Influence de la géométrie du maillage sur le schéma de Godunov appliqué à l'équation des ondes dans le régime Bas Mach (Attention aux noyaux)

Pascal Omnes
Joint work with S. Dellacherie and F. Rieper

CEA, DEN, DM2S-SFME F-91191 Gif-sur-Yvette Cedex
Laga Université Paris 13

pascal.omnes@cea.fr

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Outline of the talk

Introduction : the low Mach regime

1D advection and the upwind scheme

1D waves and the Godunov scheme

2D waves and the Godunov scheme

Perspectives

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The isentropic Euler Equations

Gas described by its density ρ, velocity u and pressure p :

$$\begin{aligned} &\partial_t \rho + \nabla \cdot \rho u = 0, \\ &\partial_t \rho u + \nabla \cdot \rho u \otimes u + \nabla p = 0, \\ &p = p(\rho). \end{aligned}$$

Nondimensionalization : $t = \bar{t} t'$, $x = \bar{x} x'$, $\rho = \bar{\rho} \rho'$, $u = \bar{u} u'$, $p = \bar{p} p'$. If one chooses $\bar{u} = \bar{x}/\bar{t}$ and $\bar{p} = p'(\bar{\rho})\,\bar{\rho}$, one gets

$$\begin{aligned} &\partial_t \rho + \nabla \cdot \rho u = 0, \\ &\partial_t \rho u + \nabla \cdot \rho u \otimes u + \frac{1}{M^2} \nabla p = 0, \end{aligned}$$

with $M = \bar{u}/c_s$ is the Mach number and $c_s = \sqrt{p'(\bar{\rho})}$ is a reference sound speed.

Low Mach asymptotics

$$\partial_t \rho + \nabla \cdot \rho u = 0, \tag{1}$$

$$\partial_t \rho u + \nabla \cdot \rho u \otimes u + \frac{1}{M^2} \nabla p = 0, \tag{2}$$

From an asymptotic expansion of (2) when $M \ll 1$, we have $p(x,t) = p^0(t) + O(M^2)$.

Then, from the state law : $\rho(x,t) = \rho^0(t) + O(M^2)$ and from the integration of (1) over Ω and periodic boundary conditions one gets that $\rho^0(t) \equiv \rho^0$, and then $p(x,t) = p^0 + O(M^2)$.

Then $u = u^0 + Mu^1$ and (1) implies that $\nabla \cdot u^0 = 0$.

It is simpler to work with a rescaling of the pressure such that $r(x,t) = (p(x,t) - p^0)/M$ (we thus have $\frac{1}{M^2}\nabla p = \frac{1}{M}\nabla r$.)

The solution is thus a constant pressure (r) field and an incompressible velocity plus a perturbation of size M.

Statement of our study

The Godunov scheme fails to reproduce this : spurious $O(\Delta x)$ waves appear. To be accurate you would have to pay for $\Delta x \leq M$.

We shall study the simpler linearized case

$$\partial_t r + \frac{1}{M}\nabla \cdot u = 0, \;\; \partial_t u + \frac{1}{M}\nabla r = 0$$

with I.C. $q^0 = (r^0, u^0)$ such that $q^0 = \widehat{q}^0 + \widetilde{q}^0 \in \mathcal{E} \overset{\perp}{\oplus} \mathcal{E}^\perp$, $||\widetilde{q}^0|| = O(M)$. The incompressible and acoustic subspaces are

$$\mathcal{E} = \{(r,u),\; r \equiv c,\; \nabla \cdot u = 0\}\;,\; \mathcal{E}^\perp = \left\{(r,u), \int_\Omega r = 0,\; u = \nabla \phi\right\}.$$

By linearity and energy conservation of the wave equation, we have $q(t) = \widehat{q}^0 + \widetilde{q}(t)$ and $||\widetilde{q}(t)|| = O(M)$.

A scheme able to reproduce this behaviour at the discrete level will be said to be accurate at low Mach number.

Outline of the talk

Basic properties of 1D advection on $\Omega =]0,1[$

$$\partial_t u + M^{-1}\partial_x u = 0$$

- Energy conservation :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx + \frac{1}{2}M^{-1}\int_\Omega \partial_x u^2(x,t)dx = 0$$

Periodicity yields $\int_\Omega \partial_x u^2(x,t)dx = u^2(1) - u^2(0) = 0$.
And thus $\int_\Omega u^2(x,t)dx = cte$.

- Invariant space : $\partial_x u = 0$, i.e. $u \in \mathcal{E}$ with

$$\mathcal{E} = \{u,\ u \equiv c,\ c \in \mathbb{R}\}\ ,\ \mathcal{E}^\perp = \left\{u,\ \int_\Omega u = 0\right\}.$$

So if $u(x,t=0) = \widehat{u}^0 + \widetilde{u}^0$ with $(\widehat{u}^0, \widetilde{u}^0) \in \mathcal{E} \times \mathcal{E}^\perp$, and $||\widetilde{u}^0|| = O(M)$, then $u(x,t) = \widehat{u}^0 + \widetilde{u}(x,t)$ and $||\widetilde{u}(t)|| = O(M)$.

The semi-discrete upwind scheme for 1D advection (1)

Mesh : $]0,1[$ devided into cells $S_i := [x_{i-1/2}, x_{i+1/2}]$ of equal size Δx. Integrating over S_i :

$$\frac{1}{\Delta x}\int_{S_i} \partial_t u(x,t)dx + \frac{M^{-1}}{\Delta x}\int_{S_i} \partial_x u(x,t)dx = 0$$

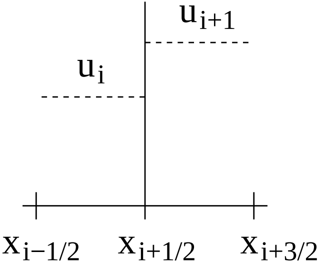
Setting $u_i(t) := \frac{1}{\Delta x}\int_{S_i} u(x,t)dx$ one gets

$$\frac{d}{dt}u_i(t) + \frac{M^{-1}}{\Delta x}\left[u(x_{i+1/2},t) - u(x_{i-1/2},t)\right] = 0.$$

If one chooses $u_i(t)$ as unknowns, one has to approach $u(x_{i+1/2},t)$ as a function of the set $(u_j(t))$.

The semi-discrete upwind scheme for 1D advection (2)

Solution of the Riemann problem



If $u(x,t) = \begin{cases} u_i & \textit{if} \quad x < x_{i+1/2} \\ u_{i+1} & \textit{if} \quad x > x_{i+1/2} \end{cases}$

Then since $M^{-1} > 0$, the caracteristics method yields that $u(x_{i+1/2}, s) = u_i$ for $t < s < t + M^{-1}\Delta x$. Upwind scheme :

$$\frac{d}{dt} u_i(t) + \frac{M^{-1}}{\Delta x} \left(u_i - u_{i-1} \right)(t) = 0.$$

Discrete invariant space

$$\frac{d}{dt}u_i(t) + \frac{M^{-1}}{\Delta x}\left(u_i - u_{i-1}\right)(t) = 0.$$

Invariant space : $(u_i - u_{i-1}) = 0$ for all i, thus

$$\mathcal{E}_h = \{(u_i),\ u_i \equiv c, \forall i, c \in \mathbb{R}\}\ ,\ \mathcal{E}_h^{\perp} = \left\{(u_i), \sum_i \Delta x\, u_i = 0\right\}.$$

What is the projection of the solution on this invariant space ?

$$\frac{d}{dt}\sum_i \Delta x\, u_i(t) + M^{-1}\sum_i \left(u_i - u_{i-1}\right)(t) = 0.$$

So, by periodicity : $\left(\sum_i \Delta x\, u_i\right)(t) = \left(\sum_i \Delta x\, u_i^0\right)$.

$$u_i(t) = \frac{1}{|\Omega|}\sum_i \Delta x\, u_i^0 + v_i(t) \text{ with } \sum_i \Delta x v_i(t) = 0.$$

Numerical diffusion towards the invariant space (1)

$$\frac{d}{dt}u_i(t) + \frac{M^{-1}}{\Delta x}\left(u_i - u_{i-1}\right)(t) = 0.$$

Truncation error :

$$u(x_{i-1}) = u(x_i) - \Delta x \partial_x u(x_i) + \frac{1}{2}\Delta x^2 \partial_{xx} u(x_i) + O(\Delta x^3)$$

so that the scheme is consistant up to Δx^2 with

$$\partial_t u + M^{-1}\partial_x u - \frac{M^{-1}\Delta x}{2}\partial_{xx} u = 0$$

convection diffusion equation with diffusion rate $M^{-1}\Delta x/2$.
Another way to see this

$$\frac{d}{dt}u_i + \frac{M^{-1}}{2\Delta x}\left(u_{i+1} - u_{i-1}\right) - \frac{M^{-1}\Delta x}{2}\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) = 0.$$

Numerical diffusion towards the invariant space (2)

With $v_i(t) := u_i(t) - \frac{1}{|\Omega|}\sum_i \Delta x\, u_i^0$ we have $\sum_i \Delta x v_i = 0$ and

$$\frac{d}{dt}v_i(t) + \frac{M^{-1}}{\Delta x}\left(v_i - v_{i-1}\right)(t) = 0.$$

Multiplying by $\Delta x\, v_i$ and sum over i, we get the discrete energy $e(t) = \sum_i \Delta x v_i^2$ evolution equation

$$\frac{1}{2}\frac{d}{dt}e(t) + \frac{a\Delta x}{2}\sum_i \Delta x\left(\frac{v_i - v_{i-1}}{\Delta x}\right)^2 = 0.$$

With the discrete Poincaré inequality : $\exists C(\Omega)$ such that for any (v_i) such that $\sum_i \Delta x v_i = 0$, we have

$$\sum_i \Delta x v_i^2 \leq C(\Omega)\sum_i \Delta x\left(\frac{v_i - v_{i-1}}{\Delta x}\right)^2.$$

And we prove that

$$e(t) \leq e(0)\exp\left(-\frac{M^{-1}\Delta x}{C(\Omega)}t\right).$$

Conclusion for 1D advection

If at the continuous level $u(x,0) = \widehat{u}^0 + \widetilde{u}^0$ with $(\widehat{u}^0, \widetilde{u}^0) \in \mathcal{E} \times \mathcal{E}^\perp$

$$\mathcal{E} = \{u,\ u \equiv c,\ c \in \mathbb{R}\}\ ,\ \mathcal{E}^\perp = \left\{u,\ \int_\Omega u = 0\right\}.$$

Then, it is possible to discretize accurately $(\widehat{u}^0, \widetilde{u}^0)$ by $(\widehat{u}^0_h, \widetilde{u}^0_h) \in \mathcal{E}_h \times \mathcal{E}_h^\perp$

$$\mathcal{E}_h = \{(u_i),\ u_i \equiv c, \forall i,\ c \in \mathbb{R}\}\ ,\ \mathcal{E}_h^\perp = \left\{(u_i), \sum_i \Delta x\, u_i = 0\right\}$$

and, like in the continous case, because $\mathcal{E}_h$ is the discrete kernel of the discrete wave operator, we have

$$u_h(t) = \widehat{u}^0_h + \widetilde{u}_h(t) \ \text{ with } ||\widetilde{u}_h(t)|| = O(M).$$

Outline of the talk

Basic properties of 1D waves

$$\partial_t r + M^{-1}\partial_x u = 0, \;\; \partial_t u + M^{-1}\partial_x r = 0$$

- Energy $e(t) := ||u(t)||^2 + ||r(t)||^2$. Conservation : $\dfrac{d}{dt}e = 0$.
- Invariant space : $\partial_x u = \partial_x r = 0$, i.e. $q := (r, u) \in \mathcal{E}$ with

$$\mathcal{E} = \left\{ q = (r, u), \; r \equiv a \, , u \equiv b, \, (a, b) \in \mathbb{R}^2 \right\},$$

$$\mathcal{E}^{\perp} = \left\{ q = (r, u), \, \int_\Omega r = \int_\Omega u = 0 \right\}.$$

So if $q(x, t = 0) = \widehat{q}^0 + \widetilde{q}^0$ with $(\widehat{q}^0, \widetilde{q}^0) \in \mathcal{E} \times \mathcal{E}^{\perp}$, and $||\widetilde{q}^0|| = O(M)$, then $q(x, t) = \widehat{q}^0 + \widetilde{q}(x, t)$ and $||\widetilde{q}(t)|| = O(M)$.

The Godunov scheme for 1D waves

$$\frac{d}{dt}r_i(t) + \frac{M^{-1}}{\Delta x}\left[u(x_{i+1/2},t) - u(x_{i-1/2},t)\right] = 0,$$
$$\frac{d}{dt}u_i(t) + \frac{M^{-1}}{\Delta x}\left[r(x_{i+1/2},t) - r(x_{i-1/2},t)\right] = 0.$$

Approximation of $(r(x_{i+1/2},t), u(x_{i+1/2},t))$ by the Riemann problem (Diagonalization into 2 independent transport equations) :

$$r(x_{i+1/2},t) \approx \frac{1}{2}(r_{i+1} + r_i) - \frac{1}{2}(u_{i+1} - u_i)$$
$$u(x_{i+1/2},t) \approx \frac{1}{2}(u_{i+1} + u_i) - \frac{1}{2}(r_{i+1} - r_i)$$

The Godunov scheme for 1D waves reads

$$\frac{d}{dt}r_i(t) + M^{-1}\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right) - \frac{\Delta x}{2M}\left(\frac{r_{i+1} - 2r_i + r_{i-1}}{\Delta x^2}\right) = 0,$$
$$\frac{d}{dt}u_i(t) + M^{-1}\left(\frac{r_{i+1} - r_{i-1}}{2\Delta x}\right) - \frac{\Delta x}{2M}\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) = 0.$$

Discrete invariant space and stability

Discrete energy : $e(t) = \sum_i \Delta x r_i^2 + \sum_i \Delta x u_i^2$. Energy variation :

$$\frac{d}{dt} e(t) = -\frac{\Delta x}{M} \left[\sum_i \Delta x \left(\frac{r_{i+1} - r_i}{\Delta x} \right)^2 + \sum_i \Delta x \left(\frac{u_{i+1} - u_i}{\Delta x} \right)^2 \right].$$

Dissipation of energy (stability)

• Invariant space : $r_{i+1} = r_i$ and $u_{i+1} = u_i$ for all i, i.e.
$q := (r, u) \in \mathcal{E}$ with

$$\mathcal{E}_h = \left\{ q = (r, u),\ r_i \equiv a\,, u_i \equiv b,\ (a, b) \in \mathbb{R}^2 \right\},$$

$$\mathcal{E}_h^{\perp} = \left\{ q = (r, u),\ \sum_i \Delta x r_i = \sum_i \Delta x u_i = 0 \right\}.$$

Conclusion for 1D waves

If at the continuous level $q(x,0) = \widehat{q}^0 + \widetilde{q}^0$ with $(\widehat{q}^0, \widetilde{q}^0) \in \mathcal{E} \times \mathcal{E}^\perp$

$$\mathcal{E} = \left\{ q = (r,u),\ r \equiv a\,, u \equiv b,\ (a,b) \in \mathbb{R}^2 \right\},$$

$$\mathcal{E}^\perp = \left\{ q = (r,u),\ \int_\Omega r = \int_\Omega u = 0 \right\}.$$

Then, it is possible to discretize accurately $(\widehat{q}^0, \widetilde{q}^0)$ by $(\widehat{q}_h^0, \widetilde{q}_h^0) \in \mathcal{E}_h \times \mathcal{E}_h^\perp$

$$\mathcal{E}_h = \left\{ q = (r,u),\ r_i \equiv a\,, u_i \equiv b,\ (a,b) \in \mathbb{R}^2 \right\},$$

$$\mathcal{E}_h^\perp = \left\{ q = (r,u),\ \sum_i \Delta x r_i = \sum_i \Delta x u_i = 0 \right\}.$$

and, like in the continous case, because $\mathcal{E}_h$ is the discrete kernel of the discrete wave operator, we have

$$q_h(t) = \widehat{q}_h^0 + \widetilde{q}_h(t) \ \text{ with } ||\widetilde{q}_h(t)|| = O(M).$$

Outline of the talk

Basic properties of 2D waves

$$\partial_t r + \frac{1}{M}\nabla\cdot\mathbf{u} = 0, \quad \partial_t \mathbf{u} + \frac{1}{M}\nabla r = 0$$

- Energy $e(t) := ||\mathbf{u}(t)||^2 + ||r(t)||^2$. Conservation : $\frac{d}{dt}e = 0$.
- Invariant space $\nabla r = 0$ and $\nabla\cdot\mathbf{u} = 0$, i.e. $q = (r,\mathbf{u}) \in \mathcal{E}$:

$$\mathcal{E} = \left\{ q = (r,\mathbf{u}),\ r \equiv c,\ \mathbf{u} = (a,b)^T + \nabla\times\psi,\ (a,b,c)\in\mathbb{R}^3 \right\},$$

$$\mathcal{E}^\perp = \left\{ (r,\mathbf{u}),\ \int_\Omega r = 0,\ \mathbf{u} = \nabla\phi \right\}.$$

If $q^0 = (r^0, u^0)$ such that $q^0 = \widehat{q}^0 + \widetilde{q}^0 \in \mathcal{E} \overset{\perp}{\oplus} \mathcal{E}^\perp$, $||\widetilde{q}^0|| = O(M)$.
Then

$q(t) = \widehat{q}^0 + \widetilde{q}(t)$ and $||\widetilde{q}(t)|| = O(M)$.

The Godunov and low-Mach Godunov schemes for 2D waves

Consider a set of cells T_i with cell-centered unknowns $q_i = (r_i, \mathbf{u}_i)^T$. The interface between T_i and T_j is called A_{ij} with unit normal vector $\mathbf{n}_{ij}$ from T_i to T_j. The Godunov ($\kappa = 1$) and low Mach Godunov ($\kappa = 0$) schemes read

$$\frac{d}{dt} q_i + \frac{\mathbb{L}^i_{\kappa,h}}{M} q = 0 \tag{3}$$

with

$$\mathbb{L}^i_{\kappa,h} q := \frac{1}{2|T_i|} \begin{pmatrix} \displaystyle\sum_{A_{ij} \subset \partial T_i} |A_{ij}| \left[(r_i - r_j) + (\mathbf{u}_i + \mathbf{u}_j) \cdot \mathbf{n}_{ij} \right] \\ \displaystyle\sum_{A_{ij} \subset \partial T_i} |A_{ij}| \left[(r_i + r_j) + \kappa (\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij} \right] \mathbf{n}_{ij} \end{pmatrix}$$

Stability and discrete invariant space

With $\langle \cdot, \cdot \rangle$ a discrete scalar product weighted by the areas of the T_is, it holds that

$$\frac{1}{2}\frac{d}{dt}e + \frac{1}{M}\langle \mathbb{L}_{\kappa,h} q, q \rangle = 0$$

$$\langle \mathbb{L}_{\kappa,h} q, q \rangle = \frac{1}{2}\sum_{A_{ij}} |A_{ij}| \left\{ (r_i - r_j)^2 + \kappa \left[(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij} \right]^2 \right\}$$

Thus, the semi-discrete scheme is stable and :

• the kernel of the Godunov scheme ($\kappa = 1$) is such that $r_i = r_j$ and $\mathbf{u}_i \cdot \mathbf{n}_{ij} = \mathbf{u}_j \cdot \mathbf{n}_{ij}$ for all neighbors i and j : constant pressure and no jump in the normal velocities.

• the kernel of the low Mach Godunov scheme ($\kappa = 0$) is such that $r_i = r_j$ for all neighbors i and j : constant pressure and moreover

$$\sum_{A_{ij} \subset \partial T_i} |A_{ij}| (\mathbf{u}_i + \mathbf{u}_j) \cdot \mathbf{n}_{ij} = \sum_{A_{ij} \subset \partial T_i} |A_{ij}| \mathbf{u}_j \cdot \mathbf{n}_{ij} = 0$$

The rectangular case – Discrete Hodge decomposition

Let $N_x \times N_y$ be the number of cells and periodicity conditions be enforced. We suppose that both N_x and N_y are odd (if not there are checkerboard modes).

Let us define the following discrete incompressible subspace :

$$\mathcal{E}_h^{\square} := \left\{ \left(r_{i,j} = c, \mathbf{u}_{i,j} = (a,b)^T + (\frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}, -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x})^T \right)^T \right\}$$

with $(a, b, c, (\psi_{i,j})) \in \mathbb{R}^3 \times \mathbb{R}^{N_x N_y}$.

The following lemma holds :

$$\left(\mathcal{E}_h^{\square}\right)^{\perp} = \left\{ \left(r \in L^2_{0,h}, \mathbf{u}_{i,j} = (\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y})^T \right)^T \right\}.$$

with $(\phi_{i,j}) \in \mathbb{R}^{N_x N_y}$ and $r \in L^2_{0,h} \Leftrightarrow \sum_{(i,j)} \Delta x \Delta y r_{i,j} = 0.$

The rectangular case – kernel structure

We have for the Godunov scheme ($\kappa = 1$) :

$$Ker\mathbb{L}_{\kappa=1,h} = \left\{ \left(r_{i,j} = c, \mathbf{u}_{i,j} = (u_j, v_i)^T \right) \right\}$$

(u constant along x and v constant along y). This implies that

$$Ker\mathbb{L}_{\kappa=1} \subsetneq \mathcal{E}_h^{\square}.$$

This subspace is too small to approach well incompressible fields.

On the other hand, for the low Mach Godunov scheme ($\kappa = 0$),

$$Ker\mathbb{L}_{\kappa=0,h} = \mathcal{E}_h^{\square}.$$

Indeed, in this case $\sum_{A_{ij} \subset \partial T_i} |A_{ij}| \mathbf{u}_j \cdot \mathbf{n}_{ij} = 0$ reduces to

$$\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = 0.$$

The rectangular case – time behaviour

Any initial condition

$$q^0 = \hat{q}^0 + \widetilde{q}^0 \text{ with } (\hat{q}^0, \widetilde{q}^0) \in \mathcal{E} \times \mathcal{E}^\perp$$

with $||\widetilde{q}^0|| = \mathcal{O}(M)$ may be accurately discretized by

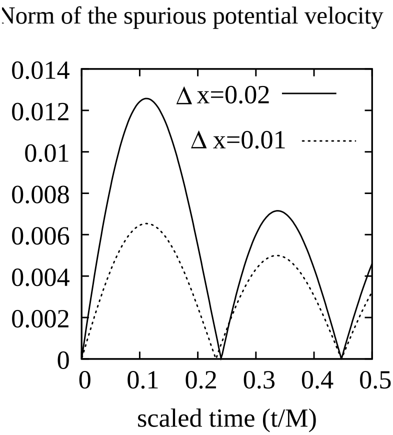
$$q_h^0 = \hat{q}_h^0 + \widetilde{q}_h^0 \text{ with } (\hat{q}_h^0, \widetilde{q}_h^0) \in \mathcal{E}_h^\square \times (\mathcal{E}_h^\square)^\perp$$

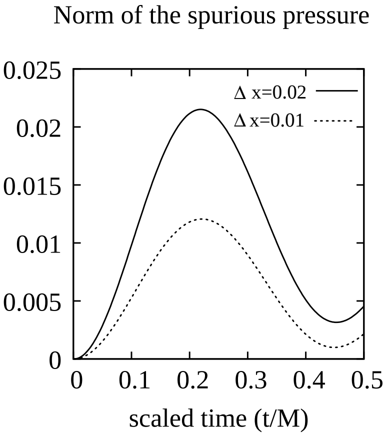
with $||\widetilde{q}_h^0|| = \mathcal{O}(M)$. By stability of the scheme, in any case ($\kappa = 0$ or 1), there holds $||\widetilde{q}_h(t)|| = \mathcal{O}(M)$.

Moreover, for the low Mach Godunov scheme ($\kappa = 0$), the discrete incompressible field $\hat{q}_h(t)$ remains forever equal to $\hat{q}_h^0$. The low Mach scheme is thus accurate (no creation of spurious acoustic waves).

The rectangular case – time behaviour

For the standard Godunov scheme $\kappa = 1$, the discrete incompressible part $\hat{q}_h^0$ is rapidly diffused (diffusion rate $\mathcal{O}(\frac{\Delta x}{M})$) to its projection on $Ker\mathbb{L}_{\kappa=1,h}$. During this diffusion process, a spurious acoustic mode is created. Its size is $\mathcal{O}(\Delta x)$. The scheme is inaccurate.





The triangular case – Discrete Hodge decomposition

Let V_h be the standard P^1 Lagrange Finite Element space

$$V_h := \left\{ \psi_h \in C_0(\overline{\Omega}),\ \psi_h \text{ periodic over } \overline{\Omega} \text{ and } (\psi_h)_{|T_i} \in P^1(T_i) \right\}.$$

Let W_h be the P^1 non conforming Crouzeix-Raviart FE space

$$W_h := \Big\{ \phi_h \in L^2(\Omega),\ \phi_h \text{ periodic over } \overline{\Omega} \text{ and } (\phi_h)_{|T_i} \in P^1(T_i) \\ \textit{and } \phi_h \text{ is continuous at the edge midpoints } \Big\}.$$

Since functions of V_h (resp. W_h) are P^1 on each cell, their curls (resp. their broken gradients ∇_h) are cell-centered constant values cell per cell.

The triangular case – Discrete Hodge decomposition

We may thus define the following subspace of $\mathbb{R}^{3N}$:

$$\mathcal{E}_h^{\Delta} = \left\{ \left(r_i = c, \mathbf{u}_i = (a,b)^T + (\nabla \times \psi_h)_{|T_i} \right)^T \right\}$$

with $(a, b, c, \psi_h) \in \mathbb{R}^3 \times V_h$.
The discrete space $\mathcal{E}_h^{\Delta}$ discretizes accurately $\mathcal{E}$

$$\mathcal{E} = \left\{ q = (r, \mathbf{u}),\ r \equiv c,\ \mathbf{u} = (a,b)^T + \nabla \times \psi,\ (a,b,c) \in \mathbb{R}^3 \right\}.$$

We may prove that (Arnold Falk, 1989)

$$\left(\mathcal{E}_h^{\Delta}\right)^{\perp} = \left\{ \left(r \in L^2_{h,0}, \mathbf{u}_i = (\nabla_h \phi_h)_{|T_i} \right)^T \right\}.$$

with $\phi_h \in W_h$ and $r \in L^2_{h,0} \Leftrightarrow \sum_i |T_i| r_i = 0$.

The discrete space $\left(\mathcal{E}_h^{\Delta}\right)^{\perp}$ discretizes accurately $\mathcal{E}^{\perp}$

$$\mathcal{E}^{\perp} = \left\{ (r, \mathbf{u}),\ \int_{\Omega} r = 0,\ \mathbf{u} = \nabla \phi \right\}.$$

The triangular case – kernel structure and time behaviour

It holds that

$$Ker\mathbb{L}_{\kappa=1,h} = \mathcal{E}_h^{\Delta} \subset Ker\mathbb{L}_{\kappa=0,h}$$

Any initial condition

$$q^0 = \hat{q}^0 + \widetilde{q}^0 \text{ with } (\hat{q}^0, \widetilde{q}^0) \in \mathcal{E} \times \mathcal{E}^{\perp}$$

with $||\widetilde{q}^0|| = \mathcal{O}(M)$ may be accurately discretized by

$$q_h^0 = \hat{q}_h^0 + \widetilde{q}_h^0 \text{ with } (\hat{q}_h^0, \widetilde{q}_h^0) \in \mathcal{E}_h^{\Delta} \times (\mathcal{E}_h^{\Delta})^{\perp}.$$

with $||\widetilde{q}_h^0|| = \mathcal{O}(M)$. By stability of the schemes, there holds $||\widetilde{q}_h(t)|| = \mathcal{O}(M)$.

Moreover, the discrete incompressible field $\hat{q}_h(t)$ remains forever equal to $\hat{q}_h^0$. The schemes are accurate (no creation of spurious acoustic waves).

Outline of the talk

Perspectives

Extend the analysis

- to other schemes / other equations (HLL / waves + convection by P.-A. Raviart)
- to other boundary conditions
- to variable cross-section equations

The discrete Hodge decompositions may help to obtain

- dissipation rates (coupled with discrete Poincaré inequalities)
- error analysis
- reinterpretation and improvement

of the schemes

Prove the stability of the fully discrete low Mach schemes