DDFV Schemes for the Euler Equations

Christophe Berthon, Yves Coudière, Vivien Desveaux

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Introduction

- Hyperbolic system of conservation laws in 2D

$$\partial_t W + \partial_x f(W) + \partial_y g(W) = 0$$

  $W : \mathbb{R}^2 \times \mathbb{R}^+ \to \Omega \subset \mathbb{R}^d$: unknown state vector
  $f, g : \Omega \to \mathbb{R}^d$: flux functions

- Ω convex set of physical states
- Objective : derive a numerical scheme
  - Second order accurate
  - Ω–preserving
  - Unstructured meshes
  - CFL restriction

Example : the 2D Euler equations

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \; f(W) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}, \; g(W) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$

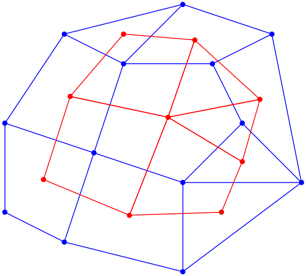
where ρ is the density, (u, v) the velocity, E the total energy and p the pressure given by the perfect gas law

$$p = (\gamma - 1)\left(E - \frac{\rho}{2}\left(u^2 + v^2\right)\right)$$

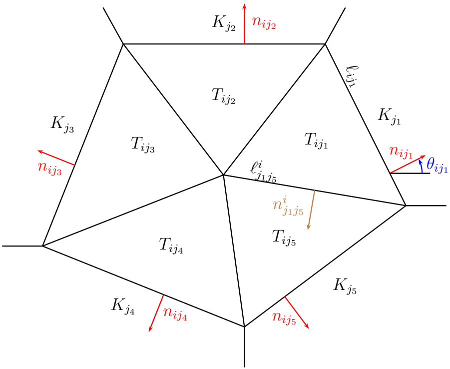
Set of physical states

$$\Omega = \left\{W \in \mathbb{R}^4; \rho > 0, (u, v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2 + v^2\right) > 0\right\}$$

Meshes notations



The primal mesh and the dual mesh



A primal or dual cell K_i

First-order scheme

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \phi\left(W_i^n, W_j^n, \theta_{ij}\right)$$

- ϕ 1D Godunov-type flux : under the CFL restriction $\frac{\Delta t}{\Delta x} \max\{\lambda^{\pm}(W_L, W_R, \theta)\} \leq \frac{1}{2}$, we have

$$\phi(W_L, W_R, \theta) = h_\theta(W_L) + \frac{\Delta x}{2\Delta t} W_L - \frac{1}{\Delta t} \int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_\theta\left(\frac{x}{\Delta t}, W_L, W_R\right) dx$$

  - $h_\theta = \cos\theta f + \sin\theta g$: flux in the direction θ
  - $\widetilde{W}_\theta$ approximate Riemann solver valued in Ω.
- Consistency : $\phi(W, W, \theta) = h_\theta(W)$
- Conservation : $\phi(W_L, W_R, \theta) = -\phi(W_R, W_L, \theta + \pi)$

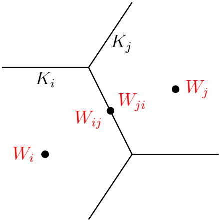
MUSCL scheme

First-order scheme on the cell K_i

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \phi\left(W_i^n, W_j^n, \theta_{ij}\right),$$

Second-order MUSCL scheme on the cell K_i

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \phi\left(W_{ij}, W_{ji}, \theta_{ij}\right),$$



W_{ij} and W_{ji} second-order approximations at the interface between K_i and K_j.

$\rightarrow$ **How to compute W_{ij} ?**

Motivations

- Usual CFL restriction on a square (first-order):

$$\frac{\Delta t}{|\ell|} \max_{j \in \nu(i)} \{\lambda^{\pm}(W_i^n, W_j^n, \theta_{ij})\} \leq \frac{1}{4}$$

- Usual CFL restriction on a quadrilateral (first-order):

$$\frac{\Delta t}{|K_i|} \max_{j \in \nu(i)} \{|\ell_{ij}| \lambda^{\pm}(W_i^n, W_j^n, \theta_{ij})\} \leq \frac{1}{8}$$

- $\Rightarrow$ Inconsistency. The CFL restriction for quadrilateral is not optimal.

First-order scheme: CFL restriction

Under the CFL restriction $\dfrac{\Delta t}{\Delta x} \max_{j\in\nu(i)} \{\lambda^{\pm}(W_i^n, W_j^n, \theta_{ij})\} \leq \dfrac{1}{2}$, we have

$$W_i^{n+1} = \left(1 - \frac{\Delta x}{2|K_i|} \sum_{j\in\nu(i)} |\ell_{ij}|\right) W_i^n - \frac{\Delta t}{|K_i|} \sum_{j\in\nu(i)} |\ell_{ij}| h_{\theta_{ij}}(W_i^n) + \frac{1}{|K_i|} \sum_{j\in\nu(i)} |\ell_{ij}| \int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_{\theta_{ij}} \left(\frac{x}{\Delta t}, W_i^n, W_j^n\right) dx$$

First-order scheme: CFL restriction

Under the CFL restriction $\dfrac{\Delta t}{\Delta x} \max_{j\in\nu(i)} \{\lambda^{\pm}(W_i^n, W_j^n, \theta_{ij})\} \leq \dfrac{1}{2}$, we have

$$W_i^{n+1} = \left(1 - \frac{\Delta x}{2|K_i|} \sum_{j\in\nu(i)} |\ell_{ij}|\right) W_i^n - \frac{\Delta t}{|K_i|} \sum_{j\in\nu(i)} |\ell_{ij}| h_{\theta_{ij}}(W_i^n)$$
$$+ \frac{1}{|K_i|} \sum_{j\in\nu(i)} |\ell_{ij}| \int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_{\theta_{ij}} \left(\frac{x}{\Delta t}, W_i^n, W_j^n\right) dx$$

$\displaystyle\sum_{j\in\nu(i)} |\ell_{ij}| h_{\theta_{ij}}(W_i^n) = \begin{pmatrix} f \\ g \end{pmatrix} (W_i^n) \cdot \sum_{j\in\nu(i)} |\ell_{ij}| n_{ij} = 0$ by Green's formula

First-order scheme: CFL restriction

Under the CFL restriction $\frac{\Delta t}{\Delta x} \max_{j \in \nu(i)} \{\lambda^{\pm}(W_i^n, W_j^n, \theta_{ij})\} \leq \frac{1}{2}$, we have

$$W_i^{n+1} = \left(1 - \frac{\Delta x}{2|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \right) W_i^n - 0$$
$$+ \frac{1}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \int_{-\frac{\Delta x}{2}}^{0} \widetilde{W}_{\theta_{ij}} \left(\frac{x}{\Delta t}, W_i^n, W_j^n \right) dx$$

We define $\mathcal{P}_i = \sum_{j \in \nu(i)} |\ell_{ij}|$ and we take $\Delta x = \frac{2|K_i|}{\mathcal{P}_i}$

$$\Rightarrow 1 - \frac{\Delta x}{2|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| = 0$$

The CFL restriction becomes

$$\frac{\Delta t}{|K_i|}\mathcal{P}_i \max_{j\in\nu(i)} \left\{\lambda^{\pm}(W_i^n, W_j^n, \theta_{ij})\right\} \leq 1$$

and we have

$$\begin{aligned} W_i^{n+1} &= \frac{1}{|K_i|}\sum_{j\in\nu(i)} |\ell_{ij}| \int_{-\frac{|K_i|}{\mathcal{P}_i}}^{0} \widetilde{W}_{\theta_{ij}}\left(\frac{x}{\Delta t}, W_i^n, W_j^n\right) dx \\ &= \frac{1}{\mathcal{P}_i}\sum_{j\in\nu(i)} |\ell_{ij}| W_{ij}^{n+1} \end{aligned}$$

with $W_{ij}^{n+1} = \dfrac{\mathcal{P}_i}{|K_i|}\displaystyle\int_{-\frac{|K_i|}{\mathcal{P}_i}}^{0} \widetilde{W}_{\theta_{ij}}\left(\frac{x}{\Delta t}, W_i^n, W_j^n\right) dx$

$W_{ij}^{n+1} \in \Omega$ as the mean value of a function valued in the convex Ω.
$W_i^{n+1} \in \Omega$ as a convex combination of the W_{ij}^{n+1}.

Theorem : Robustness of the first-order scheme

If the following hypothesis are satisfied

(i) $W_i^n \in \Omega, \forall i \in \mathbb{Z}$

(ii) We have the CFL condition

$$\Delta t \frac{\mathcal{P}_i}{|K_i|} \max_{j \in \nu(i)} \left\{ \lambda^{\pm}(W_i^n, W_j^n, \theta_{ij}) \right\} \leq 1, \forall i$$

Then the states W_i^{n+1} remain in Ω.

Remark : this CFL can be written

$$\Delta t \frac{|l_i|}{|K_i|} \max_{j \in \nu(i)} \left\{ \lambda^{\pm}(W_i^n, W_j^n, \theta_{ij}) \right\} \leq \frac{1}{p_i}$$

p_i number of edges of the cell K_i
$|l_i| = \frac{1}{p_i}\mathcal{P}_i$ mean length of the edges
$\Rightarrow$ Consistency with the CFL restriction for a square

Second-order MUSCL scheme

We define the intermediate states

$$W_{ij}^{n+1} = W_{ij} - \frac{\Delta t}{|T_{ij}|}\left(|\ell_{ij}|\phi(W_{ij}, W_{ji}, \theta_{ij}) + \sum_{k\in\nu(i,j)} |\ell_{jk}^i|\phi(W_{ij}, W_{ik}, \theta_{jk}^i)\right)$$

which are the updated states by the first-order scheme on the subcell T_{ij}. By the previous theorem, if the CFL restriction

$$\Delta t \frac{\mathcal{P}_{ij}}{|T_{ij}|} \max_{k\in\nu(i,j)} \left\{\lambda^{\pm}(W_{ij}, W_{ji}, \theta_{ij}), \lambda^{\pm}(W_{ij}, W_{ik}, \theta_{jk}^i)\right\} \leq 1$$

is satisfied, then W_{ij}^{n+1} is in Ω.

$$\frac{1}{|K_i|}\sum_{j\in\nu(i)} |T_{ij}|W_{ij}^{n+1} = \sum_{j\in\nu(i)} \frac{|T_{ij}|}{|K_i|}W_{ij} - \frac{\Delta t}{|K_i|}\sum_{j\in\nu(i)} \phi(W_{ij}, W_{ji}, \theta_{ij})$$

This state is in Ω as a convex combination of states in Ω.

Theorem : Robustness of the MUSCL scheme

If the following hypothesis are satisfied

(i) The initial states W_i^n are in Ω

(ii) The reconstructed states W_{ij} are in Ω

(iii) The reconstruction satisfies the conservation property

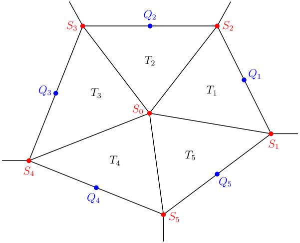
$$\sum_{j\in\nu(i)} \frac{|T_{ij}|}{|K_i|} W_{ij} = W_i^n$$

(iv) We have the CFL condition $\forall i \in \mathbb{Z}$

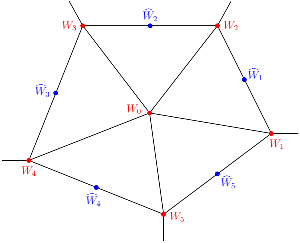
$$\Delta t \max_{j\in\nu(i)} \frac{\mathcal{P}_{ij}}{|T_{ij}|} \max_{k\in\nu(i,j)} \left\{\lambda^{\pm}(W_{ij}, W_{ji}, \theta_{ij}), \lambda^{\pm}(W_{ij}, W_{ik}, \theta^i_{jk})\right\} \leq 1$$

Then the states W_i^{n+1} remain in Ω.

Computation of the states W_{ij}



Geometry of the cell K



Known states and reconstructed states

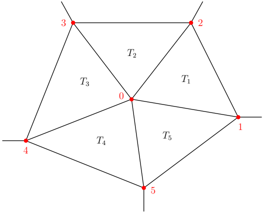
The reconstructed states have to satisfy:

- $\widehat{W}_j \in \Omega$
- $\displaystyle\sum_{j \in \nu(i)} \frac{|T_{ij}|}{|K_i|} \widehat{W}_j = W_0$

If we take $\widehat{W}_j = \widetilde{W}(Q_j)$ with $\widetilde{W}$ a linear function on K, we have

$$\sum_{j \in \nu(i)} \frac{|T_{ij}|}{|K_i|} \widehat{W}_j = W_0 \Longleftrightarrow \widetilde{W}(S_0) = W_0$$

1. Gradient reconstruction (DDFV)
We define a continuous function $\overline{W} : K \to \mathbb{R}^d$ piecewise linear on each triangle T_j and such that $\overline{W}(S_j) = W_j$, $j \in \nu(i)$.



2. Projection
For $1 \leq k \leq d$, we define

$$E_k(\nu) = \int_K \left| \overline{W}_k(X) - [(W_0)_k + \nu \cdot (X - S_0)] \right|^2 dX,$$

where the subscript k denotes the k-th component.
Let $\mu \in \mathbb{R}^d$ be the vector whose k-th component is the solution of

$$E_k(\mu_k) = \min_{\nu \in \mathbb{R}^2} E_k(\nu).$$

We define $\widetilde{W}_\mu(X) : K \to \mathbb{R}^d$ the linear function whose k-th component is $(W_0)_k + \mu_k \cdot (X - S_0)$.

3. Limitation of the slope μ

We restrict Ω to a close set Ω_ϵ. In the Euler case,

$$\Omega_\epsilon = \left\{ W \in \mathbb{R}^4; \rho \geq \epsilon, (u, v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2 + v^2\right) \geq \epsilon \right\}.$$

We define the optimal slope limiter by

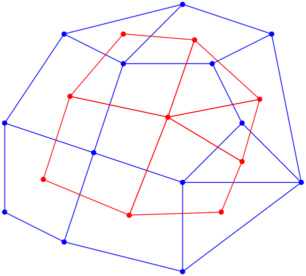
$$\alpha = \max \left\{ t \in [0, 1], \widetilde{W}_{t\mu}(Q_j) \in \Omega_\epsilon, \forall j \in \nu(i) \right\}.$$

4. Finally, the reconstructed states are given by $\widehat{W}_j = \widetilde{W}_{\alpha\mu}(Q_j)$.

$$\begin{aligned} &\text{Limitation procedure} && \Rightarrow && \widehat{W} j \in \Omega \\ &\widetilde{W}(S_0) = W_0 && \Rightarrow && \sum_{j \in \nu(i)} \frac{|T_{ij}|}{|K_i|} \widehat{W}_j = W_0 \end{aligned}$$

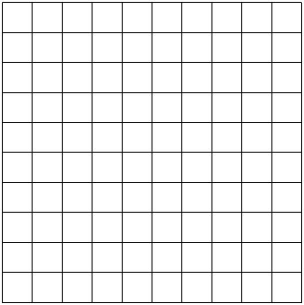
$\Rightarrow$ **The DDFV-MUSCL scheme is robust**

Computation of the states at the vertices

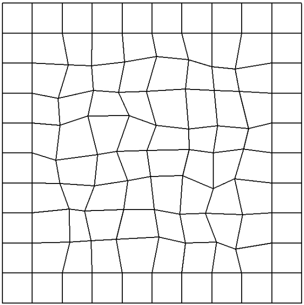


- We write a MUSCL scheme on both the primal and the dual mesh
- The states at a vertex of the dual mesh is exactly the state at the center of the associated primal cell
- We approximate the state at a vertex of the primal mesh by the state at the center of the associated dual cell

Meshes

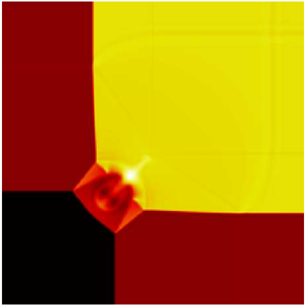


Square mesh 10×10

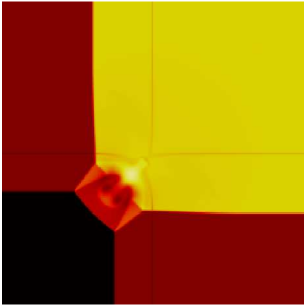


Quadrilateral mesh 10×10

Four shocks

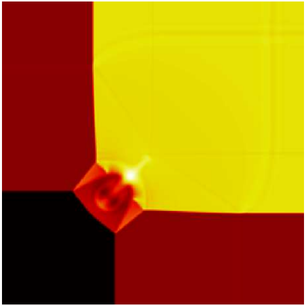


Square mesh 200×200
DDFV-MUSCL scheme

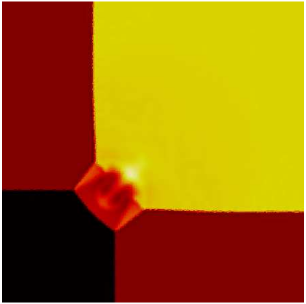


Square mesh 300×300
MUSCL scheme

Four shocks

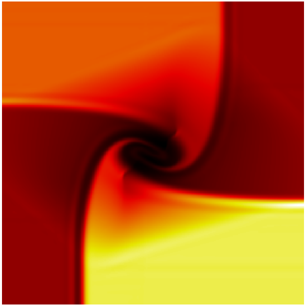


Square mesh 200×200
DDFV-MUSCL scheme

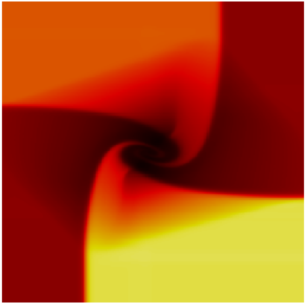


Quadrilateral mesh 200×200
DDFV-MUSCL scheme

Four contact discontinuities

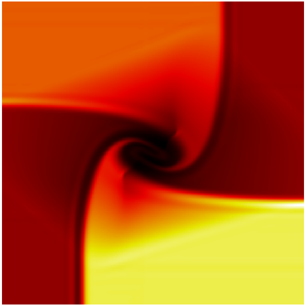


Square mesh 200 $\times$ 200
DDFV-MUSCL scheme

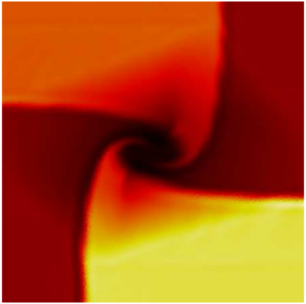


Square mesh 300 $\times$ 300
MUSCL scheme

Four contact discontinuities



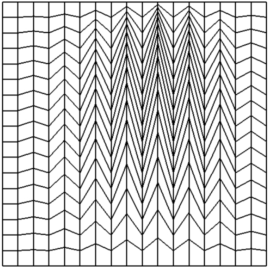
Square mesh 200×200
DDFV-MUSCL scheme



Quadrilateral mesh 200×200
DDFV-MUSCL scheme

Perspectives

- Allow non-conservative reconstructions, i.e. which don't satisfy
$$\sum_{j \in \nu(i)} \frac{|T_{ij}|}{|K_i|} W_{ij} = W_i^n$$
- Optimization of the CFL condition in the robustness theorem for the MUSCL scheme
- Better approximation of the value at the vertices of the primal mesh, especially in the case of very distorded meshes



Distorded mesh