A second-order cut-cell method for the numerical simulation of 2D flows past obstacles

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Summary

Context

Numerical simulation of incompressible fluid flows around obstacles

Conclusion and prospects

Summary

Context

N-S eq. for incompressible fluid flows

Time discretization

M.A.C. Scheme

Numerical simulation of incompressible fluid flows around obstacles

Conclusion and prospects

Navier-Stokes equations for incompressible fluid flow

Given $\mathbf{x} \in \Omega \subset \mathbb{R}^2$ and $t > 0$.
Velocity and pressure fields $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^2$, $p = p(\mathbf{x}, t) \in \mathbb{R}$ are solution of

$$
\begin{aligned}
&\partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) - \triangle \mathbf{u}/Re + \nabla p = \mathbf{f} \text{ in } \Omega, \\
&\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \\
&\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega, \\
&\mathbf{u} = \mathbf{u}_0 \text{ at } t = 0.
\end{aligned}
$$

with

$$Re = U_* L_* / \nu,$$

and

$$\nu = \mu/\rho \text{ kinematic viscosity.}$$

Backward F.D. scheme + Projection method (1/2)

Let $\delta t > 0$. Given $\mathbf{u}^k(\mathbf{x}) \approx \mathbf{u}(\mathbf{x}, t_k)$ and $p^k(\mathbf{x}) \approx p(\mathbf{x}, t_k)$, $t_k = k\delta t$,

we solve the **prediction step** :

$$\frac{3\tilde{\mathbf{u}}^{k+1} - 4\mathbf{u}^k + \mathbf{u}^{k-1}}{2\delta t} - \triangle\tilde{\mathbf{u}}^{k+1}/Re = -\nabla p^k + \mathbf{f}^{k+1}$$
$$-2\,\mathrm{div}\big(\mathbf{u}^k \otimes \mathbf{u}^k\big) + \mathrm{div}\big(\mathbf{u}^{k-1} \otimes \mathbf{u}^{k-1}\big),$$

$$\tilde{\mathbf{u}}^{k+1}|_{\partial\Omega} = \mathbf{g}.$$

then the **projection step** :

$$\mathbf{u}^{k+1} = \tilde{\mathbf{u}}^{k+1} - 2\delta t\,\nabla\big(\delta p^{k+1}\big)/3$$

$$\mathrm{div}\ \mathbf{u}^{k+1} = 0$$

$$\big(\mathbf{u}^{k+1} - \tilde{\mathbf{u}}^{k+1}\big)|_{\partial\Omega}\,.\mathbf{n} = 0.$$

with $\delta p^{k+1} = p^{k+1} - p^k$.

Backward F.D. scheme + Projection method (2/2)

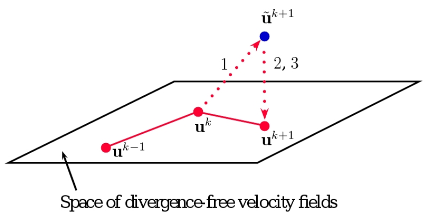
Pressure increment δp^{k+1} is solution of :

$$\triangle\left(\delta p^{k+1}\right) = 3\,\mathrm{div}\ \tilde{\mathbf{u}}^{k+1}/2\delta t$$

$$\partial_{\mathbf{n}}\left(\delta p^{k+1}\right)|_{\partial\Omega} = 0,$$

At each iteration,

1. solve prediction step,
2. solve system on pressure increment,
3. correction of velocity via projection step.



M.A.C. Scheme : position of unknowns

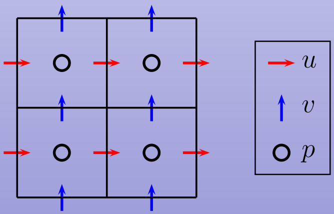
Let $\Omega = (0, L) \times (0, H)$.

We consider $\ell = L/n_\ell$, $h = H/n_h$, $x_i = i\,\ell$ and $y_j = j\,h$.

$$K_{i,j} = [x_{i-1}\ ,\ x_i] \times [y_{j-1}\ ,\ y_j]$$

$$u_{i\,j}(t) \simeq \langle u(\ .\ , t)\rangle_{K_{i+\frac{1}{2},\,j}}, \quad v_{i\,j}(t) \simeq \langle v(\ .\ , t)\rangle_{K_{i,\,j+\frac{1}{2}}},$$

$$p_{i\,j}(t) \simeq \langle p(\ .\ , t)\rangle_{K_{i,\,j}}, \quad \text{where} \quad \langle w\rangle_K = \frac{1}{|K|}\int_K w(\mathbf{x})\,d\mathbf{x}.$$



F.H. Harlow and J.E. Welch, *Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface*, Phys. Fluids **8**, 1965.

Summary

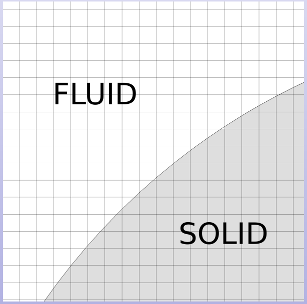
Context

Numerical simulation of incompressible fluid flows around obstacles

- Immersed boundary methods
- Numerical scheme
- Solver
- Numerical results

Conclusion and prospects

Immersed boundary methods on cartesian grid

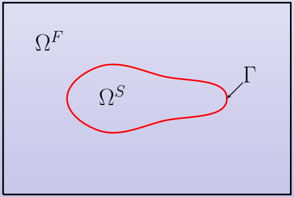


- simulation of flows in complex geometry
- in the literature, several methods exist : forcing, ghost cell, penalization, cut cell

Taking into account the obstacle

Rectangular domain Ω.

The obstacle Ω^S is bounded by a closed curve Γ.



Algebraic distance $d : \Omega \to \mathbb{R}$ is defined by :

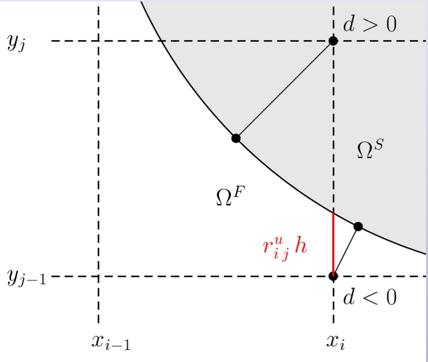
$$d(\mathbf{x}) = \begin{cases} \text{dist}(\mathbf{x}, \Gamma) & \text{if } \mathbf{x} \in \Omega^S, \\ -\text{dist}(\mathbf{x}, \Gamma) & \text{otherwise.} \end{cases}$$

Cell-face ratio

$$r^u_{i\,j} \approx \frac{|\sigma^u_{i,j} \cap \Omega^F|}{|\sigma^u_{i,j}|} \in [0,1],$$

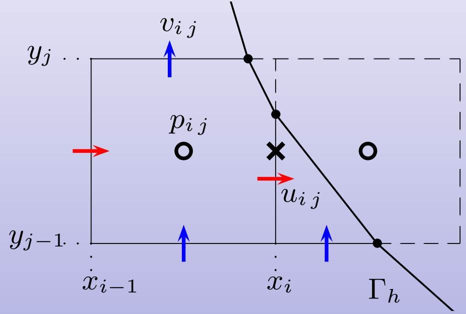
with

$$\sigma^u_{i,j} = \{x_i\} \times [y_{j-1}, y_j]\,.$$



O. Botella and Y. Cheny, *The LS-STAG method: A new immersed boundary/level-set method for the computation of incompressible viscous flows in complex moving geometries with good conservation properties*, J. Comp. Phys. **229**, 2010.

Position of unknows



- Position of velocity field well-adapted to divergence
- Interpolation of the pressure gradient

Discretization of the prediction step

Far away from the obstacle : second order centered discretization

Near the obstacle :

$$\begin{cases} \triangle \mathbf{u} & : \text{ first order Finite Difference approximation} \\ \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) & : \text{ first order Finite Volume approximation} \end{cases}$$

N. Matsunaga and T. Yamamoto, *Superconvergence of the Shortley-Weller approximation for Dirichlet problems*, J. Comp. Appl. Math. **116**, 2000.

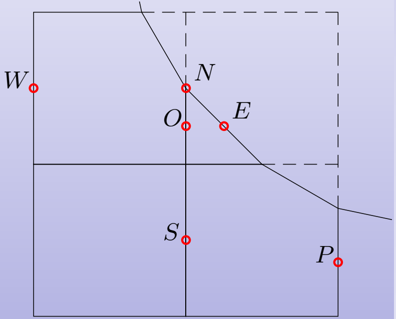
$\xrightarrow{\text{linear elliptic}}$ **Second order accurate**

Discretization of the prediction step : $\triangle u$

First-order Finite Difference approximation is exact on $\mathbb{R}_2[X, Y]$.

$\mathcal{V} = \{O, N, S, E, W, P\}$

- O the position of u_{ij},
- N, S, E, W among unknowns close to O or on the board Γ,
- P arbitrarily chosen

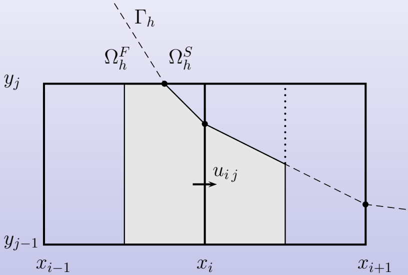


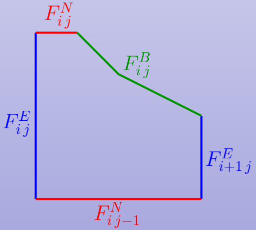
Find coefficients α_M such that :

$$\sum_{M \in \mathcal{V}} \alpha_M u(M) = \triangle u(O) + \mathcal{O}(h).$$

Discretization of the prediction step : $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$

$$\tilde{K}_{i,j} = K_{i,j} \cap \Omega^F$$





$$\begin{aligned}\mathcal{I}_{i,j} &= \int_{\tilde{K}_{i+\frac{1}{2},j}} \left(\partial_x(u^2) + \partial_y(uv)\right)\, d\mathbf{x} \\ &= \int_{\partial\tilde{K}_{i+\frac{1}{2},j}} \left(u^2 n_x + (uv) n_y\right) dS \\ &= F_{i+1,j}^E - F_{i,j}^E + F_{i,j}^N - F_{i,j-1}^N + F_{i,j}^B.\end{aligned}$$

$\rightarrow$ flux reconstruction

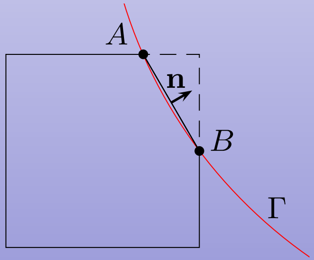
Discretization of the correction step

- div $\mathbf{u}$: Discrete divergence on cut cells
- ∇p : Interpolation of the pressure gradient

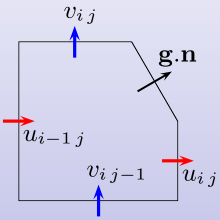
Discretization of the correction step : div $\mathbf{u}$ (1/2)

$$\iint_{\tilde{K}_{i,j}} \text{div } \mathbf{u} \, d\mathbf{x} = \int_{\partial \tilde{K}_{i,j}} \mathbf{u}.\mathbf{n} \, dS$$
$$= \int_{\sigma^u_{i,j} \cap \Omega^F} u dS - \int_{\sigma^u_{i-1,j} \cap \Omega^F} u \, dS$$
$$+ \int_{\sigma^v_{i,j} \cap \Omega^F} v dS - \int_{\sigma^v_{i,j-1} \cap \Omega^F} v \, dS + \int_{\widehat{AB}} \mathbf{u}.\mathbf{n} \, dS,$$

with $\widehat{AB} = \Gamma \cap K_{i,j}$.



Discretization of the correction step : div $\mathbf{u}$ (2/2)



- Assuming $h \ll$ radius of curvature of Γ :

$$\int_{\widehat{AB}} \mathbf{u}.\mathbf{n}\, dS \approx \int_{[AB]} \mathbf{u}.\mathbf{n}\, dS$$
$$\approx L\,\mathbf{g}\Big(\big(A+B\big)/2\Big).\mathbf{n}_{i,j}.$$

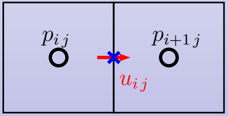
- $\displaystyle\int_{\sigma^u_{i,j}\cap\Omega^F} u\, dS \approx r^u_{i,j}\; h\; u_{i,j}$ and $\displaystyle\int_{\sigma^v_{i,j}\cap\Omega^F} v\, dS \approx r^v_{i,j}\; h\; v_{i,j}.$

$$\begin{aligned}(D_{obs}\mathbf{u})_{i,j} &= h\,(r^u_{i,j}u_{i,j} - r^u_{i-1,j}u_{i-1,j}) + h\,(r^v_{i,j}v_{i,j} - r^v_{i,j-1}v_{i,j-1}) \\ &\quad + L\,\mathbf{g}\Big(\big(A+B\big)/2\Big)\,.\,\mathbf{n}_{i,j} \\ &= (D^0_{obs}\mathbf{u})_{i,j} + D^{supp}_{i,j}\end{aligned}$$

Discretization of the correction step : $\mathcal{P}_\phi G\delta p$

Without

$$Gp = \begin{pmatrix} (p_{i+1,j} - p_{i,j})/h \\ (p_{i,j+1} - p_{i,j})/h \end{pmatrix}$$

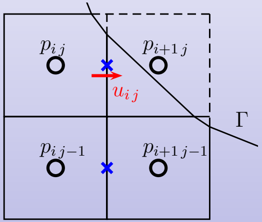


$$D(G\delta p) = \frac{3}{2}\frac{h^2}{\delta t} D(\tilde{\mathbf{u}})$$

$$\Rightarrow \ \mathbf{u} = \tilde{\mathbf{u}} - \frac{2}{3}\frac{\delta t}{h^2} G\delta p$$

$$\Rightarrow \ D(\mathbf{u}) = 0$$

With



$$D^0_{obs}(\mathcal{P}_\phi(G\delta p)) = \frac{3}{2}\frac{h^2}{\delta t} D_{obs}(\tilde{\mathbf{u}})$$

$$\Rightarrow \ \mathbf{u} = \tilde{\mathbf{u}} - \frac{2}{3}\frac{\delta t}{h^2} \mathcal{P}_\phi(G\delta p)$$

$$\Rightarrow \ D_{obs}(\mathbf{u}) = 0$$

Solver

Obstacle $\rightarrow$ **nonsymmetric** linear system

- **Iterative** methods for solving linear systems

Cut cells $\rightarrow$ ill-conditioned linear systems
$\rightarrow$ slow convergence
$\rightarrow$ simulation on coarse mesh
$\rightarrow$ simulations of flow at moderate Reynolds

- **Direct** method for solving linear systems

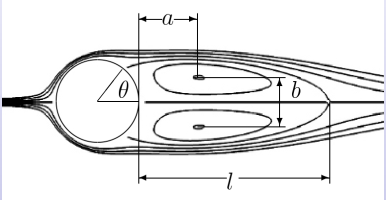
Unmoving obstacle :

- *Preprocessing step* : $\mathcal{O}(n^3)$ operations, once per simulation.
- Every iteration : $\mathcal{O}\left(n^2 \log n\right)$ operations (idem without obstacle).

B.L. Buzbee, F.W.Dorr, J.A. George and G.H. Golub, *The direct solution of the discrete Poisson equation on irregular regions*, J. Num. Anal. **8**, 1971.

Numerical results : $Re = 40$

Laminar flows



R. Bouard and M. Coutanceau, *Experimental determination of the main features of the viscous flow in the wake of a circular cylinder in uniform translation*, J. Fluid Mech. **79**, 1977.

$$\sum \text{Forces}_{/\text{obstacle}} = \frac{1}{2}\rho A u_\infty \begin{pmatrix} C_d \\ C_l \end{pmatrix}$$

Authors	$Re = 40$				
	C_d	θ	l	a	b
Bouard et al		53.8	2.13	0.76	0.59
Calhoun	1.62	54.2	2.18		
Dennis et al	1.52	53.8	2.35		
Fornberg	1.50	55.6	2.24		
Linnick et al	1.54	53.6	2.28	0.72	0.60
Taira et al	1.55	54.1		0.73	0.60
Present study	1.50	53.4	2.26	0.710	0.60

Numerical results : $Re = 9\ 500$

$\Omega = (-5, 5) \times (-2.5, 2.5)$, obstacle = disk, $D = 1$
Non-uniform grid, 3072 mesh points in each direction
Near the obstacle $h = 1.6\ 10^{-3}$
CFL stability condition $\Rightarrow \delta t = 10^{-4}$

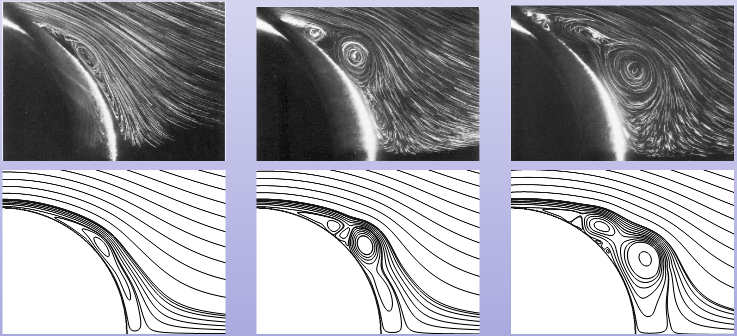


Figure: Evolution of the boundary layer : comparison with experimental results.

Numerical results : flow past a NACA airfoil

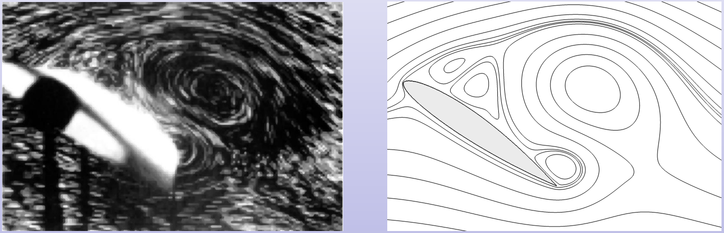


Figure: Flow behind NACA 0012 at $Re = 1\ 000$, incidence 34^o : comparison with experimental results

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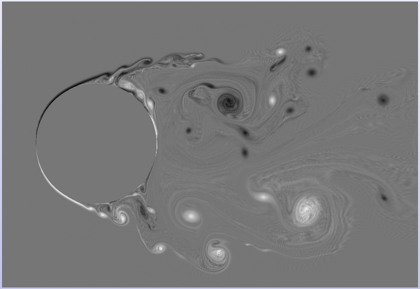
Accurate (second order) and fast (efficient solver) new cut cell method.



1. Three Dimensional flows
2. Coupling with :
   - H-box method (avoid the *small cell problem*, $\delta t \nearrow$)
   - Turbulence model (flows at high *Re*)
   - Local grid refinement
   - Domain decomposition

Thank you