Méthodes de raffinement espace-temps

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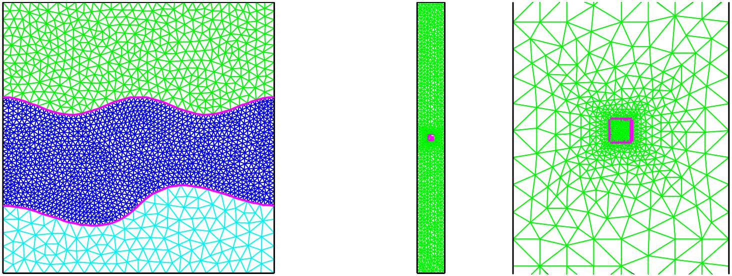
Journée GNR MOMAS / GDR Calcul, 5 Mai 2010

Coupling process

- For a given problem, split the domain : domain decomposition
- Subdomains with curved interfaces and widely differing lengths
- Coupling heterogeneous models
- Different time and space steps in different subdomains
- Application to nuclear waste disposal (GdR MoMaS)

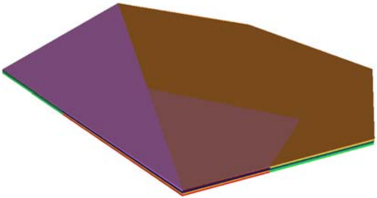


Far field 3D simulations of underground nuclear waste

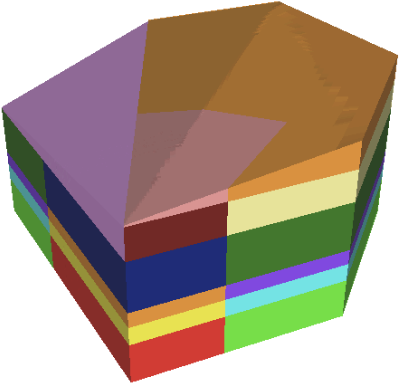
Research Group MoMaS

With Jérôme Jaffré, Michel Kern and Jean Roberts (INRIA)

Actual dimensions : $40km \times 40km \times 500m$



A blow-up in the vertical direction (30 times)



The repository is located in the red part of the bottom layer.

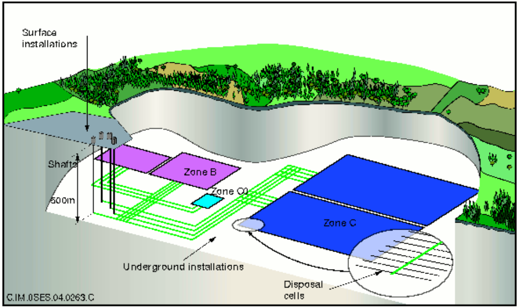
Hydrogeological data

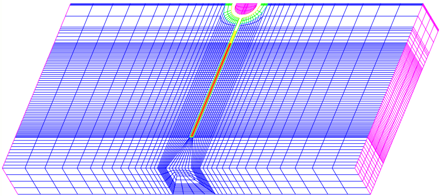
Hydrogeologic layers	Thickness [m]	Porosity [%]	Permeability [m/s]		Effective diffusion coefficient [m^2/s]	Dispersivity Coefficients [m]
			Regional	Local		
Tithonian	Variable	10	3.10^{-5}	3.10^{-5}	10^{-9}	6.0, 0.6
Kimmeridgian when it outcrops	Variable	10	3.10^{-4}	3.10^{-4}	10^{-9}	6.0, 0.6
Kimmeridgian under cover			10^{-11}	10^{-12}		
Oxfordian L2a-L2b	165	6	$2\ 10^{-7}$	10^{-9}	10^{-9}	6.0, 0.6
Oxfordian Hp1-Hp4	50	18	$6\ 10^{-7}$	$8\ 10^{-9}$	10^{-9}	1600, 30
Oxfordian C3a-C3b	60	1	10^{-10}	10^{-12}	4.10^{-12}	6.0, 0.6
Callovo-Oxfordian Cox	135	1	$K_v=10^{-14}$ $K_h=10^{-12}$		4.10^{-12}	6.0, 0.6

Near-field Computation

Research Group MoMaS

With Pascal Omnes and Paul-Marie Berthe (CEA Saclay, DEN/DM2S/SFME/MTMS)





DDM for evolution problems

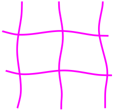
The goals

- Different time and space steps in different domains, adapted to the physics,
- Different models in different subdomains,
- Easy to use, fast and cheap.

The tools

- Work on the PDE level, globally in time,
- Use time windows for long time simulations,
- Use physical transmission conditions, transmit with optimized transmission condition,

Robin transmission conditions



Order 2 transmission conditions

- Then discretize separately.

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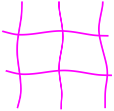


Order 2 transmission conditions

- Then discretize separately.

Optimized Schwarz Waveform Relaxation

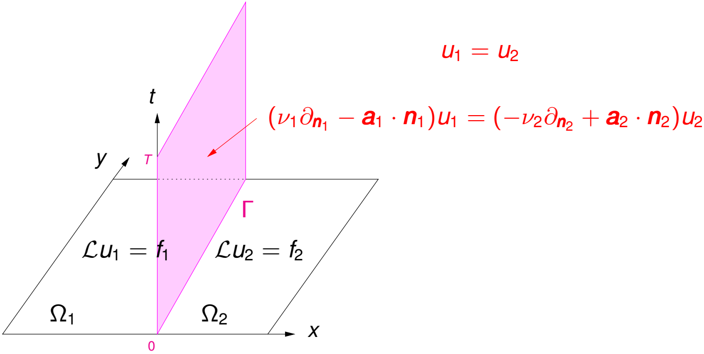
Heterogeneous advection-diffusion equation (Radionuclide transport)

$$\mathcal{L}u = \partial_t u + \nabla \cdot (\boldsymbol{a}(\boldsymbol{x})u - \nu(\boldsymbol{x})\nabla u) + cu = f \text{ in } \Omega \times [0, T]$$

$$u = 0 \quad \text{on } \partial\Omega \times [0, T], \quad u(.,0) = u_0 \text{ in } \Omega$$

$\boldsymbol{a}(\boldsymbol{x})_{|\Omega_i} = \boldsymbol{a}_i(\boldsymbol{x})$ and $\nu(\boldsymbol{x})_{|\Omega_i} = \nu_i(\boldsymbol{x})$, $\nu(\boldsymbol{x}) > 0$

the interface is placed on the jump of $\boldsymbol{a}$, ν

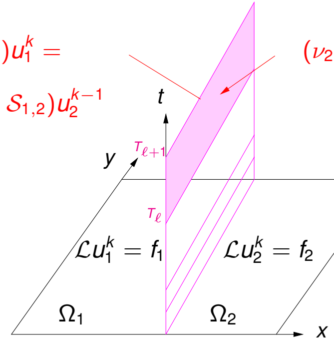


Optimized Schwarz Waveform Relaxation Method

(M. Gander, L. Halpern, F. Nataf (1998), V. Martin (2003), M. Gander, L. Halpern, M. Kern (2007), E. Blayo, L. Halpern, C. J. (2007), D. Bennequin, M. Gander, L. Halpern (2009), L. Halpern, C. J., J. Szeftel (2010))

$$(\nu_1 \partial_{\boldsymbol{n}_1} - \boldsymbol{a}_1 \cdot \boldsymbol{n}_1 + \mathcal{S}_{1,2}) u_1^k = (-\nu_2 \partial_{\boldsymbol{n}_2} + \boldsymbol{a}_2 \cdot \boldsymbol{n}_2 + \mathcal{S}_{1,2}) u_2^{k-1}$$

$$(\nu_2 \partial_{\boldsymbol{n}_2} - \boldsymbol{a}_2 \cdot \boldsymbol{n}_2 + \mathcal{S}_{2,1}) u_2^k = (-\nu_1 \partial_{\boldsymbol{n}_1} + \boldsymbol{a}_1 \cdot \boldsymbol{n}_1 + \mathcal{S}_{2,1}) u_1^{k-1}$$



Choose $\mathcal{S}_{1,2}$ and $\mathcal{S}_{2,1}$ in order to optimize the convergence factor

Optimized Schwarz Waveform Relaxation Method

$$\mathcal{S}_{1,2} = p_{1,2} + q_{1,2}(\partial_t + \partial_{\boldsymbol{\tau}_2}(r_{1,2} - s_{1,2}\,\partial_{\boldsymbol{\tau}_2}))$$
$$\mathcal{S}_{2,1} = p_{2,1} + q_{2,1}(\partial_t + \partial_{\boldsymbol{\tau}_1}(r_{2,1} - s_{2,1}\,\partial_{\boldsymbol{\tau}_1}))$$

$$(\nu_1\partial_{\boldsymbol{n}_1} - \boldsymbol{a}_1 \cdot \boldsymbol{n}_1 + \mathcal{S}_{1,2})u_1^k = (-\nu_2\partial_{\boldsymbol{n}_2} + \boldsymbol{a}_2 \cdot \boldsymbol{n}_2 + \mathcal{S}_{1,2})u_2^{k-1}$$

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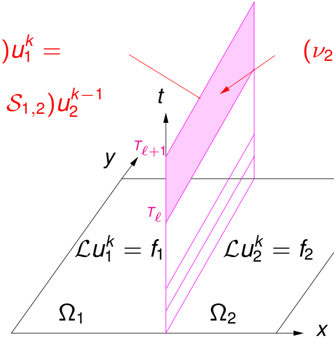

where $p_{1,2}, p_{2,1}, q_{1,2}, q_{2,1}$ are taken such that

- They optimize the convergence factor,
- The transmission conditions imply the coupling conditions at convergence,
- The subdomain problems are well-posed.

For the simulations : $r_{i,j} = \boldsymbol{a}_j \cdot \boldsymbol{\tau}_j$ and $s_{i,j} = \nu_j$

Optimized Schwarz Waveform Relaxation Method

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$$(\nu_1\partial_{\boldsymbol{n}_1} - \boldsymbol{a}_1\cdot\boldsymbol{n}_1 + \mathcal{S}_{1,2})u_1^k = (-\nu_2\partial_{\boldsymbol{n}_2} + \boldsymbol{a}_2\cdot\boldsymbol{n}_2 + \mathcal{S}_{1,2})u_2^{k-1}$$

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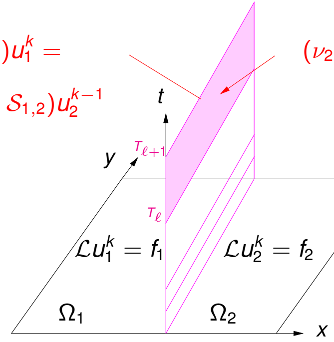

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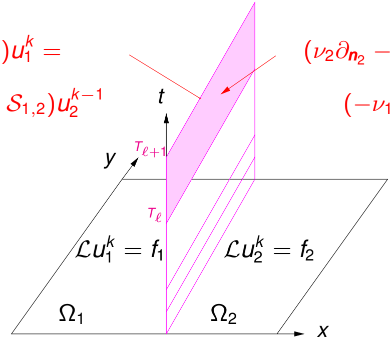

How to discretize these conditions with nonmatching space-time grids ?

Optimized Schwarz Waveform Relaxation Method

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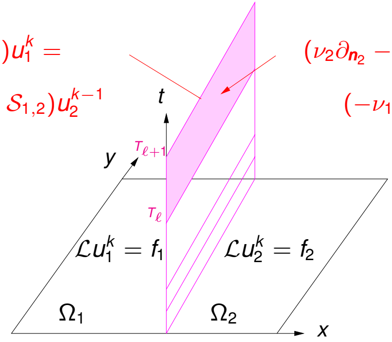


How to discretize these conditions with nonmatching space-time grids ?

Time discontinuous Galerkin method

Subdomain solver : Time Discontinuous Galerkin

(K. Eriksson, C. Johnson, V. Thomée (1985), L. Halpern, C. J. (2008))

Problem in a subdomain Ω_i, in one time window $I = (T_\ell, T_{\ell+1})$

$$\left\{\begin{array}{rcll} \partial_t u + \nabla\cdot(\boldsymbol{a}u - \nu\nabla u) + cu & = & f & \text{in } \Omega_i \times I, \\ u(\cdot, T_\ell) & = & u_0 & \text{in } \Omega_i, \\ (\nu\partial_{\boldsymbol{n}} - \boldsymbol{a}\cdot\boldsymbol{n} + p + q(\partial_t + \partial_{\boldsymbol{\tau}}(r - s\,\partial_{\boldsymbol{\tau}}))\,u & = & g & \text{on } \Gamma \times I \end{array}\right.$$

Let $H^\sigma_\sigma(\Omega_i) = \{v \in H^\sigma(\Omega_i),\ v_{|\Gamma} \in H^\sigma(\Gamma)\}, \sigma > \frac{1}{2}$, and $((u,v)) = (u,v)_{L^2(\Omega_i)} + q(u,v)_{L^2(\Gamma)}$

Weak formulation : Find u such that

$$((\partial_t u, v)) + a(u,v) = \ell(v), \quad \forall v \in H^1_1(\Omega_i)$$

with

$$a(u,v) = \int_{\Omega_i} \frac{1}{2}((\boldsymbol{a}\cdot\nabla u)v - (\boldsymbol{a}\cdot\nabla v)u)\,dx + \int_{\Omega_i} \nu\nabla u\cdot\nabla v\,dx + \int_{\Omega_i}(c + \frac{1}{2}\nabla\cdot\boldsymbol{a})uv\,dx$$
$$+ \int_\Gamma ((p - \frac{\boldsymbol{a}\cdot\boldsymbol{n}}{2} + \frac{q}{2}\partial_{\boldsymbol{\tau}}\cdot r)uv + \frac{q}{2}(\partial_{\boldsymbol{\tau}}\cdot(ru)v - \partial_{\boldsymbol{\tau}}\cdot(rv)u) + qs\partial_{\boldsymbol{\tau}}u\cdot\partial_{\boldsymbol{\tau}}v)\,d\sigma,$$

$$\ell(v) = (f,v)_{L^2(\Omega_i)} + (g,v)_{L^2(\Gamma)}$$

Discontinuous Galerkin time stepping method

Let $\mathcal{T}$ be a partition of $I = \cup_{n=0}^{N} I^n$, with $I^n = (t_n, t_{n+1}]$, and $\Delta t_n = t_{n+1} - t_n$.
Let $\varphi(t_n^{\pm}) = \lim_{t \to t_n \pm 0} \varphi(t)$. We define

$$\begin{aligned} \mathbf{P}_q(V) &= \{\varphi : \varphi(t) = \sum_{i=0}^{q} \varphi_i t^i,\ \varphi_i \in V\} \\ \mathcal{P}_q(V, \mathcal{T}) &= \{\varphi : I \to V,\ \varphi_{|I_n} \in \mathbf{P}_q(V),\ 0 \le n \le K\}. \end{aligned}$$

The discontinous Galerkin method defines recursively on I_n, an approximate solution U in $\mathcal{P}_q(H_1^1(\Omega_i), \mathcal{T})$ such that

$$\begin{cases} U(0, \cdot) = u_0, \\ \forall \varphi \in \mathcal{P}_q(H_1^1(\Omega_i), \mathcal{T}) : \quad \displaystyle\int_{I_n} \left(\left(\left(\frac{dU}{dt}, \varphi\right)\right) + a(U, \varphi) \right) dt \\ \qquad +((\, U(\cdot, t_n^+), \varphi(\cdot, t_n^+)\,)) = ((\, U(\cdot, t_n), \varphi(\cdot, t_n^+)\,)) + \displaystyle\int_{I_n} \ell(\varphi) dt \end{cases}$$

Discontinuous Galerkin time stepping method

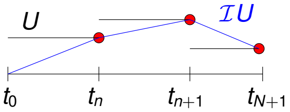
(C. Makridakis, R. Nochetto (2006))

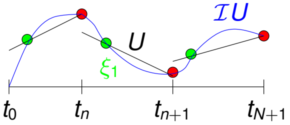
Let $0 < \xi_1, \ldots, \xi_{q+1} = 1$, the Gauss-Radau points such that

$$\int_0^1 f(t)dt \approx \sum_{j=1}^{q+1} w_j\, f(\xi_j)$$

is exact in $\mathbf{P}_{2q}$, and the interpolation operator $\mathcal{I}_n$ on $[t_n, t_{n+1}]$ at points $(t_n, t_n + \xi_1 \Delta t_n, \ldots, t_n + \xi_{q+1} \Delta t_n)$ with $\mathcal{I}_n U(t_n) = U(t_n^-)$.

Let $\mathcal{I} : \mathcal{P}_q(H_1^1(\Omega_i), \mathcal{T}) \rightarrow \mathcal{P}_{q+1}(H_1^1(\Omega_i), \mathcal{T})$ be the operator whose restriction to each subinterval is $\mathcal{I}_n$. It satisfies $\mathcal{I} U(t_n^+) = U(t_n^-)$.





Discontinuous Galerkin time stepping method

The discrete equation can be rewritten as

$$\int_{I_n} \left(\left(\left(\frac{d\mathcal{I}U}{dt}, V \right)\right) + a(U,V) \right) dt = \int_{I_n} L(V)\, dt$$

or in the strong formulation

$$\begin{gathered} \partial_t(\mathcal{I}U) + \nabla \cdot (\boldsymbol{a}U - \nu \nabla U) + cU = Pf \text{ in } \Omega_i \times I \\ (\nu\, \partial_{\boldsymbol{n}} - \boldsymbol{a} \cdot \boldsymbol{n}) U + p\, U + q(\partial_t(\mathcal{I}U) + \partial_\tau(rU - s\, \partial_\tau U)) = Pg \text{ on } \Gamma \times I \end{gathered}$$

where P is the L^2 projection in time on $\mathcal{P}_q(V, \mathcal{T})$

Time Nonconforming Domain Decomposition Method

Let $\mathcal{T}_i$ be the partition of the time window $(T_\ell, T_{\ell+1})$ in subdomain Ω_i, with $N_i + 1$ intervals I_n^i. We define interpolation operators $\mathcal{I}^i$ and projection operators P^i.

Let $m \geq 1$ small in order to make very few iterations in each time window. Let V_i^m be a discrete approximation of u_i in $\Omega_i \times (T_{\ell-1}, T_\ell)$ at step m of the method.

Then, the next time window's solution U_i is obtained after m DG-OSWR iterations, starting with a given initial guess $(g_{i,j})$ in $\mathcal{P}_d(L^2(\Gamma_{i,j}), \mathcal{T}_i)$

$$
\begin{aligned}
&\partial_t(\mathcal{I}^i U_i^k) + \nabla \cdot (\boldsymbol{a} U_i^k - \nu_i \nabla U_i^k) + c_i\, U_i^k = P^i f \text{ in } \Omega_i \times (0, T), \\
&\left(\nu_i \partial_{\boldsymbol{n}_i} - \boldsymbol{a}_i \cdot \boldsymbol{n}_i\right) U_i^k + S_{i,j} U_i^k = \\
&\qquad\qquad P^i \left((\nu_j \partial_{\boldsymbol{n}_i} - \boldsymbol{a}_j \cdot \boldsymbol{n}_i)\, U_j^{k-1} + \widetilde{S}_{i,j} U_j^{k-1} \right) \text{ on } \Gamma_{i,j} \times (0, T), \\
&U_i^k(\cdot, T_\ell) = V_i^m(\cdot, T_\ell),
\end{aligned}
$$

with

$$
\begin{aligned}
S_{i,j}\, U &= p_{i,j}\, U + q_{i,j}\, (\partial_t(\mathcal{I}^i U) + \partial_\tau (r_{i,j} U - s_{i,j} \partial_\tau U)) \\
\widetilde{S}_{i,j}\, U &= p_{i,j}\, U + q_{i,j}\, (\partial_t(\mathcal{I}^j U) + \partial_\tau (r_{i,j} U - s_{i,j} \partial_\tau U))
\end{aligned}
$$

Theorem (L. Halpern, C. J., J. Szeftel (DD18,2009))

(Well-posedness and convergence) The coupled discret problem in time has a unique solution and the discret Schwarz algorithm is convergent for

- $q_{i,j} = 0$, $p_{j,i} - p_{i,j} - \mathbf{a}_i \cdot \mathbf{n}_i = 0$, $p_{i,j} > \frac{\mathbf{a}_i \cdot \mathbf{n}_i}{2}$, $\nu_i \neq \nu_j$, $\mathbf{a}_i \neq \mathbf{a}_j$, *and rectangular subdomains*
- $p_{i,j} = p_{j,i} = p > 0$, $q_{i,j} = q_{j,i} = q > 0$, $\nu_i = \nu_j$, $s_{i,j} = s > 0$, $\mathbf{a}_i = 0$, $r_{i,j} = 0$, *and strips*

(Error estimates) If $\nabla \cdot \mathbf{a} = 0$, $p_{i,j} - \frac{\mathbf{a}_i \cdot \mathbf{n}_i}{2} = p_{j,i} - \frac{\mathbf{a}_j \cdot \mathbf{n}_j}{2} = p > 0$, $q_{i,j} = 0$, $\Delta t = max_n \Delta t_n$, *then*

$$\sum_{i=1}^{I} \|u - U_i\|^2_{L^\infty(0,T,L^2(\Omega_i))} \leq C \Delta t^{2(q+1)} \|\partial_t^{q+1} u\|^2_{L^2(0,T;H^2(\Omega))}.$$

Theorem (L. Halpern, C. J., J. Szeftel (DD19,2010))

(Well-posedness) For $\mathbf{a}_i$, ν_i, $r_{i,j}$, $s_{i,j}$, $p_{i,j}$ *and* $q_{i,j}$ *sufficiently smooth with* $q_{i,j} \geq 0$, $s_{i,j} > 0$ *a.e., the subdomain problem has a unique solution.*

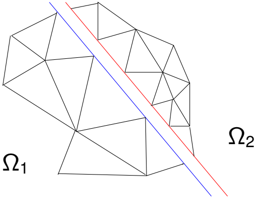
(Convergence) The continuous Schwarz algorithm converges for

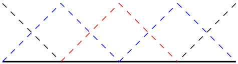
- $q_{i,j} = q_{j,i} = 0$, $p_{i,j} \neq p_{j,i}$, $\nu_i \neq \nu_j$, $\mathbf{a}_i \neq \mathbf{a}_j$ *and general decomposition*
- *if* $q_{i,j} = q_{j,i} = q > 0$, $p_{i,j} \neq p_{j,i}$, $\nu_i \neq \nu_j$, $\mathbf{a}_i \neq \mathbf{a}_j$ *and decomposition into bands.*

The proof relies on energy estimates and Gronwall Lemma.
Generalization to **general geometries and piecewise smooth coefficients** of results in D. Bennequin, M. Gander, L. Halpern (2009)

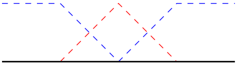
Space discretization with nonconforming space grids

(M. Gander, C. J., Y. Maday, F. Nataf (2004))
For stationary elliptic problems and Robin transmission conditions
Finite element space : $V = \prod_{i=1}^{K} X_i^h$





Basis functions of X_i^h



Basis functions of W_i^h

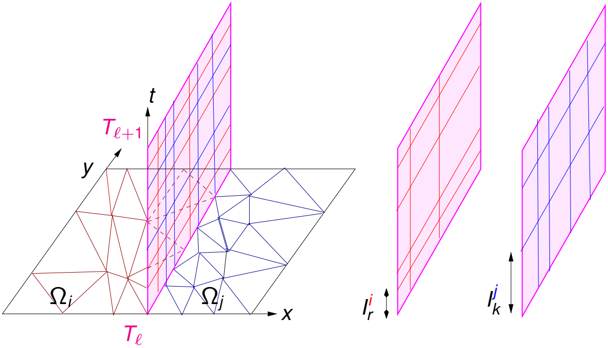
$$\int_\Gamma (\nu_1 \partial_{\mathbf{n}_1} - \mathbf{a}_1 \cdot \mathbf{n}_1 + \mathcal{S}_{1,2}) u_1 \, \psi_1 = \int_\Gamma (-\nu_2 \partial_{\mathbf{n}_2} + \mathbf{a}_2 \cdot \mathbf{n}_2 + \mathcal{S}_{1,2}) u_2 \, \psi_1, \quad \forall \psi_1 \in W_h^1$$
$$\int_\Gamma (\nu_2 \partial_{\mathbf{n}_2} - \mathbf{a}_2 \cdot \mathbf{n}_2 + \mathcal{S}_{2,1}) u_2 \, \psi_2 = \int_\Gamma (-\nu_1 \partial_{\mathbf{n}_1} + \mathbf{a}_1 \cdot \mathbf{n}_1 + \mathcal{S}_{2,1}) u_1 \, \psi_2, \quad \forall \psi_2 \in W_h^2$$

Extension to time advection-diffusion problems and order 2 transmission conditions

An efficient way to perform the projections between space-time grids

(Martin J. Gander, C. J. (2009))

Generalization in 2d and 3d of the algorithm in M. Gander, L. Halpern, F. Nataf (2003)



Let V_k^i (resp. V_ℓ^j) the shape functions of $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_i)$ (resp. $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_j)$)

How to compute $M_{k,\ell} = \int_I V_k^i V_\ell^j$?

$\Rightarrow$ **Algorithm with linear complexity and without an additional grid**

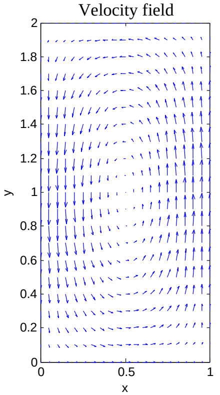
www.unige.ch/∼gander

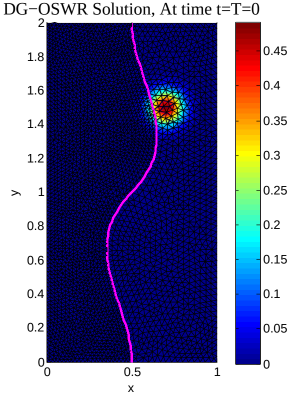
Numerical results. Extension to a transport equation with discontinuous porosity ω

$$\mathcal{L}u = \omega \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{a}(\mathbf{x})u - \nu(\mathbf{x})\nabla u) + cu = f$$

Initial condition $u_0(x) = \frac{1}{2} e^{(-100((x-0.7)^2+(y-1.5)^2))}$

$\mathbf{a}_1 = (-sin(\frac{\pi}{2}(y-1))cos(\pi(x-\frac{1}{2})), 3cos(\frac{\pi}{2}(y-1))sin(\pi(x-\frac{1}{2}))), \nu_1 = 0.003, \omega_1 = 0.1$,
$\mathbf{a}_2 = \mathbf{a}_1, \nu_2 = 0.01$, $\omega_2 = 1$, final time $T = 1.5$
Boundary conditions : homogeneous Dirichlet (S,W), Neumann (N,E)

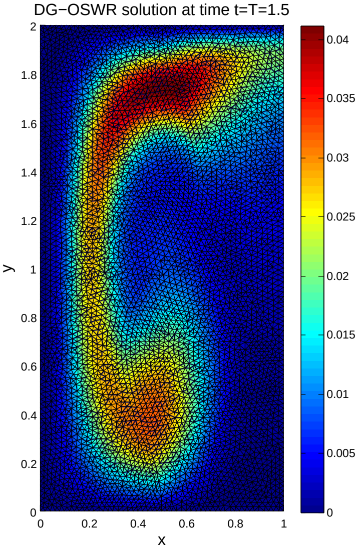


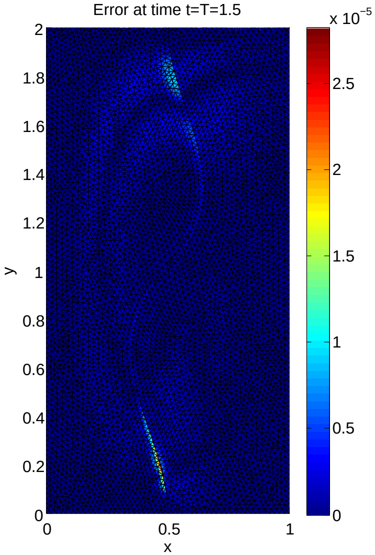


Time nonconforming DG-OSWR solution (after 5 iterations)

Domain 1 (left), 180 time steps, Domain 2 (right), 100 time steps

On the right : error with the one domain solution

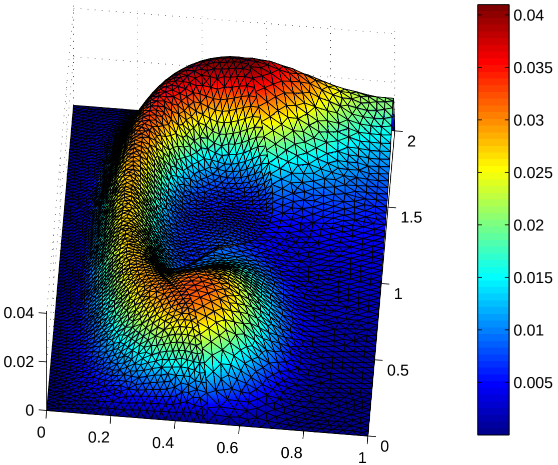




Space-time nonconforming DG-OSWR solution (after 5 iterations)

Domain 1 (left), mesh size $h_1 = 1/66$, 180 time steps

Domain 2 (right), mesh size $h_2 = 1/44$, 100 time steps

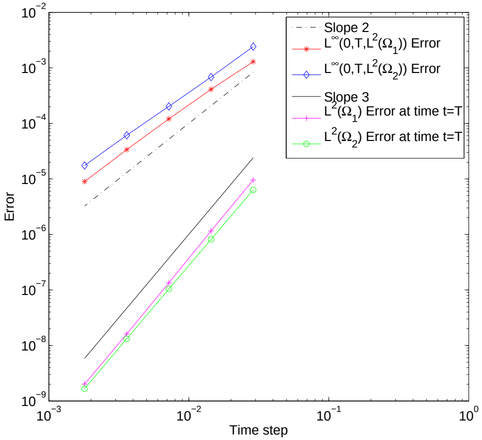


Relative error versus the time step

One domain : $\Delta t = \frac{1}{7680}$

domain 1 : $\Delta t_1 = \frac{1}{120}$, domain 2 : $\Delta t_2 = \frac{1}{26}$
then divide by 2 the time steps Δt_1 and Δt_2

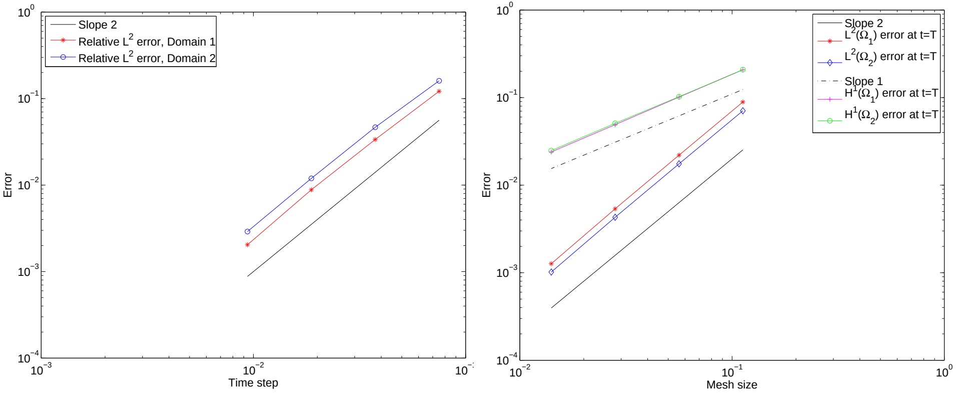
The converged solution is such that the residual is smaller than 10^{-12}



Relative error versus the mesh size and time step

One domain : $\Delta t = \frac{1}{960}, h = 3.5\,10^{-3}$

domain 1 : $\Delta t_1 = \frac{1}{60},\ h_1 = 0.056$, domain 2 : $\Delta t_2 = \frac{1}{20},\ h_2 = 0.11$
then divide by 2 the time steps and mesh sizes



Application to Porous Media
With Pascal Omnes (CEA)

Time domain : $(0, T)$, $T = 10^{11} s \approx 10^4$ years

time steps : $10^9\ s$ in the clay (host rock)

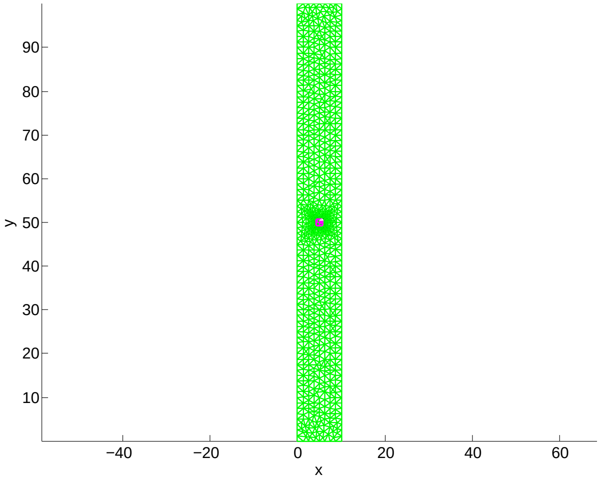
$2.10^8\ s$ in the repository

mesh size : $h = 0.15\ m$ in the repository

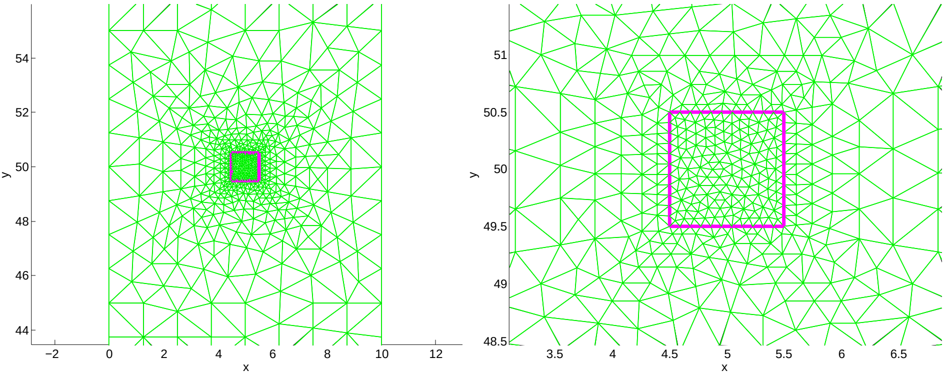
$h = 2\ m$ in the clay

- Isotropic case with $\nu = 6.10^{-13}\ m^2/s,\ \omega = 0.06$ in the clay
  $\nu = 10^{-11}\ m^2/s,\ \omega = 1$ in the repository
- computational domain (in meters) : $(0, 10) \times (0, 100)$ for the clay
  $(4.5, 5.5) \times (49.5, 50.5)$ for the repository
- Initial condition : $u_0 = 0$ in the clay
  $u_0 = 1$ in the repository
- Boundary conditions : homogeneous Neumann at $x = 0$ and $x = 10$, homogeneous Dirichlet at $y = 0$ and $y = 100$.

Computational domain



Computational domain (zoom)

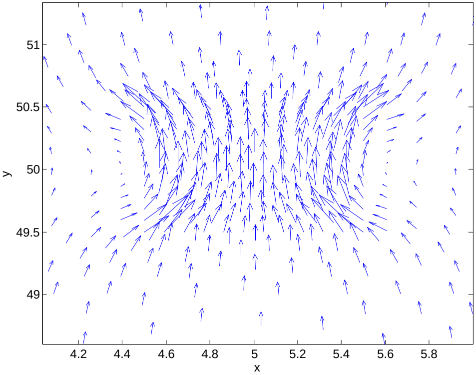


Darcy flow

$$\nabla \cdot \boldsymbol{a} = 0$$
$$\boldsymbol{a} = -K \nabla h$$

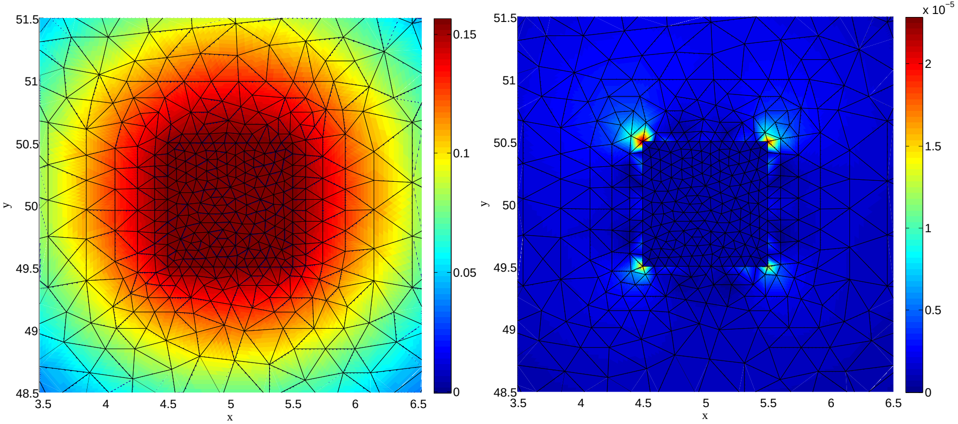
with $K = k\, Id$, and $k = 10^{-8}$ in the repository and $k = 10^{-13}$ in the clay

Boundary conditions : homogeneous Neumann at $x = 0$ and $x = 10$ and Dirichlet conditions with $h = 100$ at $y = 0$ and $h = 0$ at $y = 100$



Time nonconforming DG-OSWR solution
With 10 time windows and 4 iterations per window

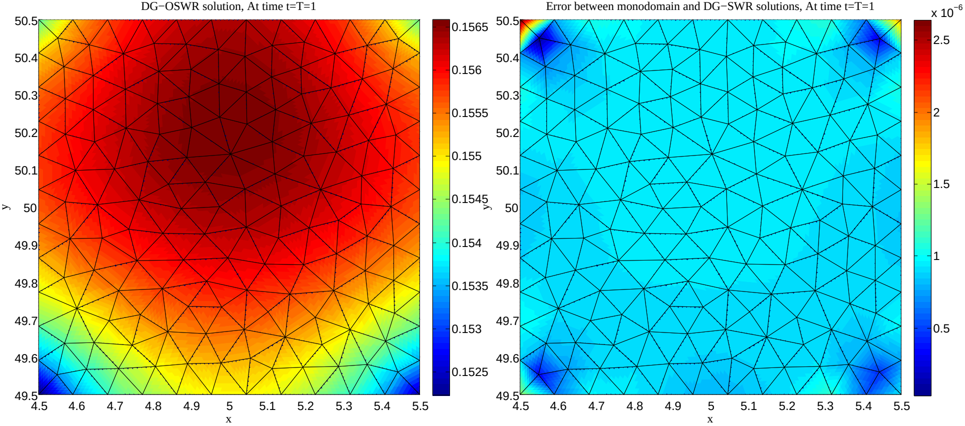
(Left : DG-OSWR solution. Right : error with the one domain solution)



Time nonconforming DG-OSWR solution (repository)

With 10 time windows and 4 iterations per window

(Left : DG-OSWR solution. Right : error with the one domain solution)

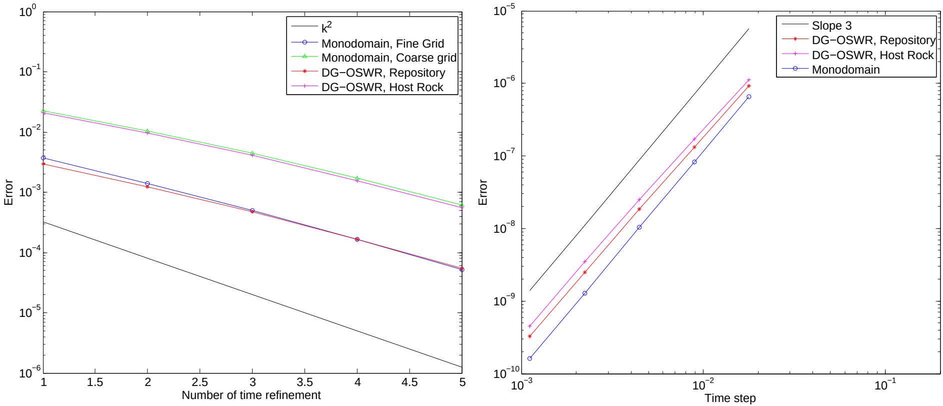


Relative error versus the time step

One domain : $\Delta t = \frac{1}{8192}$

domain 1 : $\Delta t_1 = \frac{1}{128}$, domain 2 : $\Delta t_2 = \frac{1}{28}$
then divide by 2 the time steps Δt_1 and Δt_2

The converged solution is such that the residual is smaller than 10^{-12}

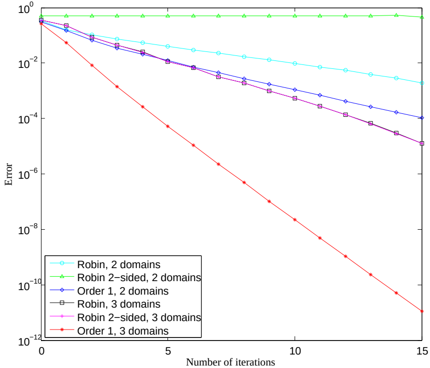


Choice of the parameters : an example in 1d

(The Order 2 conditions reduce to $\mathcal{S}_{i,j} = p_{i,j} + q_{i,j}\partial_t$)

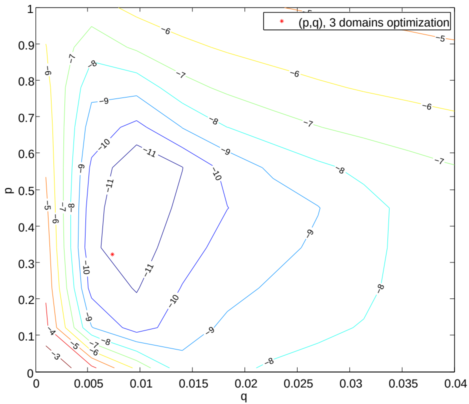
L. Halpern, C. J., P. Omnes (ECCOMAS CFD, 2010)
Optimization of the convergence factor : Fourier analysis + domains with highly variable length (classical analysis : two half-spaces case).

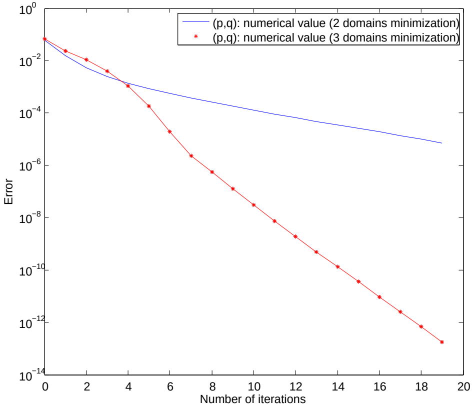
$\Omega_1 = (0, 0.4965)$, $\Omega_2 = (0.4965, 0.5035)$ (repository), $\Omega_3 = (0.5035, 1)$, final time $T = 0.04$.
We take $u_0 = 1$ in the repository, $u_0 = 0$ in the clay, $f = 0$, $c = 0$, $a = 1$, $\nu_2 = 1$, $\omega_2 = 1$,
$\nu_1 = \nu_3 = 0.06$, $\omega_1 = \omega_3 = 0.06$,
$\Delta t_2 = \frac{T}{500}$, $\Delta t_1 = \Delta t_3 = \frac{T}{100}$, $h_2 = \frac{0.007}{20}$ and $h_1 = h_3 = \frac{0.4965}{200}$.



Convergence in the 2d conforming case

Constant parameters $p_{1,2} = p_{2,1} = p$, $q_{1,2} = q_{2,1} = q$ (mean value of the (p, q) obtained by optimization of the convergence factor)





Work in progress

- Extension to the MoMaS approach (Mixte Finite Elements), 3d benchmark
  (with Jérôme Jaffré, Michel Kern and Jean Roberts)
- Extension to finite volume methods and general decomposition (with corners) for Order 2 transmission conditions, 3d simulations
  (with P-M. Berthe and P. Omnes)
- Analysis of the fully discrete method
- Analysis of the influence of the decomposition in time windows
- Numerical and mathematical analysis of the convergence factor
  (with M.J. Gander)

Nonoverlapping algorithms

$$\partial_t u - \nu \Delta u + f(u) = 0. \quad f\ \mathcal{C}^2, f(0) = 0.$$

F. Caetano, M. Gander, L.H, J. Szeftel, DD19.
Well-posedness and convergence for linear Robin TC (common existence time for all iterates by energy estimates) to appear in NTMC 09.
New : nonlinear TC (DD19)

Robin : $B_i(u) = \dfrac{\partial u}{\partial n_i} + p(u)u.$

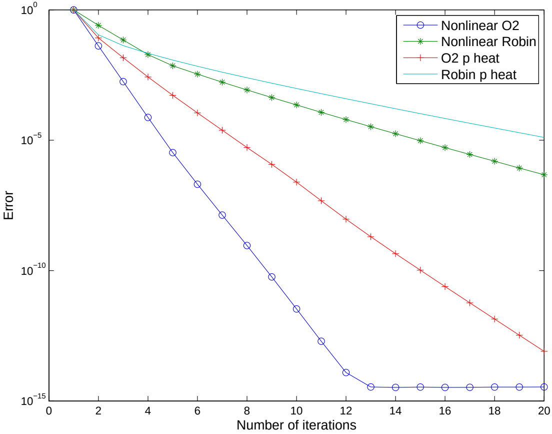
Order 2 : $B_i(u) = \dfrac{\partial u}{\partial n_i} + p(u)u + q(u)(\partial_t u - \nu \Delta_y u).$

$$p(u) = p^*(\vec{0}, \nu, f'(u)), \quad q(u) = q^*(\vec{0}, \nu, f'(u))$$

p^* and q^* asymptotic formulas.

A simple example

$f(u) = 10(\exp(u) - 1)$, two subdomains, $T = 1$.



Convergence history.

Overlapping algorithms

TRAN Minh Binh.

New study of well-posedness and convergence,

- Classical Schwarz algorithm for the semilinear heat equation, with explosion permitted : f is in $\mathcal{C}^1(\mathbb{R})$ and

$$|f'(x)| \leq C_f |x|^{p-1}, \quad p \geq 1$$

- Extension to systems of stochastic differential equations.