\mathrm{div}^{\mathcal{T}} : \mathbb{R}^{\mathfrak{D}} \to \mathbb{R}^{\mathcal{T}}$ is then naturally :

$$\int_{\mathbb{C}} \mathrm{div}^{\mathcal{T}} \xi^{\mathcal{T}} = \sum_{\mathrm{F} \in \partial \mathbb{C}} \int_{\mathrm{F}} \xi^{\mathcal{T}} \cdot n_{\mathbb{C}}$$

Bibliography

- **Gradient reconstruction in 2D**
  Confrontation of these schemes : 2D anisotropic benchmark FVCA5
  Herbin, H. (09)
  - "Cell centered" scheme
    Andreianov, Gutnic, Wittbold (04), Andreianov, Boyer, H. (04), (05), (06), (07)
    MPFA **Aavatsmark (98)(04), Lepotier (05),...**
    Diamond scheme **Coudière (99)**
    DG scheme
  - Mixte and hybrid FV scheme
    Droniou, Eymard (06)
    Eymard, Gallouët, Herbin (06), (07) ...
    Mimetic schemes **Brezzi, Lipnikov &** ***al*** **(05), Manzini &** ***al*** **(08)...**
  - DDFV schemes. **Hermeline (00), Domelevo & Omnès (05), Pierre (06), Delcourte & al (06), Andreianov, Boyer, H. (07), ...**
- **Gradient reconstruction in 3D**
  Confrontation of these schemes : 3D anisotropic benchmark FVCA6
  Eymard, Herbin, Henry, H., Kloefkorn, Manzini (11)

Outline

Outline

Assumptions on the continuous problem

- DDFV scheme (Discrete Duality Finite Volume) fo

$$\begin{cases} -\mathrm{div}\,(\varphi(z, \nabla u_e(z))) = f(z), \quad \text{in } \Omega, \\ u_e = 0, \ \text{on } \partial\Omega, \end{cases}$$

  - Ω is a polygonal open set of $\mathbb{R}^2$.
  - $u \mapsto -\mathrm{div}(\varphi(\cdot, \nabla u_e))$ is monotonous coercitive (of Leray-Lions type).

Assumptions on φ

  - Let $p \in]1, \infty[$, $p' = \frac{p}{p-1}$ and $f \in L^{p'}(\Omega)$. ▶ $p \geq 2$ to simplify.
  - $\varphi : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Caratheodory function such that :

$$(\varphi(z,\xi), \xi) \geq C_\varphi \left(|\xi|^p - 1\right), \tag{$\mathcal{H}_1$}$$

$$|\varphi(z,\xi)| \leq C_\varphi \left(|\xi|^{p-1} + 1\right). \tag{$\mathcal{H}_2$}$$

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq \frac{1}{C_\varphi}|\xi - \eta|^p. \tag{$\mathcal{H}_3$}$$

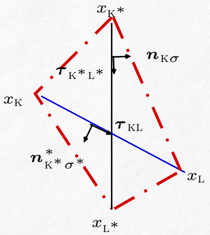
$$|\varphi(z,\xi) - \varphi(z,\eta)| \leq C_\varphi \left(1 + |\xi|^{p-2} + |\eta|^{p-2}\right)|\xi - \eta|. \tag{$\mathcal{H}_4$}$$

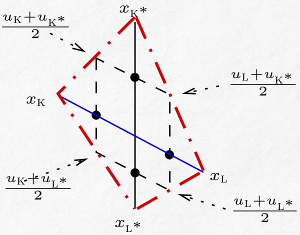
  - φ is lipschitz continuous w.r.t z.

Construction of a discrete gradient

The discrete gradient

Coudière & *al* 99, Omnès & *al* 05...





We look for $\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}$ such that

$$\begin{cases} \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_L - x_K) = u_L - u_K, \\ \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{L^*} - x_{K^*}) = u_{L^*} - u_{K^*}. \end{cases}$$

then

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{\sin \alpha_{\mathcal{D}}} \left(\frac{u_L - u_K}{|\sigma^*|} \boldsymbol{n}_{K\sigma} + \frac{u_{L^*} - u_{K^*}}{|\sigma|} \boldsymbol{n}_{K^*\sigma^*} \right), \quad \forall \, \text{diamond} \, \mathcal{D}.$$

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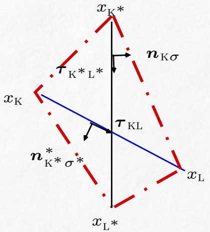
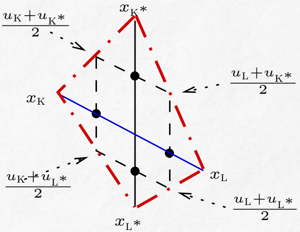
Coudière & *al* 99, Omnès & *al* 05...





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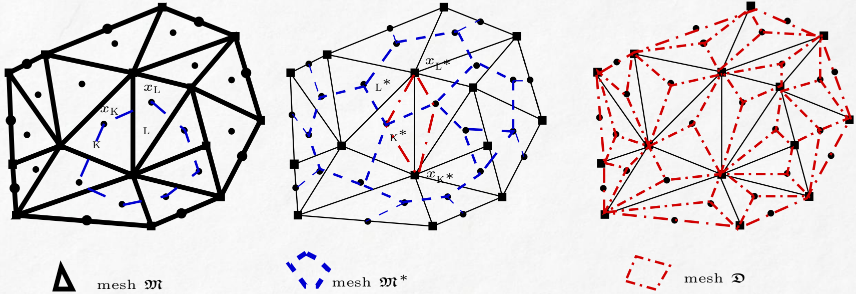
then

$$\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{2|\mathcal{D}|}\left((u_L - u_K)|\sigma|\boldsymbol{n}_{K\sigma} + (u_{L^*} - u_{K^*})|\sigma^*|\boldsymbol{n}_{K^*\sigma^*}\right), \quad \forall \, \text{diamond} \, \mathcal{D}.$$

DDFV meshes

primal, dual and “diamond”.



Primal cells
$\rightsquigarrow u^{\mathfrak{M}} = (u_K)_{K\in\mathfrak{M}}$

dual cells
$\rightsquigarrow u^{\mathfrak{M}^*} = (u_{K^*})_{K^*\in\mathfrak{M}^*}$

Diamond cells
$\rightsquigarrow$ gradient discret

Construction of the DDFV scheme

The discrete gradient

Coudière & *al* 99, Omnès & *al* 05...

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{|\mathcal{D}|}\left((u_{\mathrm{L}} - u_{\mathrm{K}})|\sigma|\boldsymbol{n}_{\mathrm{K}\sigma} + (u_{\mathrm{L}^*} - u_{\mathrm{K}^*})|\sigma^*|\boldsymbol{n}_{\mathrm{K}^*\sigma^*}\right), \quad \forall\, \text{diamond}\, \mathcal{D}.$$

The standart DDFV scheme

$$-\sum_{\sigma\in\mathcal{E}_{\mathrm{K}}} |\sigma|\left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathrm{K}\sigma}\right) = \int_{\mathrm{K}} f(z)\, dz, \; \forall \mathrm{K} \in \mathfrak{M}$$

$$-\sum_{\sigma^*\in\mathcal{E}_{\mathrm{K}^*}} |\sigma^*|\left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathrm{K}^*\sigma^*}\right) = \int_{\mathrm{K}^*} f(z)\, dz, \; \forall \mathrm{K}^* \in \mathfrak{M}^*$$

with

$$\varphi_{\mathcal{D}}(\xi) = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \varphi(z,\xi)\, dz, \quad \text{approximate flux on the diamond cell}$$

Construction of the DDFV scheme

The discrete gradient **Coudière & *al* 99, Omnès & *al* 05...**

$$\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{|\mathcal{D}|}\left((u_{\mathrm{L}} - u_{\mathrm{K}})|\sigma|\boldsymbol{n}_{\mathrm{K}\sigma} + (u_{\mathrm{L}^*} - u_{\mathrm{K}^*})|\sigma^*|\boldsymbol{n}_{\mathrm{K}^*\sigma^*}\right), \quad \forall \,\text{diamond}\, \mathcal{D}.$$

The standart DDFV scheme The classical FV strategy.

$$-|\mathrm{K}|\mathrm{div}_{\mathrm{K}}^{\mathcal{T}}\left(\varphi^{\mathcal{T}}(\nabla^{\mathcal{T}} u^{\mathcal{T}})\right) \stackrel{def}{=} -\sum_{\sigma\in\mathcal{E}_{\mathrm{K}}} |\sigma|\left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathrm{K}\sigma}\right) = \int_{\mathrm{K}} f(z)\,dz, \ \forall \mathrm{K} \in \mathfrak{M}$$

$$-|\mathrm{K}^*|\mathrm{div}_{\mathrm{K}^*}^{\mathcal{T}}\left(\varphi^{\mathcal{T}}(\nabla^{\mathcal{T}} u^{\mathcal{T}})\right) \stackrel{def}{=} -\sum_{\sigma^*\in\mathcal{E}_{\mathrm{K}^*}} |\sigma^*|\left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathrm{K}^*\sigma^*}\right) = \int_{\mathrm{K}^*} f(z)\,dz, \ \forall \mathrm{K}^* \in \mathfrak{M}^*$$

or

$$\boxed{-\mathrm{div}^{\mathcal{T}}\varphi^{\mathcal{T}}(\nabla^{\mathcal{T}} u^{\mathcal{T}}) = f^{\mathcal{T}}}$$

Well-possedness of the scheme

Fondamental tool (Discrete duality)

$$-\int_\Omega \left(\mathrm{div}^{\mathcal{T}} \xi^{\mathfrak{D}}\right) v^{\mathcal{T}} = \int_\Omega \xi^{\mathfrak{D}} \nabla^{\mathcal{T}} u^{\mathcal{T}}$$

$\Rightarrow$ (Variationnal formulation) :

$$\sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}), \nabla^{\mathcal{T}}_{\mathcal{D}} v^{\mathcal{T}}\right) = \frac{1}{2}\left(\int_\Omega f v^{\mathfrak{M}} dz + \int_\Omega f v^{\mathfrak{M}^*} dz\right), \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}.$$

Consequences

- Existence and uniqueness of a solution for the scheme. ▸ Sketch of proof
- Preservation of the variational structure if $\varphi = \nabla_\xi \Phi$.

Convergence of the DDFV

▸ Sketch of proof

Theorem

Let $f \in L^{p'}(\Omega)$ *and a family of meshes* $\mathcal{T}_n$ *whose mesh size tends to* 0 *with*

$$\mathrm{reg}(\mathcal{T}_n) = \max\left(\max_{\mathcal{D}\in\mathfrak{D}} \frac{\mathrm{d}_{\mathcal{D}}}{\sqrt{|\mathcal{D}|}}, \max_{\mathrm{K}\in\mathfrak{M}} \frac{\mathrm{d}_{\mathrm{K}}}{\sqrt{|\mathrm{K}|}}, \max_{\mathrm{K}^*\in\mathfrak{M}^*} \frac{\mathrm{d}_{\mathrm{K}^*}}{\sqrt{|\mathrm{K}^*|}}, \dots \right) \text{ bounded.}$$

Then

- $u^{\mathcal{T}_n} \underset{n\to\infty}{\longrightarrow} \overline{u}$ *strongly in* $L^p(\Omega)$.
- $\nabla^{\mathcal{T}_n} u^{\mathcal{T}_n} \underset{n\to\infty}{\longrightarrow} \nabla\overline{u}$ *strongly in* $L^p(\Omega)$.
- $\varphi(\cdot, u^{\mathcal{T}_n}) \underset{n\to\infty}{\longrightarrow} \varphi(\cdot, \overline{u})$ *strongly in* $L^{p'}(\Omega)$.

where $\overline{u}$ *solves* $-\mathrm{div}(\varphi(., \nabla\overline{u})) = f$.

Andreianov, Boyer & H. (Num. Meth. for PDEs, 07)

Error estimates for the DDFV scheme

Sketch of proof

Smooth diffusion

- Laplacian (i.e. $\varphi(\xi) = \xi$ *i.e.* $p = 2$) :

  Domelevo & Omnès (M^{2}AN, 05)

  $\Rightarrow$ Estimation in $O(h)$ under few restriction on the meshes
- General case :

  Andreianov, Boyer & H. (Num. Meth. for PDEs, 07)

Theorem ($p \geq 2$)

If $\overline{u} \in W^{2,p}(\Omega)$ *and if*

$$\varphi \textit{ est Lip. on } \Omega, \textit{ with } \left|\frac{\partial \varphi}{\partial z}(z,\xi)\right| \leq C_\varphi \left(1 + |\xi|^{p-1}\right), \quad \forall \xi \in \mathbb{R}^2, \qquad (\mathcal{H}_5)$$

Then

$$\|\overline{u} - u^{\mathfrak{M}}\|_{L^p} + \|\overline{u} - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla \overline{u} - \nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^p} \leq C\, h^{\frac{1}{p-1}}.$$

Implementation

Creation of a diamond structure

- diam(i,_K),diam(i,_L),diam(i,_K*),diam(i,_L*) references to the vertices of the diamond cell
- diam(i,_MESD) measure of the diamond cell
- diam(i,_nKL), diam(i,_nK*L*) the two normal
- diam(i,_MES SIG),diam(i,_MES DSIG) measure of the two edges σ and d_σ.

The linear system $AU = b$

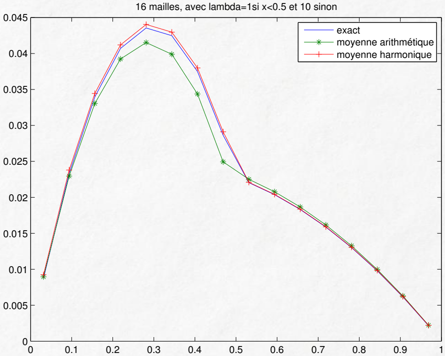
- Creation of the matrix A going through the diamond structure.
  An adge $\sigma = \mathrm{K}|\mathrm{L}$ in the equation relative to the volumes K, L, K^* and L^* :
  - Equation for K : $\ldots + (u_{\mathrm{K}} - u_{\mathrm{L}}) \frac{|\sigma|^2}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{KL}} \cdot \boldsymbol{n}_{\mathrm{KL}} + (u_{\mathrm{K}^*} - u_{\mathrm{L}^*}) \frac{|\sigma^*||\sigma|}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} \cdot \boldsymbol{n}_{\mathrm{KL}} + \ldots$
  - Equation for L : $\ldots - (u_{\mathrm{K}} - u_{\mathrm{L}}) \frac{|\sigma|^2}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{KL}} \cdot \boldsymbol{n}_{\mathrm{KL}} - (u_{\mathrm{K}^*} - u_{\mathrm{L}^*}) \frac{|\sigma^*||\sigma|}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} \cdot \boldsymbol{n}_{\mathrm{KL}} + \ldots.$
  - Equation for K^* : $\ldots + (u_{\mathrm{K}} - u_{\mathrm{L}}) \frac{|\sigma||\sigma^*|}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{KL}} \cdot \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} + (u_{\mathrm{K}^*} - u_{\mathrm{L}^*}) \frac{|\sigma^*|^2}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} \cdot \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} \ldots$
  - Equation for L^* : $\ldots - (u_{\mathrm{K}} - u_{\mathrm{L}}) \frac{|\sigma||\sigma^*|}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{KL}} \cdot \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} - (u_{\mathrm{K}^*} - u_{\mathrm{L}^*}) \frac{|\sigma^*|^2}{2|\mathcal{D}|} \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} \cdot \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} + \ldots$

What to do in case of discontinuous permeability ?

In presence of discontinuities, the classical scheme converges but the order of convergence depends on the choice of k_σ :

Cas 1D : $-\frac{d}{dx}\left(k(x)\frac{d}{dx}u_e\right) = f$, with $k(x) = \begin{cases} k^+ & \text{if } x > 0.5 \\ k^- & \text{if } x < 0.5 \end{cases}$

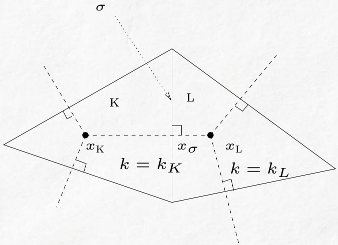
- k_σ arithmetic mean value : order $\frac{1}{2}$
- k_σ harmonic mean value : order 1



The problem of discontinuous coefficients

$$-\mathrm{div}(k(z)\nabla u) = f,$$

$$k(z) \in \mathbb{R}.$$



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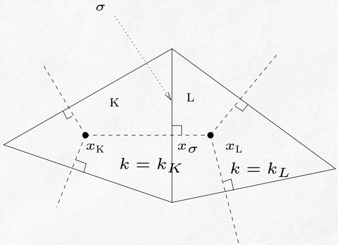


If k is smooth, the finite volume FV4 writes :

$$\int_K f = \int_K -\mathrm{div}(k(z)\nabla u)\, dz = -\sum_{\sigma \subset \partial K} \int_\sigma \underbrace{(k(s)\nabla u)}_{=\text{flux}} \cdot \boldsymbol{n} ds \approx \sum_{\sigma \subset \partial K} |\sigma| k_\sigma \frac{u_L - u_K}{d_{KL}},$$

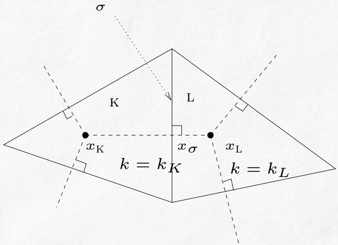
where k_σ is an approximation of k on the edge σ

$$k_\sigma = k(x_\sigma) \quad \text{where} \quad k_\sigma = \frac{1}{|\sigma|} \int_\sigma k(s)\, ds.$$

The problem of discontinuous coefficients

$$-\mathrm{div}(k(z)\nabla u) = f,$$

$$k(z) \in \mathbb{R}.$$



If k is discontinuous across σ : k_{K} and k_{L} on K et L :
How to write the scheme ? We look for k_σ such that

$$|\sigma| k_\sigma \frac{u_{\mathrm{L}} - u_{\mathrm{K}}}{d_{\mathrm{KL}}} \approx \int_\sigma (k(s)\nabla u(s)) \cdot \boldsymbol{n} ds.$$

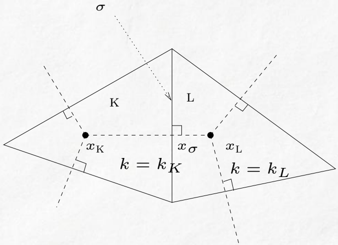
The *simple* choices of $k_\sigma = k_{\mathrm{K}}$, $k_\sigma = k_{\mathrm{L}}$ where $k_\sigma = \frac{1}{2}(k_{\mathrm{K}} + k_{\mathrm{L}})$ lead to non consistent fluxes.

Indeed $\nabla u \cdot \boldsymbol{n}$ is discontinuous across σ !

The problem of discontinuous coefficients

$$-\mathrm{div}(k(z)\nabla u) = f,$$

$$k(z) \in \mathbb{R}.$$



Take a new unknown u_σ on the edge σ :

Write the continuity of the approximate fluxes across σ.

$$F_{\mathrm{KL}} \stackrel{\mathrm{def}}{=} |\sigma| k_{\mathrm{L}} \frac{u_{\mathrm{L}} - u_\sigma}{d_{\mathrm{L}\sigma}} = |\sigma| k_{\mathrm{K}} \frac{u_\sigma - u_{\mathrm{K}}}{d_{\mathrm{K}\sigma}}.$$

Eliminate the fictive unknown u_σ :

$$u_\sigma = \frac{k_{\mathrm{L}} d_{\mathrm{K}\sigma} u_{\mathrm{L}} + k_{\mathrm{K}} d_{\mathrm{L}\sigma} u_{\mathrm{K}}}{k_{\mathrm{L}} d_{\mathrm{K}\sigma} + k_{\mathrm{K}} d_{\mathrm{L}\sigma}}$$

$$\Longrightarrow \quad F_{\mathrm{KL}} = |\sigma| k_\sigma \frac{u_{\mathrm{L}} - u_{\mathrm{K}}}{d_{\mathrm{KL}}}, \text{ with } k_\sigma = \frac{k_{\mathrm{K}} k_{\mathrm{L}} (d_{\mathrm{K}\sigma} + d_{\mathrm{L}\sigma})}{k_{\mathrm{L}} d_{\mathrm{K}\sigma} + k_{\mathrm{K}} d_{\mathrm{L}\sigma}}, \text{ harmonic mean value.}$$

Why the m-DDFV scheme ?

(Boyer & H., SIAM JNA 08)

If φ is discontinuous in z

- The DDFV scheme converges slowly (typically $h^{\frac{1}{2}}$ if $p = 2$).
- The solution $\overline{u}$ **can not** be in $W^{2,p}(\Omega)$.
- The consistency of the normal fluxes is lost.

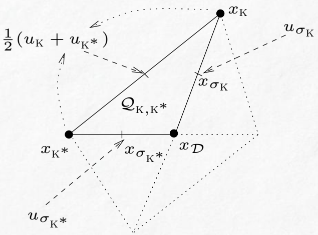
Assume that φ is piecewise Lip.

We can to recover the consistency of the normal fluxes.

The m-DDFV scheme

- $\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}}$ is constant per quarter of diamond cell

$$\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}} = \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} 1_{\mathcal{Q}} \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}},$$



$$\nabla_{\mathcal{Q}_{K,K^*}}^{\mathcal{N}} u^{\mathcal{T}} = \frac{2}{\sin\alpha_{\mathcal{D}}}\left(\frac{u_{\sigma_{K^*}} - \frac{1}{2}(u_K + u_{K^*})}{|\sigma_K|}\boldsymbol{n}_{KL} + \frac{u_{\sigma_K} - \frac{1}{2}(u_K + u_{K^*})}{|\sigma_{K^*}|}\boldsymbol{n}_{K^*L^*}\right)$$

$$\rightsquigarrow \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}} = \nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}}\delta^{\mathcal{D}}, \ \forall \mathcal{Q} \subset \mathcal{D}.$$

- $B_{\mathcal{Q}}$ is a 2×4 matrix depending on geometry.
- $\delta^{\mathcal{D}} = (\delta_K, \delta_L, \delta_{K^*}, \delta_{L^*})^t$ four additional unknowns to be determined by imposing the continuity of the normal fluxes.
- $B_{\mathcal{Q}_{K,K^*}} = \frac{1}{|\mathcal{Q}_{K,K^*}|}(|\sigma_K|\boldsymbol{n}_{K^*L^*}, 0, |\sigma_{K^*}|\boldsymbol{n}_{KL}, 0)$

The new scheme

$$\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \varphi_{\mathcal{Q}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})), \quad (2)$$

$$\varphi_{\mathcal{Q}}(\xi) = \int_{\mathcal{Q}} \varphi(z, \xi)\, d\mu_{\bar{\mathcal{Q}}}(z).$$

FV formulation

$$\begin{aligned} -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathrm{K} \neq \emptyset} |\sigma| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathrm{KL}} \right) &= \int_{\mathrm{K}} f(z)\, dz, \quad \forall \mathrm{K} \in \mathfrak{M} \\ -\sum_{\mathcal{D}_{\sigma,\sigma^*} \cap \mathrm{K}^* \neq \emptyset} |\sigma^*| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \boldsymbol{n}_{\mathrm{K}^*\mathrm{L}^*} \right) &= \int_{\mathrm{K}^*} f(z)\, dz, \ \forall \mathrm{K}^* \in \mathfrak{M}^* \end{aligned} \quad (3)$$

The new scheme

$$\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} |\mathcal{Q}| \varphi_{\mathcal{Q}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}})), \tag{2}$$

$$\varphi_{\mathcal{Q}}(\xi) = \int_{\mathcal{Q}} \varphi(z, \xi)\, d\mu_{\bar{\mathcal{Q}}}(z).$$

Variational formulation :

$$\begin{aligned} 2 \sum_{\mathcal{D} \in \mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}^{\mathcal{N}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}\right) &= 2 \sum_{\mathcal{Q} \in \mathfrak{Q}} |\mathcal{Q}| \left(\varphi_{\mathcal{Q}}(\nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}}), \nabla_{\mathcal{Q}}^{\mathcal{N}} v^{\mathcal{T}}\right) \\ &= \int_{\Omega} f v^{\mathfrak{M}} dz + \int_{\Omega} f v^{\mathfrak{M}^*} dz, \quad \forall v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}. \end{aligned}$$

m-DDFV for linear operator

- If φ **linear** : $\varphi(z,\xi) = A(z)\xi$.
- **constant on primal cells**, $A(z) = A_{\rm K}$ sur $\rm K$.

We obtain the scheme developped by **Hermeline (03)**.

$$(\varphi_{\mathcal{D}}^{\mathcal{N}}\boldsymbol{n},\boldsymbol{n}) = \frac{(|\sigma_{\rm K}|+|\sigma_{\rm L}|)(A_{\rm K}\boldsymbol{n},\boldsymbol{n})(A_{\rm L}\boldsymbol{n},\boldsymbol{n})}{|\sigma_{\rm L}|(A_{\rm K}\boldsymbol{n},\boldsymbol{n})+|\sigma_{\rm K}|(A_{\rm L}\boldsymbol{n},\boldsymbol{n})},$$

$$(\varphi_{\mathcal{D}}^{\mathcal{N}}\boldsymbol{n}^*,\boldsymbol{n}^*) = \frac{|\sigma_{\rm L}|(A_{\rm L}\boldsymbol{n}^*,\boldsymbol{n}^*)+|\sigma_{\rm K}|(A_{\rm K}\boldsymbol{n}^*,\boldsymbol{n}^*)}{|\sigma_{\rm K}|+|\sigma_{\rm L}|} - \frac{|\sigma_{\rm K}||\sigma_{\rm L}|}{|\sigma_{\rm K}|+|\sigma_{\rm L}|}\frac{((A_{\rm K}\boldsymbol{n},\boldsymbol{n}^*)-(A_{\rm L}\boldsymbol{n},\boldsymbol{n}^*))^2}{|\sigma_{\rm L}|(A_{\rm K}\boldsymbol{n},\boldsymbol{n})+|\sigma_{\rm K}|(A_{\rm L}\boldsymbol{n},\boldsymbol{n})},$$

$$(\varphi_{\mathcal{D}}^{\mathcal{N}}\boldsymbol{n},\boldsymbol{n}^*) = \frac{|\sigma_{\rm L}|(A_{\rm L}\boldsymbol{n},\boldsymbol{n}^*)(A_{\rm K}\boldsymbol{n},\boldsymbol{n})+|\sigma_{\rm K}|(A_{\rm K}\boldsymbol{n},\boldsymbol{n}^*)(A_{\rm L}\boldsymbol{n},\boldsymbol{n})}{|\sigma_{\rm L}|(A_{\rm K}\boldsymbol{n},\boldsymbol{n})+|\sigma_{\rm K}|(A_{\rm L}\boldsymbol{n},\boldsymbol{n})}.$$

Bibliography on DDFV in 2D

- Linear case **Omnés et *al* 05'**
- Nonlinear case **Andreianov, Boyer, H. 07'**
- Discontinuous operator **Boyer, H. 08'**

2D extensions

- Extension to Stokes problems **Delcourte et *al* 08'**
- Extension to Stokes problem with variable viscosity**Krell 09', 10'**
- Extension to convection-diffusion operators **Coudière-Manzini 09'**
- Extension to general boundary condition **Boyer, H., Krell 08'**

Comparison of more than 20 schemes

Herbin, H. "2D Anisotropy benchmark", FVCA5 2008

http ://www.latp.univ-mrs.fr/latp_numerique/ ?q=node/3

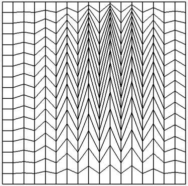
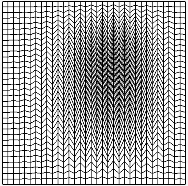
Some numerical results in 2D

Mild anisotropy

$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

with $\mathbf{K} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$ and $u(x,y) = 16x(1-x)y(1-y)$.

In L^2 norm , order = 2. In H^1 norm, order = 1.

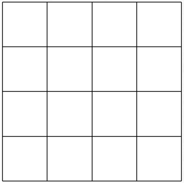
Mesh4_1   Mesh4_2

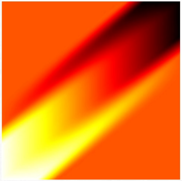
	mesh 4_1		mesh 4_2	
	umin	umax	umin	umax
CMPFA	9.95E-03	1.00E+00	2.73E-03	9.99E-01
DDFV-BHU	1.33E-02	9.96E-01	3.63E-03	9.99E-01
FVSYM	7.34E-03	9.59E-01	2.33E-03	9.89E-01
MFD-BLS	8.54E-03	9.55E-01	2.44E-03	9.87E-01
NMFV	1.30E-02	**1.11E+00**	3.61E-03	**1.04E+00**
SUSHI	7.64E-03	8.88E-01	2.33E-03	9.61E-01

Herbin, H. "2D Anisotropy benchmark"

http ://www.latp.univ-mrs.fr/latp_numerique/ ?q=node/3

Some numerical results in 2D





	umin_i	umax_i	i
CMPFA	6.90E-02	9.31E-01	1
	9.83E-04	9.99E-01	7
DDFV-BHU	**-4.72E-03**	1.00E+00	1
	-5.31E-04	1.00E+00	7
FVSYM	6.85E-02	9.32E-01	1
	4.92E-04	9.99E-01	8
MFD-BLS	6.09E-02	9.39E-01	1
	1.29E-03	9.99E-01	7
MFE	3.12E-02	9.69E-01	1
	5.08E-04	9.99E-01	8
NMFV	1.11e-01	8.88e-01	1
	1.28E-03	9.99E-01	7
SUSHI	6.03E-02	9.40E-01	1
	8.52E-04	9.99E-01	7

$$-\mathrm{div}(\mathbf{K}\nabla u) = 0 \text{ in } \Omega$$

with $\mathbf{K} = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 10^{-3} \end{pmatrix} R_\theta^{-1}$, $\theta = 40^\circ$

Piecewise linear boundary condition $\bar{u}$ on $\partial\Omega$:

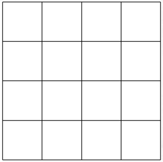
$$\bar{u}(x,y) = \begin{cases} 1 & \text{on } (0,.2)\times\{0.\}\cup\{0.\}\times(0,.2) \\ 0 & \text{on } (.8,1.)\times\{1.\}\cup\{1.\}\times(.8,1.) \\ \frac{1}{2} & \text{on } ((.3,1.)\times\{0\}\cup\{0\}\times(.3,1.) \\ \frac{1}{2} & \text{on } (0.,.7)\times\{1.\}\cup\{1.\}\times(0.,0.7) \end{cases}$$

Herbin, H. “2D Anisotropy benchmark”

http ://www.latp.univ-mrs.fr/latp_numerique/ ?q=node/3

Some numerical results in 2D

Rotating anisotropy



$$-\mathrm{div}(\mathbf{K}\nabla u) = f \text{ in } \Omega$$

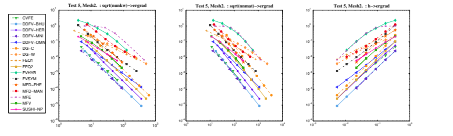
with

$$\mathbf{K} = \frac{1}{(x^2+y^2)}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

for $u(x,y) = \sin \pi x \sin \pi y$.

Maximum principle on a coarse grid

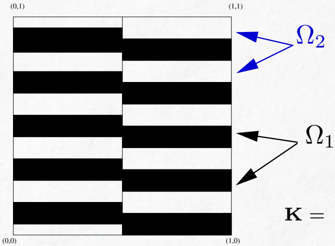
	umin	umax
CMPFA	**-1.06E-01**	**1.09E+00**
DDFV-BHU	0.00E+00	1.00E+00
DG-W	**-7.68E-02**	**1.06E+00**
FEQ1	0.00E+00	**1.05E+00**
MFE	**-1.62E+00**	**1.90E+01**



Herbin, H. "2D Anisotropy benchmark"

http ://www.latp.univ-mrs.fr/latp_numerique/ ?q=node/3

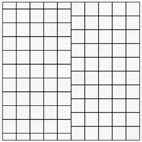
Some numerical results in 2D



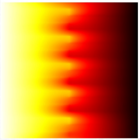
$$-\mathrm{div}(\mathbf{K}\nabla u) = 0 \text{ in } \Omega$$

$$u = \bar{u} = 1 - x \text{ on } \partial\Omega$$

$$\mathbf{K} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ with } \begin{cases} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ in } \Omega_1, \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^{-2} \\ 10^{-3} \end{pmatrix} \text{ in } \Omega_2 \end{cases}$$



	ener1 mesh5	eren mesh5	ener1 mesh5_ref	eren mesh5_ref
DDFV-BHU	42.1	3.65E-02	43.2	1.27E-03
MFD-BLS	33.9	7.93E-14	43.2	2.84E-12
SUSHI	39.1	6.67E-02	43.1	8.88E-04



	flux0 mesh5	flux0 mesh5_ref	flux1 mesh5	flux1 mesh5_ref	fluy0 mesh5	fluy0 mesh5_ref	fluy1 mesh5
DDFV-BHU	-40.0	-42.1	41.8	44.4	-1.81	-2.33	9.08E-04
MFD-BLS	-32.3	-42.1	36.2	44.4	-3.94	-2.33	1.22E-03
SUSHI	-40.9	-42.1	43.1	44.4	-2.21	-2.33	6.94E-04

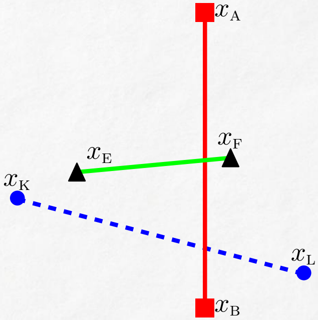
Herbin, H. "2D Anisotropy benchmark"

http ://www.latp.univ-mrs.fr/latp_numerique/ ?q=node/3

Outline

Reconstruction of a gradient en 3D

In 3D, let us consider 6 points of $\mathbb{R}^3$ that define 3 independant directions !

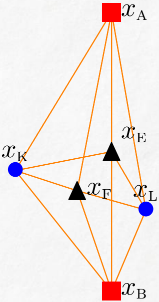


We look for $\nabla^{\mathcal{T}} u^{\mathcal{T}} \in \mathbb{R}^3$ that satisfies

$$\begin{aligned}
\nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_L - x_K) &= u_L - u_K \\
\nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_A - x_B) &= u_A - u_B \\
\nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_F - x_E) &= u_F - ue
\end{aligned}$$

Reconstruction of a gradient in 3D

In 3D, let us consider 6 points of $\mathbb{R}^3$ that define 3 independant directions !



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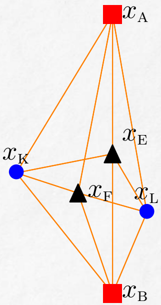
\nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_L - x_K) &= u_L - u_K \\
\nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_A - x_B) &= u_A - u_B \\
\nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_F - x_E) &= u_F - u_E
\end{aligned}$$

We call **diamond cell** the polyhedron whose face are (x_K, x_A, x_F), (x_L, x_A, x_F), (x_K, x_B, x_F), (x_L, x_B, x_F), (x_K, x_A, x_E), (x_L, x_A, x_E), (x_K, x_B, x_E), (x_L, x_B, x_E).

Gradient reconstruction in 3D

▸ Proof

Diamond cells : polyhedron whose faces $\left(\begin{pmatrix} x_K \\ x_L \end{pmatrix}, \begin{pmatrix} x_A \\ x_B \end{pmatrix}, \begin{pmatrix} x_F \\ x_E \end{pmatrix}\right)$.



The unique vector $\nabla^{\mathcal{T}} u^{\mathcal{T}}$ of $\mathbb{R}^3$ that satisfies

$$\begin{aligned} \nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_L - x_K) &= u_L - u_K \\ \nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_A - x_B) &= u_A - u_B \\ \nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_F - x_E) &= u_F - u_E \end{aligned}$$

is given by

$$\nabla^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{3|\mathcal{D}|}\Big((u_L - u_K)N_{KL} + (u_B - u_A)N_{AB} + (u_F - u_E)N_{EF}\Big)$$

where

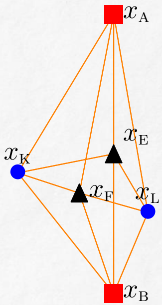
$$|\mathcal{D}| = \frac{1}{6}\det(x_L - x_K, x_A - x_B, x_F - x_E)(> 0), N_{KL} = \frac{1}{2}(x_A - x_B) \wedge (x_F - x_E)$$

$$N_{AB} = \frac{1}{2}(x_F - x_E) \wedge (x_L - x_K),\ N_{EF} = \frac{1}{2}(x_L - x_K) \wedge (x_A - x_B)$$

Gradient reconstruction in 3D

▸ Proof

Diamond cells : polyhedron whose faces $\left(\begin{pmatrix} x_K \\ x_L \end{pmatrix}, \begin{pmatrix} x_A \\ x_B \end{pmatrix}, \begin{pmatrix} x_F \\ x_E \end{pmatrix}\right)$.



The unique vector $\nabla^{\mathcal{T}} u^{\mathcal{T}}$ of $\mathbb{R}^3$ that satisfies

$$\begin{aligned} \nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_L - x_K) &= u_L - u_K \\ \nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_A - x_B) &= u_A - u_B \\ \nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_F - x_E) &= u_F - u_E \end{aligned}$$

is given by

$$\nabla^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{3|\mathcal{D}|}\Big((u_L - u_K)N_{KL} + (u_B - u_A)N_{AB} + (u_F - u_E)N_{EF}\Big)$$

Choice of the points

- We naturally take
  - x_K,x_L centers of two neighbouring cells
  - x_A, x_B two common vertices to K, L
- Several choice available for the points x_F and x_E.

The different strategies

In all the approaches, we take x_K, x_L as centers of two neighbouring cells, and x_A, x_B two common vertices to K and L.

- Unknowns at the centers of the cells and at vertices **(Coudière-Pierre 07')** **(Andreianov and *al* 08')**
  $\Rightarrow x_F, x_E$ are also vertices of the cells
- Unknowns at centers of the cells, at vertices and at the center of the faces **(Hermeline 07')**
  $\Rightarrow x_F$ is the center of the face between K and L, x_E is the center of the edge $x_A x_B \subset \partial F$.
- Unknowns at center of the cells, at the vertices, at the center of the faces, at the centers of the edges. **(Coudière-H. 09')**
  $\Rightarrow x_F$ is the center of the face between K and L, x_E is the center of the edge $x_A x_B \subset \partial F$.

Strategy of the scheme

- Associate to each unknown $u_K, u_L, u_A, u_B, u_F, u_E$, a control volume.

$\Rightarrow$ We get three family of meshes

  - The initial mesh (primal mesh)
  - The mesh associated to the vertices (node mesh)
  - The mesh associated to the couple face/edges (face/edge mesh)

- Integrate the equation to each control volume and use the discrete divergence and the discrete gradient.

$$\begin{aligned} -\operatorname{div}_K \varphi^{\mathfrak{D}}(\nabla^{\mathcal{T}} u^{\mathcal{T}}) &= f_K \\ -\operatorname{div}_A \varphi^{\mathfrak{D}}(\nabla^{\mathcal{T}} u^{\mathcal{T}}) &= f_A \\ -\operatorname{div}_F \varphi^{\mathfrak{D}}(\nabla^{\mathcal{T}} u^{\mathcal{T}}) &= f_F, \quad -\operatorname{div}_E \varphi^{\mathfrak{D}}(\nabla^{\mathcal{T}} u^{\mathcal{T}}) = f_E \\ +BC & \end{aligned}$$

Construction of the three meshes

The primal mesh $\mathfrak{M}$

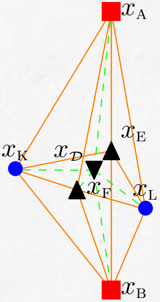
- Definition of K in term of small tetrahedra :

$$\mathrm{K} = \cup_{\mathcal{D}\in\mathfrak{D}_{\mathrm{K}}} \,\mathrm{hull}\left(x_{\mathrm{K}}, x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix}\right)$$

The node mesh $\mathcal{N}$

- Definition of A in term of small tetrahedra :

$$\mathrm{A} = \cup_{\mathcal{D}\in\mathfrak{D}_{\mathrm{K}}} \,\mathrm{hull}\left(x_{\mathrm{A}}, x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix}\right)$$



The face/edge mesh $\mathcal{FE}$

- Definition of F/E in term of small tetrahedra :

$$\mathrm{F} = \cup_{\mathcal{D}\in\mathfrak{D}_{x_{\mathrm{F}}}} \,\mathrm{hull}\left(x_{\mathrm{F}}, x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}\right)$$

$$\mathrm{E} = \cup_{\mathcal{D}\in\mathfrak{D}_{x_{\mathrm{E}}}} \,\mathrm{hull}\left(x_{\mathrm{E}}, x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}\right)$$

Construction of the three meshes

The primal mesh $\mathfrak{M}$

- The faces of K included in the diamond $\mathcal{D}$:

$$\cup \left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix} \right)$$

The node mesh $\mathcal{N}$

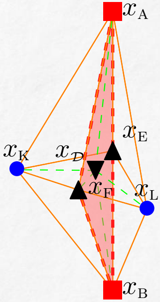
- The faces of A included in $\mathcal{D}$:

$$\cup \left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix} \right)$$

The face/edge mesh $\mathcal{FE}$

- The faces of F or E included in $\mathcal{D}$:

$$\cup \left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix} \right)$$



Construction of the three meshes

The primal mesh $\mathfrak{M}$

- The faces of K included in the diamond $\mathcal{D}$:

$$\cup \left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix} \right)$$

The node mesh $\mathcal{N}$

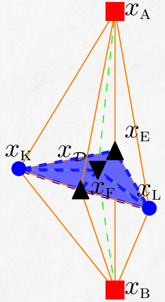
- The faces of A included in $\mathcal{D}$:

$$\cup \left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix} \right)$$

The face/edge mesh $\mathcal{FE}$

- The faces of F or E included in $\mathcal{D}$:

$$\cup \left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix} \right)$$



Construction of the three meshes

The primal mesh $\mathfrak{M}$

- The faces of K included in the diamond $\mathcal{D}$:

$$\cup\left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix}\right)$$

The node mesh $\mathcal{N}$

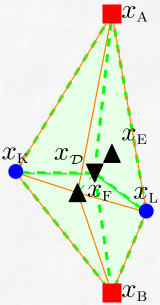
- The faces of A included in $\mathcal{D}$:

$$\cup\left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{E}} \\ x_{\mathrm{F}} \end{pmatrix}\right)$$

The face/edge mesh $\mathcal{FE}$

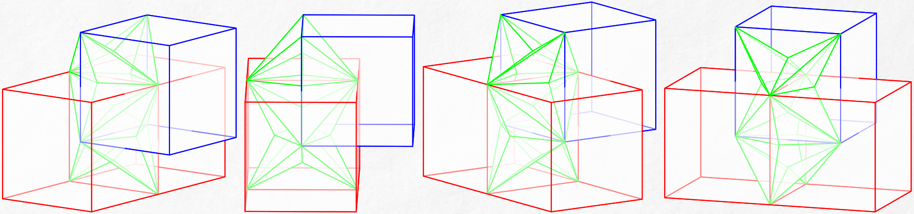
- The faces of F or E included in $\mathcal{D}$:

$$\cup\left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}\right)$$

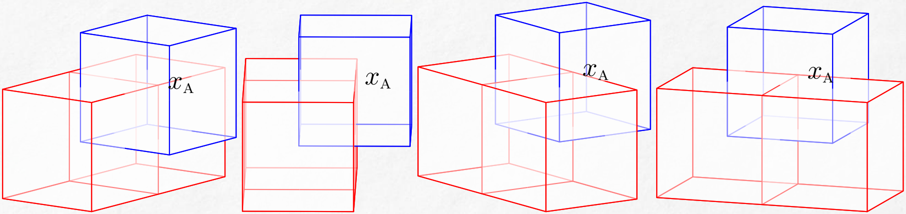


Example of cubic meshes

The three meshes

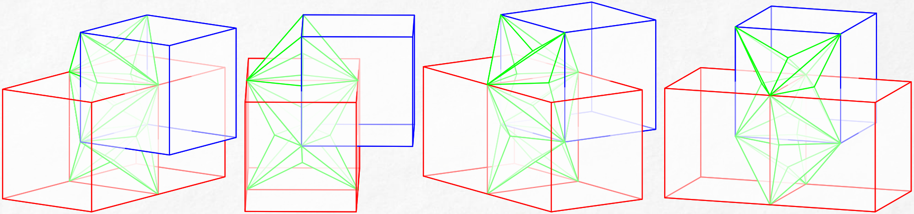


The primal mesh and the control volume A

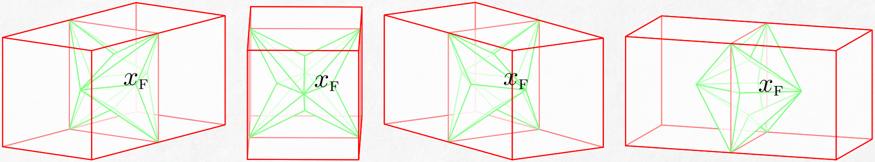


Example of cubic meshes

The three meshes

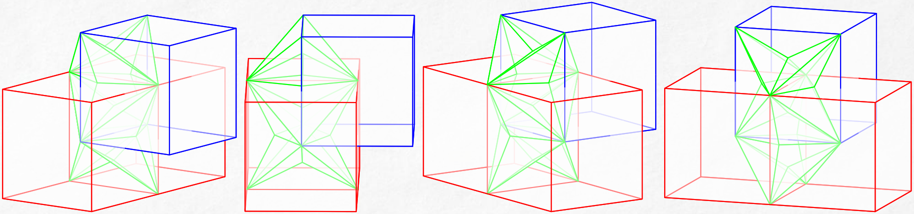


The primal mesh and the control volume F

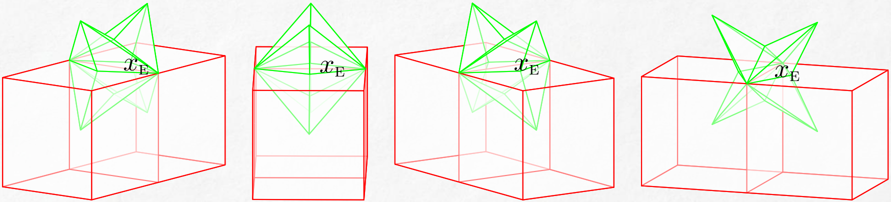


Example of cubic meshes

The three meshes

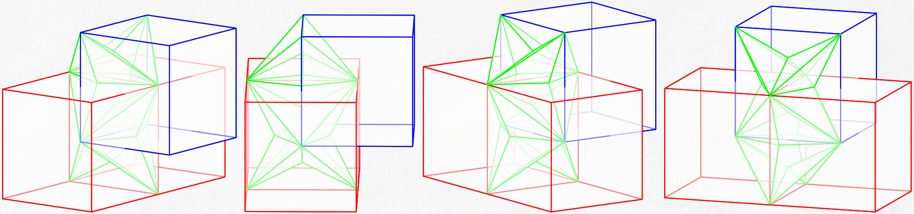


The primal mesh and the control volume E

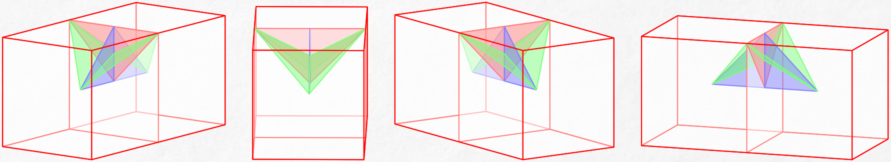


Example of cubic meshes

The three meshes



The faces of the three meshes included in a diamond cell



The discrete operators

The scheme

$$-\operatorname{div}^{\mathcal{T}}(\varphi^{\mathfrak{D}}(\nabla^{\mathcal{T}} u^{\mathcal{T}})) = f^{\mathcal{T}}$$

The discrete gradient

$$\forall \mathcal{D} \in \mathfrak{D}, \quad \nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{3|\mathcal{D}|}\left((u_{\mathrm{L}} - u_{\mathrm{K}})N_{\mathrm{KL}} + (u_{\mathrm{B}} - u_{\mathrm{A}})N_{\mathrm{AB}} + (u_{\mathrm{F}} - u_{\mathrm{E}})N_{\mathrm{EF}}\right).$$

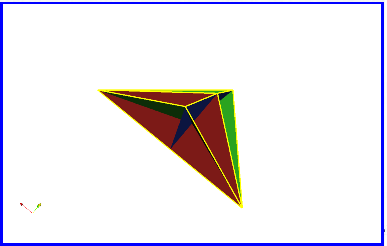
The discrete divergence

$$\operatorname{div}^{\mathcal{T}} = \Big((\operatorname{div}_{\mathrm{K}})_{\mathrm{K}\in\mathfrak{M}}, (\operatorname{div}_{\mathrm{A}})_{\mathrm{A}\in\mathcal{N}}, (\operatorname{div}_{\mathrm{E}}, \operatorname{div}_{\mathrm{F}})_{\mathrm{E},\mathrm{F}\in\mathcal{FE}}\Big)$$

$$\mathrm{K}\operatorname{div}_{\mathrm{K}}\xi^{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathcal{D}_{\mathrm{K}}} \xi_{\mathcal{D}} N_{\mathrm{KL}}, \quad |\mathrm{A}|\operatorname{div}_{\mathrm{A}}\xi^{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathcal{D}_{\mathrm{A}}} \xi_{\mathcal{D}} N_{\mathrm{AB}},$$

$$|\mathrm{E}|\operatorname{div}_{\mathrm{E}}\xi^{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathcal{D}_{\mathrm{E}}} \xi_{\mathcal{D}} N_{\mathrm{EF}}, \ |\mathrm{F}|\operatorname{div}_{\mathrm{F}}\xi^{\mathfrak{D}} = -\sum_{\mathcal{D}\in\mathcal{D}_{\mathrm{F}}} \xi_{\mathcal{D}} N_{\mathrm{EF}}$$

$$\begin{aligned}
N_{\mathrm{KL}} &= \frac{1}{2}(x_{\mathrm{B}} - x_{\mathrm{A}}) \times (x_{\mathrm{F}} - x_{\mathrm{E}}) = \int_{\bar{\mathrm{K}}\cap\bar{\mathrm{L}}\cap\mathcal{D}} n_{\mathrm{KL}}\, ds \\
N_{\mathrm{AB}} &= \frac{1}{2}(x_{\mathrm{F}} - x_{\mathrm{E}}) \times (x_{\mathrm{L}} - x_{\mathrm{K}}) = \int_{\bar{\mathrm{A}}\cap\bar{\mathrm{B}}\cap\mathcal{D}} n_{\mathrm{AB}}\, ds \\
N_{\mathrm{EF}} &= \frac{1}{2}(x_{\mathrm{L}} - x_{\mathrm{K}}) \times (x_{\mathrm{B}} - x_{\mathrm{A}}) = \int_{\bar{\mathrm{E}}\cap\bar{\mathrm{F}}\cap\mathcal{D}} n_{\mathrm{EF}}\, ds
\end{aligned}$$



The discrete duality

$\Rightarrow$ For homogeneous Dirichlet conditions

$$-[\![\mathrm{div}^{\mathcal{T}}\,\xi^{\mathfrak{D}}, u^{\mathcal{T}}]\!] = \sum_{\mathcal{D}\in\mathfrak{D}} |_{\mathcal{D}}|\xi^{\mathcal{D}}\nabla^{\mathcal{T}}u^{\mathcal{T}}$$

with

$$[\![u^{\mathcal{T}}, v^{\mathcal{T}}]\!] = \frac{1}{3}\left(\sum_{\mathrm{K}\in\mathfrak{M}} |\mathrm{K}|u_{\mathrm{K}}v_{\mathrm{K}} + \sum_{\mathrm{A}\in\mathcal{N}} |\mathrm{A}|u_{\mathrm{A}}v_{\mathrm{A}} + \sum_{x_{\mathrm{F}}\in\mathcal{F}} |\mathrm{F}|u_{\mathrm{F}}v_{\mathrm{F}} + \sum_{x_{\mathrm{E}}\in e^{\mathcal{T}}} |\mathrm{E}|u_{\mathrm{E}}v_{\mathrm{E}}\right)$$

Remarks on the implementation

- Neither $\mathcal{N}$ nor $\mathcal{FE}$ need to be constructed.
- We just require a diamond cell structure that contains
  - The reference to the points $x_{\text{A}}, x_{\text{B}}, x_{\text{K}}, x_{\text{L}}, x_{\text{E}}, x_{\text{F}}$.
  - The values $N_{\text{KL}}, N_{\text{AB}}, N_{\text{EF}}$.
  - The measure of 8 tetrahedrons : $(x_{\mathcal{D}}, x_{\text{K}}, x_{\text{A}}, x_{\text{F}})$,$(x_{\mathcal{D}}, x_{\text{K}}, x_{\text{A}}, x_{\text{E}})$, ...
- In the linear case $\varphi(x,\xi) = \mathbf{K}(x)\xi$, the matrix of the system is made of terms like $\mathbf{K}^{\mathcal{D}} N_{\text{KL}} \cdot N_{\text{KL}}$, $\mathbf{K}^{\mathcal{D}} N_{\text{AB}} \cdot N_{\text{KL}}, \cdots$
- Source terms are evaluated diamond cell by diamond cell thanks to the measure of the 8 tetrahedrons $(x_{\mathcal{D}}, x_{\text{K}}, x_{\text{A}}, x_{\text{F}})$,$(x_{\mathcal{D}}, x_{\text{K}}, x_{\text{A}}, x_{\text{E}})$, ...

Properties of the scheme

The scheme

$$-\operatorname{div}^{\mathcal{T}}(\varphi^{\mathfrak{D}}(\nabla^{\mathcal{T}} u^{\mathcal{T}})) = f^{\mathcal{T}}$$

- Number of unknowns = # control volumes +# interior vertices + # interior faces + # interior edges.
- Monotonicity and coercivity preserved.
- Existence and uniqueness.
- Variationnal structure preserved if $\varphi = \nabla_\xi \Phi$.
- Convergence for $f \in L^{p'}(\Omega)$.
- Error estimates in h^{p-1} if $p \geq 2$, as soon as $u_e \in W^{2,p}(\Omega)$.

Properties of the scheme

The "good points"

- Easy implementation.
- Theoritical robustness.
- Works on general meshes !

The "bad points"

- Large number of unknowns.

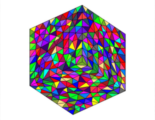
Extension

- Natural extension to the discontinuous case !
  - Discrete gradient are constructed by addind a fictitious unknowns on some point $x_{\mathcal{D}} \in \mathrm{F}$.
  - This fictitious unknown is then eliminated thanks to the continuity of the normal flux on the face F.
- Extension to convection diffusion problems (work in progress with Y. Coudière and G. Manzini)
- Extension to stokes problems (S. Krell and G. Manzini)

Numerical examples

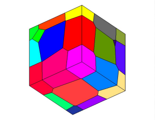
⇝ Extracted from the 3D anisotropic benchmark
`http://www.latp.univ-mrs.fr/latp_numerique/`

- Contribution of R. Eymard, G. Henry, R. Herbin, R. Kloefkorn, G. Manzini, ...
- Last issue for FVCA6 june 2011 in Praha.



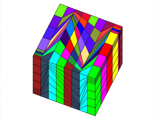
Tetraedric mesh

G. Manzini - tetgen



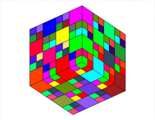
Voronoï mesh

G. Manzini



Kershaw mesh

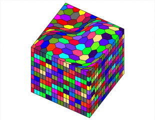
K. Lipnikov



Checkerboard mesh

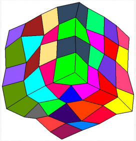
S. Minjeaud -

PELICANS IRSN



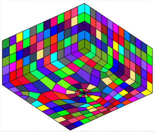
Prism mesh

G. Manzini



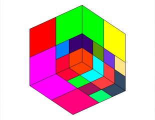
Random mesh

C. Guichard - IFP



Well mesh

J. Brac - IFP



Locally refined mesh

S. Minjeaud -

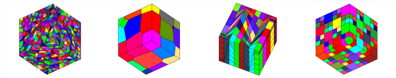
PELICANS IRSN

Numerical results

Test 1 : mild anisotropy

$$-\operatorname{div}(\mathbf{K}(x,y,z)\nabla u) = f + DirichletBC$$

$$\mathbf{K}(x,y,z) = \begin{pmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix}$$



Tetra **Voronoi** **Kershaw** **Checkerboard**

$$u(x,y,z) = 1 + \sin(\pi x)\sin\left(\pi\left(y+\frac{1}{2}\right)\right)\sin\left(\pi\left(z+\frac{1}{3}\right)\right)$$

Rate of convergence

	tetra mesh	voronoi mesh	kershaw mesh	checkerboard mesh
L^2 norm	2.02	1.65	1.73	1.90
H^1 norm	1.02	1.01	1.243	0.923

Maximum principle (reference values $\min(u) = 0$, $\max(u) = 2$)

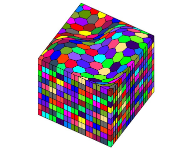
	Tetra Mesh		Checkerboard Mesh	
#	$u_{\min}$	$u_{\max}$	$u_{\min}$	$u_{\max}$
Coarse grid	0.017	2.000	0.015	2.000
Fine grid	0.001	2.000	0.003	2.000

Numerical results

Test 2 : Anisotropy and heterogeneity

Lipnikov

$$-\operatorname{div}(\mathbf{K}(x,y,z)\nabla u) = f + DirichletBC$$



Prism mesh

$$\mathbf{K}(x,y,z) = \begin{pmatrix} y^2+z^2+1 & -xy & -xz \\ -xy & x^2+z^2+1 & -yz \\ -xz & -yz & x^2+y^2+1 \end{pmatrix}$$

$$u(x,y,z) = x^3y^2z + x\sin(2\pi xz)\sin(2\pi xy)\sin(2\pi z)$$

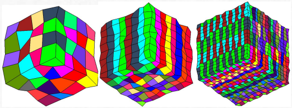
	Prism Mesh			
#	$\|\cdot\|_2$	Rate	$\|\cdot\|_{H_1}$	Rate
1	0.39e-01	–	0.81e-01	–
2	0.11e-02	1.85	0.39e-01	1.05
3	0.5e-02	1.91	0.25e-01	1.04

Numerical results

Test 3 : Flow on random meshes

IFP

$$-\operatorname{div}(\mathbf{K}(x,y,z)\nabla u) = f + DirichletBC$$



Random meshes

$$\mathbf{K}(x,y,z) = \begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & \varepsilon \end{pmatrix} \text{ with } \varepsilon = 10^2 \text{ or } \varepsilon = 10^3$$

$$u(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$$

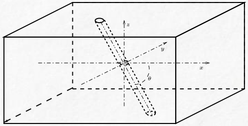
Rate of convergence : 2.06 in L^2 norm and 0.983 in H^1 norm.
Maximum principle : on Random32 mesh $u_{\min} = -1.19$ and $u_{\max} = 1.16$.
No problem on the primal or node cells.

Numerical results

Test 4 : Flow around a well

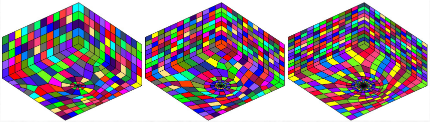
I. Aavatsmark and R.A. Klausen, SPE Journal, 2003

$$-\operatorname{div}(\mathbf{K}(x,y,z)\nabla u) = f + DirichletBC$$



The domain $\Omega =]-15,15[\times]-15,15[\times]-7.5,7.5[\backslash W$

W **a slanted circular cylindar with radius** $r_w = 0.1$

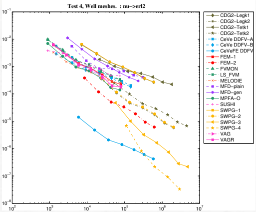


Well meshes (J. Brac, IFP)

$$\mathbf{K}(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix} \text{ with } \tau = 0.2$$

- **Source term :** $f = 0.$
- **Explicit analytical solution** s.t.

$$u(x,y,z) = 0 \text{ on } \partial W$$



Outline

Conclusions

- We can obtain all classical theoritical results for DDFV scheme.
- The implementation of the scheme is easy even if the description of the differents meshes may looks scary.
- DDFV methods are particularly precise as far the gradients are concerned !

Conclusion

Thanks for your attention !

Edge structure in matlab

Return

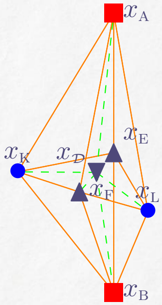
- Create the edges going through volume information (t)
  temp(1, :)=[t(1, :) t(1,1+nbt :2nbt)=t(2, :) t(1,1+2nbt :3nbt)=t(3, :) f(1, :)]
  temp(2, :)=[t(2, :) t(1,1+nbt :2nbt)=t(3, :) t(1,1+2nbt :3nbt)=t(1, :) f(2, :)]
  temp(3, :)=[1 :nbt 1 :nbt 1 :nbt 0*[1 :nbf]]
  Each edge appears twice in the table temp
- Eventually exchange temp(1,i) and temp(2,i) in such a way that temp(1,i)¡temp(2,i)
- Finally, use your favorite sort algorithm
  temp → tempnew
  with
  tempnew(1, :)= [P1(1) P1(1) P1(2) P1(2).....]
  tempnew(2, :)= [P2(1) P2(1) P2(2) P2(2).....]
  tempnew(3, :)= [_L(1) _K(1) _L(2) _K(2).....]

The discrete gradient in 3D (Proof)

- Volume of the diamond cell

  Case of convex diamond cell $\mathcal{D}$. Let $x_{\mathcal{D}} \in \mathring{\mathcal{D}}$.

  The polyhedron $\mathcal{D}$ is the union of 8 tetrahedron $\left(x_{\mathcal{D}}, \begin{pmatrix} x_{\mathrm{K}} \\ x_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{A}} \\ x_{\mathrm{B}} \end{pmatrix}, \begin{pmatrix} x_{\mathrm{F}} \\ x_{\mathrm{E}} \end{pmatrix}\right)$.



> The volume of the tetrahedron (A_0, A_1, A_2, A_3) is equal to
>
> $$\pm\frac{1}{6}\det(A_1 - A_0, A_2 - A_0, A_3 - A_0)$$
>
> depending on the orientation of the points.

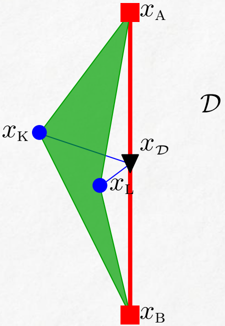
So,

$$\begin{aligned} 6|\mathcal{D}| &= \det(x_{\mathrm{L}} - x_{\mathcal{D}}, x_{\mathrm{A}} - x_{\mathcal{D}}, x_{\mathrm{F}} - x_{\mathcal{D}}) - \det(x_{\mathrm{K}} - x_{\mathcal{D}}, x_{\mathrm{A}} - x_{\mathcal{D}}, x_{\mathrm{F}} - x_{\mathcal{D}}) \\ &\quad \det(x_{\mathrm{L}} - x_{\mathcal{D}}, x_{\mathrm{B}} - x_{\mathcal{D}}, x_{\mathrm{F}} - x_{\mathcal{D}}) - \det(x_{\mathrm{K}} - x_{\mathcal{D}}, x_{\mathrm{B}} - x_{\mathcal{D}}, x_{\mathrm{F}} - x_{\mathcal{D}}) \\ &\quad -\det(x_{\mathrm{L}} - x_{\mathcal{D}}, x_{\mathrm{A}} - x_{\mathcal{D}}, x_{\mathrm{E}} - x_{\mathcal{D}}) + \det(x_{\mathrm{K}} - x_{\mathcal{D}}, x_{\mathrm{A}} - x_{\mathcal{D}}, x_{\mathrm{E}} - x_{\mathcal{D}}) \\ &\quad -\det(x_{\mathrm{L}} - x_{\mathcal{D}}, x_{\mathrm{B}} - x_{\mathcal{D}}, x_{\mathrm{E}} - x_{\mathcal{D}}) + \det(x_{\mathrm{K}} - x_{\mathcal{D}}, x_{\mathrm{B}} - x_{\mathcal{D}}, x_{\mathrm{E}} - x_{\mathcal{D}}) \\ &= \det(x_{\mathrm{L}} - x_{\mathrm{K}}, x_{\mathrm{A}} - x_{\mathrm{B}}, x_{\mathrm{F}} - x_{\mathrm{E}}) \end{aligned}$$

The discrete gradient in 3D (Proof)

- Volume of the diamond cell
  Non convex case : same demonstration but the measure of the tetra can be non positive !
  In 2D, in the non convex case we had :

$$2|\mathcal{D}| = \underbrace{\mathcal{A}rea(x_{\mathcal{D}}, x_{\mathrm{K}}, x_{\mathrm{A}})}_{\geq 0} + \underbrace{\mathcal{A}rea(x_{\mathcal{D}}, x_{\mathrm{K}}, x_{\mathrm{B}})}_{\geq 0} + \underbrace{\mathcal{A}rea(x_{\mathcal{D}}, x_{\mathrm{L}}, x_{\mathrm{A}})}_{\leq 0} + \underbrace{\mathcal{A}rea(x_{\mathcal{D}}, x_{\mathrm{L}}, x_{\mathrm{B}})}_{\leq 0}$$



The discrete gradient in 3D (Proof)

Return

- Definition of the gradient

If

$$\nabla^{\mathcal{T}} u^{\mathcal{T}} = \frac{1}{3|\mathcal{D}|}\left((u_{\mathrm{L}} - u_{\mathrm{K}})N_{\mathrm{KL}} + (u_{\mathrm{A}} - u_{\mathrm{B}})N_{\mathrm{AB}} + (u_{\mathrm{F}} - u_{\mathrm{E}})N_{\mathrm{FE}}\right)$$

with

$$|\mathcal{D}| = \frac{1}{6}\det(x_{\mathrm{L}} - x_{\mathrm{K}}, x_{\mathrm{A}} - x_{\mathrm{B}}, x_{\mathrm{F}} - x_{\mathrm{E}})(> 0),\ N_{\mathrm{KL}} = \frac{1}{2}(x_{\mathrm{A}} - x_{\mathrm{B}}) \wedge (x_{\mathrm{F}} - x_{\mathrm{E}})$$

$$N_{x_{\mathrm{A}}x_{\mathrm{B}})} = \frac{1}{2}(x_{\mathrm{F}} - x_{\mathrm{E}}) \wedge (x_{\mathrm{L}} - x_{\mathrm{K}}),\ N_{\mathrm{EF}} = \frac{1}{2}(x_{\mathrm{L}} - x_{\mathrm{K}}) \wedge (x_{\mathrm{A}} - x_{\mathrm{B}})$$

then

$$\nabla^{\mathcal{T}} u^{\mathcal{T}} \cdot (x_{\mathrm{L}} - x_{\mathrm{K}}) = \frac{1}{3|\mathcal{D}|}\Bigg((u_{\mathrm{L}} - u_{\mathrm{K}})\underbrace{N_{\mathrm{KL}} \cdot (x_{\mathrm{L}} - x_{\mathrm{K}})}_{=3|\mathcal{D}|}$$

$$+(u_{\mathrm{A}} - u_{\mathrm{B}})\underbrace{N_{\mathrm{AB}} \cdot (x_{\mathrm{L}} - x_{\mathrm{K}})}_{=0} + (u_{\mathrm{F}} - u_{\mathrm{E}})\underbrace{N_{\mathrm{FE}} \cdot (x_{\mathrm{L}} - x_{\mathrm{K}})}_{=0}\Bigg)$$

Existence and uniqueness

Return

2D case, Homogeneous BC

Properties of the mapping

$a_{\mathcal{T}} : u^{\mathcal{T}} \in X \mapsto -\operatorname{div}^{\mathcal{T}}(\varphi_{\mathcal{T}}(\nabla^{\mathcal{T}} u^{\mathcal{T}})) - f^{\mathcal{T}} = \mathfrak{a}(u^{\mathcal{T}}) - f^{\mathcal{T}} \in X$:

$$[\![\mathfrak{a}(u^{\mathcal{T}}), v^{\mathcal{T}}]\!] = \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}(\nabla^{\mathcal{T}}_{\mathcal{D}} u^{\mathcal{T}}), \nabla^{\mathcal{T}}_{\mathcal{D}} v^{\mathcal{T}}\right)$$

where $[\![u^{\mathcal{T}}, v^{\mathcal{T}}]\!] = \frac{1}{2}\sum_{\mathrm{K}} |\mathrm{K}| u_{\mathrm{K}} v_{\mathrm{K}} + \frac{1}{2}\sum_{\mathrm{K}^*} |\mathrm{K}^*| u_{\mathrm{K}^*} v_{\mathrm{K}^*}$.

- Continuity $((\mathcal{H}_2))$

$$\|\mathfrak{a}(u^{\mathcal{T}}) - f^{\mathcal{T}}\| \leq C_1 \|\nabla^{\mathcal{T}} u^{\mathcal{T}}\|^{p-1} + C(f)$$

- Coercitivity $((\mathcal{H}_1))$

$$[\![\mathfrak{a}(u^{\mathcal{T}}) - f^{\mathcal{T}}, u^{\mathcal{T}}]\!] \geq C_2 \|\nabla^{\mathcal{T}} u^{\mathcal{T}}\|^{p} - C(f)$$

- Monotony $((\mathcal{H}_3))$

$$[\![\mathfrak{a}(u^{\mathcal{T}}) - \mathfrak{a}(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}}]\!] > 0, \text{ if } u^{\mathcal{T}} \neq v^{\mathcal{T}}$$

$\Rightarrow$ Existence by Minty Browder theorem.
$\Rightarrow$ Uniqueness thanks to strict monotonicity

Convergence theorem

A priori estimate (Coercivity + Poincaré lemma)

$$\|\nabla^{\mathcal{T}^n} u^{\mathcal{T},n}\|_{L^p} \text{ is bounded}$$

Compacity theorem

Theorem

Let $u^{\mathcal{T},n}$ defined on $\mathcal{T}_n$ such as $\mathrm{reg}(\mathcal{T}^n)$ is bounded and h^n tends towards 0. If $\|\nabla^{\mathcal{T}^n} u^{\mathcal{T},n}\|_{L^p}$ is bounded, then there exists $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} u^{\mathcal{T},n} &\xrightarrow[n\to\infty]{} u \quad \textit{in } L^p(\Omega) \\ \nabla^{\mathcal{T}^n} u^{\mathcal{T},n} &\xrightarrow[n\to\infty]{} \nabla u \ \textit{weakly in } \left(L^p(\Omega)\right)^2 \end{aligned}$$

up to a subsequence.

- We still need to prove that u is solution of $-\mathrm{div}(\varphi(.,\nabla u)) = f$ and the strong convergence of the gradient.

Convergence theorem

◂ Return

The Minty trick

- First step :

$$\varphi^{\mathcal{T}}(\nabla^{\mathcal{T}^n} u^{\mathcal{T},n}) \xrightarrow{n\to\infty} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2$$

- Second step : for $v^{\mathcal{T},n} = \mathbb{P}^{\mathcal{T}}\theta$, $\theta \in \mathcal{C}_0^\infty(\Omega)$

$$\llbracket \mathfrak{a}(u^{\mathcal{T}}), v^{\mathcal{T}} \rrbracket = \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}\right) = \frac{1}{2}\int_\Omega f(v^{\mathfrak{M}_n} + v^{\mathfrak{M}^*_n})$$

- One can pass to the limit and get

$$\int_\Omega (\zeta, \nabla\theta)\, dx = \int_\Omega f\theta\, dx,\ \forall \theta \in \mathcal{C}_c^\infty(\Omega) \text{ thus } \int_\Omega (\zeta, \nabla v)\, dx = \int_\Omega f v\, dx,\ \forall v \in W_0^{1,p}(\Omega)$$

- We use the monotonicity

$$\llbracket \mathfrak{a}(u^{\mathcal{T}}) - \mathfrak{a}(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}} \rrbracket \geq 0$$

and the fact

$$\llbracket \mathfrak{a}(u^{\mathcal{T}}), u^{\mathcal{T}} \rrbracket = \frac{1}{2}\int_\Omega f(u^{\mathfrak{M}_n} + u^{\mathfrak{M}^*_n}) \to \int_\Omega f u$$

Finally

$$\int_\Omega f u\, dx - \int_\Omega (\zeta, \nabla\theta) - \int_\Omega (\varphi(\nabla\theta), \nabla u - \nabla\theta)\, dx \geq 0$$

Convergence theorem

◂ Return

The Minty trick

- First step :

$$\varphi^{\mathcal{T}}(\nabla^{\mathcal{T}^n} u^{\mathcal{T},n}) \xrightarrow{n\to\infty} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2$$

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$$[\![\mathfrak{a}(u^{\mathcal{T}}), u^{\mathcal{T}}]\!] = \frac{1}{2}\int_\Omega f(u^{\mathfrak{M}_n} + u^{\mathfrak{M}^*_n}) \to \int_\Omega f u$$

Finally

$$\int_\Omega (\zeta - \varphi(\nabla\theta), \nabla u - \nabla\theta)\,dx \geq 0\ \forall\theta\in\mathcal{C}_c^\infty(\Omega)$$

Convergence theorem

Return

The Minty trick

- First step :

$$\varphi^{\mathcal{T}}(\nabla^{\mathcal{T}^n} u^{\mathcal{T},n}) \xrightarrow{n\to\infty} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2$$

- Second step : for $v^{\mathcal{T},n} = \mathbb{P}^{\mathcal{T}}\theta$, $\theta \in \mathcal{C}_0^\infty(\Omega)$

$$\left[\!\left[\mathfrak{a}(u^{\mathcal{T}}), v^{\mathcal{T}}\right]\!\right] = \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}}\right) = \frac{1}{2}\int_\Omega f(v^{\mathfrak{M}_n} + v^{\mathfrak{M}^*{}_n})$$

- One can pass to the limit and get

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- We use the monotonicity

$$\left[\!\left[\mathfrak{a}(u^{\mathcal{T}}) - \mathfrak{a}(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}}\right]\!\right] \geq 0$$

and the fact

$$\left[\!\left[\mathfrak{a}(u^{\mathcal{T}}), u^{\mathcal{T}}\right]\!\right] = \frac{1}{2}\int_\Omega f(u^{\mathfrak{M}_n} + u^{\mathfrak{M}^*{}_n}) \to \int_\Omega f u$$

Finally

$$\int_\Omega (\zeta - \varphi(\nabla v), \nabla u - \nabla v)\, dx \geq 0\ \forall v\in W^{1,p}(\Omega)$$

Convergence theorem

Return

The Minty trick

- First step :

$$\varphi^{\mathcal{T}}(\nabla^{\mathcal{T}^n} u^{\mathcal{T},n}) \xrightarrow{n\to\infty} \zeta \text{ weakly in } \left(L^{p'}(\Omega)\right)^2$$

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$$\left[\!\left[\mathfrak{a}(u^{\mathcal{T}}), v^{\mathcal{T}} \right]\!\right] = \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \left(\varphi_{\mathcal{D}}(\nabla_{\mathcal{D}}^{\mathcal{T}} u^{\mathcal{T}}), \nabla_{\mathcal{D}}^{\mathcal{T}} v^{\mathcal{T}} \right) = \frac{1}{2}\int_\Omega f(v^{\mathfrak{M}n} + v^{\mathfrak{M}^*n})$$

- One can pass to the limit and get

$$\int_\Omega (\zeta, \nabla\theta)\, dx = \int_\Omega f\theta\, dx,\ \forall \theta \in \mathcal{C}_c^\infty(\Omega) \text{ thus } \int_\Omega (\zeta, \nabla v)\, dx = \int_\Omega f v\, dx,\ \forall v \in W_0^{1,p}(\Omega)$$

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$$\left[\!\left[\mathfrak{a}(u^{\mathcal{T}}) - \mathfrak{a}(v^{\mathcal{T}}), u^{\mathcal{T}} - v^{\mathcal{T}} \right]\!\right] \geq 0$$

  and the fact

$$\left[\!\left[\mathfrak{a}(u^{\mathcal{T}}), u^{\mathcal{T}} \right]\!\right] = \frac{1}{2}\int_\Omega f(u^{\mathfrak{M}n} + u^{\mathfrak{M}^*n}) \to \int_\Omega f u$$

  Finally

$$\int_\Omega (\zeta - \varphi(\nabla v), \nabla u - \nabla v)\, dx \geq 0\ \forall v \in W^{1,p}(\Omega)$$

  Conclusion with $v = u + t\psi$ and $t \to 0$.

Error estimates

Consistency error

$$R_{\mathcal{D}}(z) = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \left(\varphi(z', \nabla_{\mathcal{D}}^{\mathcal{T}}\mathbb{P}^{\mathcal{T}}\overline{u}) - \varphi(z, \nabla\overline{u}(z))\right) dz'$$

can be split into

$$R_{\mathcal{D}}(z) = R_{\mathcal{D}}^{\text{grad}} + R_{\mathcal{D}}^{\varphi}(z),$$

with

$$R_{\mathcal{D}}^{\text{grad}} = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \left(\varphi\left(z', \nabla_{\mathcal{D}}^{\mathcal{T}}\mathbb{P}^{\mathcal{T}}\overline{u}\right) - \varphi\left(z', \nabla\overline{u}(z')\right)\right) dz'$$

$$R_{\mathcal{D}}^{\varphi}(z) = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \left(\varphi\left(z', \nabla\overline{u}(z')\right) - \varphi\left(z, \nabla\overline{u}(z)\right)\right) dz'$$

- The consistency error of the flux $R_{\mathcal{D}}(u)$ is also controlled by the consistency of the gradient by use of $(\mathcal{H}_4)$, $(\mathcal{H}_5)$.
- The consistency of the gradient reads

$$\|\nabla v - \nabla^{\mathcal{T}} v^{\mathcal{T}}\|_{L^p(\Omega)} \leq Ch\|v\|_{W^{2,p}(\Omega)}, \; \|\nabla^{\mathcal{T}} v^{\mathcal{T}}\|_{L^p(\Omega)} \leq C\|v\|_{W^{2,p}(\Omega)}$$

Error estimates

Return

$$-\operatorname{div}_{\mathcal{T}}\left(\varphi_{\mathcal{T}}(\nabla^{\mathcal{T}} u^{\mathcal{T}}) - \varphi_{\mathcal{T}}(\nabla^{\mathcal{T}} \mathbb{P}^{\mathcal{T}} \overline{u})\right) = -\operatorname{div}_{\mathcal{T}}\left(R_{\mathcal{D}}(u)\right),$$

- Assumption $(\mathcal{H}_3)$ yields to

$$(c_3)^{p'} \int_\Omega |\nabla^{\mathcal{T}}(\mathbb{P}^{\mathcal{T}}\overline{u} - u^{\mathcal{T}})|^p \le \int_\Omega |R_{\mathcal{T}}(u)|^{p'}$$

- $\int_\Omega |\nabla^{\mathcal{T}}(\mathbb{P}^{\mathcal{T}}\overline{u} - u^{\mathcal{T}})|^p$ controls

$$\|\overline{u} - u^{\mathfrak{M}}\|_{L^p} + \|\overline{u} - u^{\mathfrak{M}^*}\|_{L^p} + \|\nabla\overline{u} - \nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^p}$$

thanks to Poincaré inequality and consistency of the gradient.