De calculer juste
à calculer au plus juste

Introduction à l’école PRCN

Florent de Dinechin
AriC project
The AriC project @ École Normale Supérieure de Lyon: Arithmetic and Computing at large

- Hardware and software
- From addition to linear algebra
- Fixed point, floating-point, multiple-precision, finite fields, ....
- Pervasive concern of performance, numerical quality and validation
- Interactions with computing at large
Outline

Floating-point in your machine

Accuracy versus reproductibility

Performance versus accuracy

Conclusion: It’s the Hardware, Stupid

Space-filling advertising: hardware computing just right
Floating-point in your machine

Accuracy versus reproducibility

Performance versus accuracy

Conclusion: It’s the Hardware, Stupid

Space-filling advertising: hardware computing just right
We have a nice floating-point standard

It is called IEEE-754, and you will hear a lot about it. For instance,

**Correct rounding to the nearest**

The basic operations (noted \( \oplus, \ominus, \otimes, \oslash \)), and the square root should return the FP number closest to the mathematical result.
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Nice properties:

- If $a + b$ is a FP number, then $a \oplus b$ returns it
- Rounding is monotonic
- Rounding does not introduce any statistical bias
Let us compile the following C program:

```c
float ref, index;

ref = 169.0 / 170.0;

for (i = 0; i < 250; i++) {
    index = i;
    if (ref == (index / (index + 1.0))) break;
}

printf("i=%d\n", i);
```
Equality test between FP variables is dangerous.

Or,

If you can replace \( a == b \) with \( (a-b) < \epsilon \) in your code, do it!
Equality test between FP variables is dangerous.

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A physical point of view

Given two coordinates $(x, y)$ on a snooker table, the probability that the ball stops at position $(x, y)$ is always zero.
First conclusion

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If you can replace \( a==b \) with \( (a-b)<\varepsilon \) in your code, do it!

A physical point of view

Given two coordinates \((x, y)\) on a snooker table, the probability that the ball stops at position \((x, y)\) is always zero.

Still, on this expensive laptop, FP computing is not straightforward, even within such a small program.
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Or,

If you can replace `a==b` with `(a-b)<epsilon` in your code, do it!

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Go fetch me the person in charge
Who is in charge of floating-point?

- The processor
  - has internal FP registers,
  - performs basic FP operations,
  - raises exceptions,
  - writes results to memory.
Who is in charge of floating-point?

- The processor
- The operating system

- handles exceptions
- computes functions/operations not handled directly in hardware
  - most elementary functions (sine/cosine, exp, log, ...),
  - divisions and square roots on recent processors
  - subnormal numbers
- handles floating-point status: precision, rounding mode, ...
  - older processors: global status register
  - more recent FPUs: rounding mode may be encoded in the instruction
Who is in charge of floating-point?

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- The operating system
- The programming language
  - should have a well-defined semantic
Who is in charge of floating-point?

- The processor
- The operating system
- The **programming language**
  - should have a well-defined semantic,
  - ... (detailed in some arcane 1000-pages document)
Who is in charge of floating-point?

- The processor
- The operating system
- The programming language
- The **compiler**
  - has hundreds of options
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  - but probably not by default:
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  - has hundreds of options
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  - but probably not by default:
  - Marketing says: default should be optimize for speed!
Who is in charge of floating-point?

- The processor
- The operating system
- The programming language
- The compiler
- The programmer
  - ... is in charge in the end.
Who is in charge of floating-point?

- The processor
- The operating system
- The programming language
- The compiler
- The **programmer**
  - ... is in charge in the end.

Of course, eventually, the programmer will get the blame.
The common denominator of modern processors

- Hardware support for
  - addition/subtraction and multiplication
  - in single-precision (binary32) and double-precision (binary64)
  - SIMD versions: two binary32 operations for one binary64
  - various conversions and memory accesses

Typical performance (for one SIMD way):
- 3-7 cycles for addition and multiplication, pipelined (1 op/cycle)
- 15-50 cycles for division and square root, hard or soft, not pipelined (1 op / \(n\) cycles).
- 50-500 cycles for elementary functions (soft)
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Keep clear from the legacy IA32/x87 FPU

- It is slower than the (more recent) SSE2 FPU
- It is more accurate (“double-extended” 80 bit format), but at the cost of entailing horrible bugs in well-written programs
- the bane of floating-point between 1985 and 2005
A funny horror story

(real story, told by somebody at CERN)

- Use the (robust and tested) standard sort function of the STL C++ library
- to sort objects by their radius: according to $x^2 + y^2$.
- Sometimes (rarely) segfault, infinite loop...
- Why? Because the sort algorithm works under the following naive assumption: if $A \preceq B$, then, later, $A \succeq B$
  - $x^2 + y^2$ inlined and compiled differently at two points of the program,
  - computation on 64 or 80 bits, depending on register allocation
  - enough to break the assumption (horribly rarely).

We will see there was no programming mistake.
And it is very difficult to fix.
The SSE2 unit of current IA32 processors

- Available for all recent x86 processors (AMD and Intel)
- An additional set of 128-bit registers
- An additional FP unit able of
  - 2 identical binary64 FP operations in parallel, or
  - 4 identical binary32 FP operations in parallel.
- clean and standard implementation
  - subnormals trapped to software, or flushed to zero
  - depending on a compiler switch (gcc has the safe default)

And soon AVX: multiply all these numbers by 2
(256-bit registers, etc)
Quickly, the Power family

Power and PowerPC processors, also in IBM mainframes and supercomputers

- No floating-point adders or multipliers
- Instead, one or two FMA: Fused Multiply-and-Add
- Compute \( o(a \times b + c) \):
  - faster: roughly in the time of a FP multiplication
  - more accurate: only one rounding instead of two
  - enable efficient implementation of division and square root
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- Standardized in IEEE-754-2008
  - but not yet in your favorite language
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  ARM, Power, IA64, all GPGPUs, and even latest Intel and AMD processors.
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- All the modern FPUs are built around the FMA:
  ARM, Power, IA64, all GPGPUs, and even latest Intel and AMD processors.

- enables classical operations, too...
  - Addition: $\circ (a \times 1 + c)$
  - Multiplication: $\circ (a \times b + 0)$
Using it breaks some expected mathematical properties:

- Loss of symmetry in $\sqrt{a^2 + b^2}$
- Worse: $a^2 - b^2$, when $a = b$:
  $\circ(\circ(a \times a) - a \times a)$
- Worse: if $b^2 \geq 4ac$ then (...) $\sqrt{b^2 - 4ac}$
\( \circ (a \times b + c) \)

Using it breaks some expected mathematical properties:

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  \[ \circ ( \circ (a \times a) - a \times a ) \]
- Worse: if \( b^2 \geq 4ac \) then \((...)\) \( \sqrt{b^2 - 4ac} \)

Do you see the sort bug lurking?

By default, gcc disables the use of FMA altogether (except as + and \( \times \))

(compiler switches to turn it on)
Reproductibility begins with predictability

When you write

$$\sqrt{b^2-4ac}$$

do you know how it is going to be compiled?
Consider the following program, whatever the language

```plaintext
float a, b, c, x;
x = a + b + c + d;
```

Two questions:
- In which order will the three addition be executed?
- What precision will be used for the intermediate results?
Consider the following program, whatever the language

```c
float a, b, c, x;
x = a + b + c + d;
```

Two questions:

- In which order will the three addition be executed?
- What precision will be used for the intermediate results?

Fortran, C and Java have completely different answers.
float a, b, c, x;
x = a + b + c + d;

- In which order will the three addition be executed?
  - With two FPUs (dual FMA, or SSE2, ...),
    \((a + b) + (c + d)\) faster than \(((a + b) + c) + d\)
float a,b,c,x;
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  - With two FPUs (dual FMA, or SSE2, ...),
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  - If \(a, c, d\) are constants, \((a + c + d) + b\) faster.
float a, b, c, x;

x = a + b + c + d;

- In which order will the three addition be executed?
  - With two FPUs (dual FMA, or SSE2, ...),
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  - (here we should remind that FP addition is not associative

Consider \(2^{100} + 1 - 2^{100}\)
float a, b, c, x;

\[ x = a + b + c + d; \]

- In which order will the three addition be executed?

  - With two FPUs (dual FMA, or SSE2, ...),
    \( (a + b) + (c + d) \) faster than \( ((a + b) + c) + d \)
  - If \( a, c, d \) are constants, \( (a + c + d) + b \) faster.
  - (here we should remind that FP addition is not associative
    Consider \( 2^{100} + 1 - 2^{100} \))
  - Is the order fixed by the language, or is the compiler free to choose?
float a, b, c, x;
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  - (here we should remind that FP addition is not associative
    Consider \(2^{100} + 1 - 2^{100}\))
  - Is the order fixed by the language, or is the compiler free to choose?
  - Similar issue: should multiply-additions be fused in FMA?
float a, b, c, x;

In which order will the three addition be executed?

What precision will be used for the intermediate results?

- **Bottom up precision**: (here all `float`)
  - elegant (context-independent)
  - portable
  - sometimes dangerous: compare $C = (F - 32) \times \frac{5}{9}$ and $C = (F - 32) \times \frac{5}{9}$

In C, variable types refer to memory locations

More accurate result

Is the precision fixed by the language, or is the compiler free to choose?
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- Use the maximum precision available which is no slower
  - in C, variable types refer to memory locations
  - more accurate result
float a, b, c, x;

\[ x = a + b + c + d; \]

- In which order will the three addition be executed?
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  - Use the maximum precision available which is no slower
    - in C, variable types refer to memory locations
    - more accurate result
  - Is the precision fixed by the language, or is the compiler free to choose?
Fortran's philosophy (1)


The FORmula TRANslator translates *mathematical* formula into computations.
The FORmula TRANslator translates **mathematical** formula into computations.

*Any difference between the values of the expressions* \((1./3.)*3.\) *and* 1. *is a computational difference, not a mathematical difference. The difference between the values of the expressions* \(5/2\) *and* \(5./2.\) *is a mathematical difference, not a computational difference.*
Fortran's philosophy (2)

Fortran respects mathematics, and only mathematics.

(...) the processor may evaluate any mathematically equivalent expression, provided that the integrity of parentheses is not violated. Two expressions of a numeric type are mathematically equivalent if, for all possible values of their primaries, their mathematical values are equal. However, mathematically equivalent expressions of numeric type may produce different computational results.
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Remark: This philosophy applies to both order and precision.
X,Y,Z of any numerical type, A,B,C of type real or complex, I, J of integer type.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Allowable alternative form</th>
</tr>
</thead>
<tbody>
<tr>
<td>X+Y</td>
<td>Y+X</td>
</tr>
<tr>
<td>X*Y</td>
<td>Y*X</td>
</tr>
<tr>
<td>-X + Y</td>
<td>Y-X</td>
</tr>
<tr>
<td>X+Y+Z</td>
<td>X + (Y + Z)</td>
</tr>
<tr>
<td>X-Y+Z</td>
<td>X - (Y - Z)</td>
</tr>
<tr>
<td>X*A/Z</td>
<td>X * (A / Z)</td>
</tr>
<tr>
<td>X<em>Y-X</em>Z</td>
<td>X * (Y - Z)</td>
</tr>
<tr>
<td>A/B/C</td>
<td>A / (B * C)</td>
</tr>
<tr>
<td>A / 5.0</td>
<td>0.2 * A</td>
</tr>
</tbody>
</table>

Consider the last line:

- \( A/5.0 \) is actually more accurate \( 0.2*A \). Why?
- This line is valid if you replace 5 by 4, but not by 3. Why?
The Patriot bug

In 1991, a Patriot anti-missile failed to intercept a Scud missile. 28 people were killed.

- The code worked with time increments of 0.1 s.
- But 0.1 is not representable in binary.
- In the 24-bit format used, the number stored was 0.099999904632568359375
- The error was 0.0000000953.
- After 100 hours = 360,000 seconds, time is wrong by 0.34s.
- In 0.34s, a Scud moves 500m

Test: which of the following increments should you use?

<table>
<thead>
<tr>
<th>10</th>
<th>5</th>
<th>3</th>
<th>1</th>
<th>0.5</th>
<th>0.25</th>
<th>0.2</th>
<th>0.125</th>
<th>0.1</th>
</tr>
</thead>
</table>

Florent de Dinechin, projet AriC (ex-Arénaire)
Fortunately, Fortran respects your parentheses.

*In addition to the parentheses required to establish the desired interpretation, parentheses may be included to restrict the alternative forms that may be used by the processor in the actual evaluation of the expression. This is useful for controlling the magnitude and accuracy of intermediate values developed during the evaluation of an expression.*

(this was the solution to the last FP bug of LHC@Home at CERN)
X,Y,Z of any numerical type, A,B,C of type real or complex, I, J of integer type.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Forbidden alternative form</th>
</tr>
</thead>
<tbody>
<tr>
<td>I/2</td>
<td>0.5 * I</td>
</tr>
<tr>
<td>X*I/J</td>
<td>X * (I / J)</td>
</tr>
<tr>
<td>I/J/A</td>
<td>I / (J * A)</td>
</tr>
<tr>
<td>(X + Y) + Z</td>
<td>X + (Y + Z)</td>
</tr>
<tr>
<td>(X * Y) - (X * Z)</td>
<td>X * (Y - Z)</td>
</tr>
<tr>
<td>X * (Y - Z)</td>
<td>X<em>Y-X</em>Z</td>
</tr>
</tbody>
</table>
You have been warned.

The inclusion of parentheses may change the mathematical value of an expression. For example, the two expressions $A*\frac{I}{J}$ and $A*(\frac{I}{J})$ may have different mathematical values if $I$ and $J$ are of type integer.

Difference between $C=(F-32)*(5/9)$ and $C=(F-32)*\frac{5}{9}$. 
Enough standard, the rest is in the manual

(yes, you should read the manual of your favorite language and also that of your favorite compiler)
The C philosophy

The “C11” standard:

- Contrary to Fortran, the standard imposes an order of evaluation
  - Parentheses are always respected,
  - Otherwise, left to right order with usual priorities
  - If you write $x = a/b/c/d$ (all FP), you get 3 (slow) divisions.

- Consequence: little expressions rewriting
  - Only if the compiler is able to prove that the two expressions always return the same FP number, including in exceptional cases
Morceaux choisis from appendix F.8.2 of the C11 standard:

- Commutativities are OK

\[ \frac{x}{2} \text{ may be replaced with } 0.5 \times x, \]
\[ \frac{x}{5.0} \text{ may not be replaced with } 0.2 \times x \] (C won't introduce the Patriot bug)

\[ x \times 1 \text{ and } \frac{x}{1} \text{ may be replaced with } x \]

\[ x - x \text{ may not be replaced with } 0 \text{ unless the compiler is able to prove that } x \text{ will never be } \infty \text{ nor NaN} \]

Worse:

\[ x + 0 \text{ may not be replaced with } x \text{ unless the compiler is able to prove that } x \text{ will never be } -0 \]

because \((-0) + (+0) = (+0) \) and not \((-0)\)

On the other hand

\[ x - 0 \text{ may be replaced with } x \text{ if the compiler is sure that rounding mode will be to nearest.} \]

\[ x == x \text{ may not be replaced with } true \text{ unless the compiler is able to prove that } x \text{ will never be NaN.} \]
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Morceaux choisis from appendix F.8.2 of the C11 standard:

- Commutativities are OK
- $x/2$ may be replaced with $0.5*x$, because both operations are always exact in IEEE-754.
- but $x/5.0$ may not be replaced with $0.2*x$
  
  (C won't introduce the Patriot bug)

```c
x*1 and x/1 may be replaced with x
x-x may not be replaced with 0 unless the compiler is able to prove that x will never be \infty nor NaN
```

```c
Worse:

- x+0 may not be replaced with x unless the compiler is able to prove that x will never be −0
  because (−0) + (+0) = (+0) and not (−0)

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- Worse: \(x + 0\) may not be replaced with \(x\) unless the compiler is able to prove that \(x\) will never be \(-0\) because \((-0) + (+0) = (+0)\) and not \((-0)\)
C in the gory details

*Morceaux choisis* from appendix F.8.2 of the C11 standard:

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- \(x-x\) may not be replaced with 0  
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- Worse: \(x+0\) may not be replaced with \(x\)  
  unless the compiler is able to prove that \(x\) will never be \(-0\)  
  because \((-0) + (0) = (+0)\) and not \((-0)\)
- On the other hand \(x-0\) may be replaced with \(x\)  
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- $x == x$ may not be replaced with true  
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Therefore, default behaviour of commercial compiler tend to ignore this part of the standard...
Therefore, **default** behaviour of commercial compiler tend to ignore this part of the standard...
But there is always an option to enable it.
So, perfect determinism wrt order of evaluation

Strangely, intermediate precision is not determined by the standard: it defines a bottom-up minimum precision, but invites the compiler to take the largest precision which is larger than this minimum, and no slower.

Idea:

- If you wrote float somewhere, you probably did so because you thought it would be faster than double.
- If the compiler gives you long double for the same price, you won’t complain.
Small drawback

- Before SSE, float was almost always double or double-extended
- With SSE, float should be single precision (2-4× faster)
- Or, on a newer PC, the same computation became much less accurate!
Drawbacks of C philosophy

- Small drawback
  - Before SSE, \texttt{float} was almost always double or double-extended
  - With SSE, \texttt{float} should be single precision (2-4\times faster)
  - Or, on a newer PC, the same computation became much less accurate!

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  - Thus, sometimes a value is rounded twice, which may be even less accurate than the target precision
  - And sometimes, the same computation may give different results at different points of the program.

The sort bug explained (because `double` promoted to 80 bits)
Integrist approach to determinism: *compile once, run everywhere*

- float and double only.
- Evaluation semantics with **fixed order and precision**.
- No sort bug.
- Performance impact, but...
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The great Kahan doesn’t like it.
- Many numerical unstabilities are solved by using a larger precision
- Look up *Why Java hurts everybody everywhere* on the Internet

I tend to disagree with him here. We can’t allow the sort bug.
Floating point numbers

*These represent machine-level double precision floating point numbers. You are at the mercy of the underlying machine architecture (and C or Java implementation) for the accepted range and handling of overflow.*

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Python does not support single-precision floating point numbers; the savings in processor and memory usage that are usually the reason for using these is dwarfed by the overhead of using objects in Python, so there is no reason to complicate the language with two kinds of floating point numbers.
Conclusion of this part: A historical perspective

- Before 1985, floating-point was an ugly mess
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and then arrives the multicore mess
Don’t worry, things are improving

- SSE2 has cleaned up IA32 floating-point
- Soon (AVX/SSE5) we have an FMA in virtually any processor and we may use the `fma()` to exploit it safely and portably
- The 2008 revision of IEEE-754 addresses the issues of
  - reproducibility versus performance
  - precision of intermediate computations
  - etc
- but it will take a while to percolate to your programming environment
Accuracy versus reproducibility

Floating-point in your machine

Accuracy versus reproducibility

Performance versus accuracy

Conclusion: It’s the Hardware, Stupid

Space-filling advertising: hardware computing just right
Accuracy is important
Is reproducibility important?

- Let us review a few use cases where people wanted numerical reproducibility.
- For each of these use cases, consider these two questions:

The question people ask
What is the cost of reproducibility?
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The question they should ask
Will the focus on reproducibility lead to good, or to evil?
Blender is a 3D authoring tool

- It includes blenderplayer: render Blender animations/games in real time

- Competition of animations using this tool

- I am going to show one of the winning entries
What is the cost of reproducibility?

I don’t know, I didn’t try. More on this on next slide.
The blenderplayer case

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Is the focus on reproducibility good or evil?
- Would you design the launch system of a satellite this way?
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- Would you design the launch system of a satellite this way?
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Conclusion: in such a case,
- A programmer that would insist on reproducibility would be an idiot
- What we need here is tools that make computing even less reproducible:
  let me advertise stochastic arithmetic.
By the way, what do we call reproducibility?

- Some kind of predictability, because we have read the standards?
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- Two runs on the same computer with the same OS? But
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- Two runs on the same computer with different OS’s? But
  - different mathematical libraries, policies WRT exceptions, default behaviours...
The serious version of the Blender use case

Algorithmic geometry problems:

- Example: compute the determinant of two vectors to decide their relative orientation
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Here we have a mathematical reference

We know what the code is supposed to compute.
Solution: write a test that detects if rounding may lead to wrong result, and recompute with higher accuracy in this (hopefully rare) case.
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Minor in execution time, high in coffee consumption.
Let me advertise Gappa, a tool that will reduce coffee consumption.
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Is the focus on reproducibility good or evil?

In CAD tools, I guess it is good.
In games, performance (WCET) is more important.
If she moves fast enough you won't notice the bugs
Use case: CERN’s LHC@home

Objective: simulate various configurations of the superconducting magnets (before building them)

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- The simulated phenomenon is known chaotic
- Computation distributed on a large number of untrusted PCs.
- Confidence by redundancy:
  *if two PCs return the exact same result, it is trusted.*

  - that is, the computation on each PC is trusted,
  - not its physical significance: the computation is chaotic.
Maybe I am biased on this one.

Here we don’t have a mathematical reference
... not even a physical one: we are trying to frame it.

What is the cost of reproducibility?
Performance benefit (more PCs can be exploited)

Coffee consumption: several engineer-months.

Recipe:
chose a portable compiler,
add parentheses to Fortran
replace elementary functions with correctly-rounded ones

Is the focus on reproducibility good or evil?
Mostly good ... but cost/benefit disputable
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Here also we have a mathematical reference! Why not use it?
We shouldn’t care about reproducibility. What matters is accuracy.

- Perfectly accurate results are reproducible (correct rounding)
  - Reproducibility by specification is good
  - Reproducibility of poorly understood code is dangerous.
Conclusion on this part

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In other words: reproducible accuracy
What is an error? What is accuracy?

The most important sentence of this talk

An error is a difference (absolute or relative) between two values, one being a reference for the other.

Examples:
- error of the FP addition is with reference of the real sum (easy)
- error of the polynomial is with reference to the function (easy)
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Never say “the error of this term is ...”: it doesn’t mean anything without the reference. If you are not able to define the reference value, you will not be able to know how accurate you compute.

Florent de Dinechin, projet AriC (ex-Arénaire)
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Performance versus accuracy

Floating-point in your machine

Accuracy versus reproductibility

Performance versus accuracy

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Space-filling advertising: hardware computing just right
Common wisdom

The more accurate you compute, the more expensive it gets
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## In practice

- We (hopefully) notice it when our computation is not accurate enough.
- But do we notice it when it is too accurate for our needs?
Common wisdom
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Reconciling performance and accuracy?
Or, regain performance by computing just right?
Double precision spoils us

The standard binary64 format (formerly known as double-precision) provides roughly 16 decimal digits.

Why should anybody need such accuracy?

Count the digits in the following:

- Definition of the second: the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.
- Definition of the metre: the distance travelled by light in vacuum in 1/299,792,458 of a second.
- Most accurate measurement ever (another atomic frequency) to 14 decimal places
- Most accurate measurement of the Planck constant to date: to 7 decimal places
- The gravitation constant $G$ is known to 3 decimal places only
Parenthesis: then why binary64?

This PC computes $10^9$ operations per second (1 gigaflops).

An allegory due to Kulisch

print the numbers in 100 lines of 5 columns double-sided:

1000 numbers/sheet

1000 sheets

$\approx$ a heap of $10^9$ flops

$\approx$ heap height speed of 100m/s, or 360km/h

A teraflops ($10^{12}$ op/s) prints to the moon in one second

Current top 500 computers reach the petaflop ($10^{15}$ op/s)

each operation may involve a relative error of $10^{-16}$,

and they accumulate.

Doesn't this sound wrong?

We would use these 16 digits just to accumulate garbage in them?

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- IEEE-754 floating-point single or double-precision
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  - same standard of quality as +, ×, /, √
One example of performance by computing just right

Correctly rounded elementary functions

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- Elementary functions: sin, cos, exp, log, implemented in the “standard mathematical library” (libm)
- Correctly rounded: As perfect as can be, considering the finite nature of floating-point arithmetic
  - same standard of quality as +, ×, /, √
- Now recommended by the IEEE754-2008 standard, but long considered too expensive because of the Table Maker’s Dilemma
Finite-precision algorithm for evaluating $f(x)$
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Approximation + rounding errors $\rightarrow$ overall error bound $\varepsilon$. 
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The Table Maker’s Dilemma

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Dilemma if this interval contains a midpoint between two FP numbers
The first digital signature algorithm

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De calculer juste à calculer au plus juste
I want 12 significant digits

Florent de Dinechin, projet AriC (ex-Arénaire)

De calculer juste à calculer au plus juste
The first digital signature algorithm

- I want 12 significant digits
- I have an approximation scheme that provides 14 digits
I want 12 significant digits

I have an approximation scheme that provides 14 digits

or,

\[ y = \log(x) \pm 10^{-14} \]

The first table-makers rounded these cases randomly, and recorded them to confound copiers.

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"Usually" that's enough to round

\[ y = x, \quad \text{xxxxx} \quad \text{17} \pm 10^{-14} \]

\[ y = x, \quad \text{xxxxx} \quad \text{83} \pm 10^{-14} \]
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Dilemma when

\[ y = x, \xxxxxxx50 \pm 10^{-14} \]

Florent de Dinechin, projet AriC (ex-Arénaire)

De calculer juste à calculer au plus juste

60
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De calculer juste à calculer au plus juste
Ziv’s onion peeling algorithm

1. Initialisation: \( \varepsilon = \varepsilon_1 \)
Ziv’s onion peeling algorithm

1. Initialisation: $\varepsilon = \varepsilon_1$

2. Compute $y$ such that $f(x) = y \pm \varepsilon$
Solving the table maker’s dilemma

\[ y \pm \varepsilon_1 \]

Ziv’s onion peeling algorithm

1. Initialisation: \( \varepsilon = \varepsilon_1 \)
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3. Does \( y \pm \varepsilon \) contain the middle point between two FP numbers?
Ziv’s onion peeling algorithm

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   - If no, return $\text{RN}(y)$
**Ziv’s onion peeling algorithm**

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\[
y \pm \varepsilon_1 \quad y \pm \varepsilon_1
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It is a \textit{while} loop...

- Lefèvre and Muller: compute just right the precision at which it terminates.
Accuracy versus performance

When we know that the loop terminates...

CRLibm: 2-step approximation process

- first step fast but accurate to $\varepsilon_1$
- (rarely) second step slower but always accurate enough
Accuracy versus performance

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$$T_{\text{avg}} = T_1 + p_2 T_2$$
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$$T_{\text{avg}} = T_1 + p_2 T_2$$

For each step, we want to prove a tight bound $\bar{\varepsilon}$ such that

$$\left| \frac{F(x) - f(x)}{f(x)} \right| \leq \bar{\varepsilon}$$
Accuracy versus performance

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$$\left| \frac{F(x) - f(x)}{f(x)} \right| \leq \bar{\varepsilon}$$

- Overestimating $\bar{\varepsilon}_2$ degrades $T_2$! (common wisdom)
Accuracy versus performance

When we know that the loop terminates...

**CRLibm: 2-step approximation process**

- first step fast but accurate to $\varepsilon_1$
  - sometimes not accurate enough
- (rarely) second step slower but always accurate enough

$$T_{avg} = T_1 + p_2 T_2$$

For each step, we want to prove a tight bound $\varepsilon$ such that

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- Overestimating $\varepsilon_1$ degrades $p_2$ !
First correctly rounded elementary function in CRLibm

- \( \exp \) by David Defour
- worst-case time \( T_2 \approx 10,000 \) cycles
- complex, hand-written proof
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Conclusion was:
- performance and memory consumption of CR elem function is OK
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- worst-case time $T_2 \approx 10,000$ cycles
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Conclusion was:

- performance and memory consumption of CR elem function is OK
- problem now is: performance and coffee consumption of the programmer
C. Lauter at the end of his PhD,

- development time for \text{sinpi}, \text{cospi}, \text{tanpi}:
C. Lauter at the end of his PhD,

- development time for \text{sinpi}, \text{cospi}, \text{tanpi}: 2 \text{ days}
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- development time for sinpi, cospi, tanpi: 2 days
- worst-case time $T_2 \approx 1,000$ cycles

(but as a result of three more PhDs)
Summary of the progress made

\[ T_{\text{avg}} = T_1 + p_2 T_2 \]

- Reduction of \( T_1 \) by learning from Intel
- Reduction of \( p_2 \) by automating the computation of tight \( \varepsilon_1 \)
  \( (p_2 \text{ is proportional to } \varepsilon_1) \)
- Reduction of \( T_2 \) by computing just right
- Reduction of coffee consumption by automating the whole thing
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- Reduction of coffee consumption by automating the whole thing

The MetaLibm vision

Automate libm expertise so that a new, correct libm can be written for a new processor/context in minutes instead of months.
Conclusion:
It’s the Hardware, Stupid

Floating-point in your machine

Accuracy versus reproductibility

Performance versus accuracy

Conclusion: It’s the Hardware, Stupid

Space-filling advertising: hardware computing just right
Let us end this talk with the introduction of another one

Doug Burger (Microsoft research) keynote at HiPEAC 2013.
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- *Nothing in our careers has been as fundamental as this transition*
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- tomorrow, transistors still get smaller, but we can’t even use them all together (dark silicon)
- Nothing in our careers has been as fundamental as this transition

The way out according to Doug Burger

We could still “get more” by specializing the hardware.
Meanwhile, at Intel

Jeff Arnold (Intel) says:

... and shows the following slide from his colleagues at ISSCC 2012.
Jeff Arnold (Intel) says:

*Single precision gives you 7 decimal digits. Do you really need this accuracy to compute *Angry birds* trajectories entered with your fat fingers?*
Meanwhile, at Intel

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... and shows the following slide from his colleagues at ISSCC 2012.
The ISSCC 2012 paper

• notion of “uncertainty”, a power of two attached to inputs and outputs
• technically, computing a center-radius interval
• if uncertainty allows, compute center on 6 or 12 bits only.
• this saves a lot of power.
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Absolutely no use case here... Is this chip usable for real?
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• technically, computing a center-radius interval
• if uncertainty allows, compute center on 6 or 12 bits only.
• this saves a lot of power.

Absolutely no use case here... Is this chip usable for real?

What software environment will it need?
You (probably) came here to learn how to compute right.
You (probably) came here to learn how to compute right.

This is half the work to compute just right.
You (probably) came here to learn how to compute right.

This is half the work to compute just right.

What you will learn here might help you address the hardware industry’s grand challenge.
Space-filling advertising: hardware computing just right

Floating-point in your machine

Accuracy versus reproducibility

Performance versus accuracy

Conclusion: It’s the Hardware, Stupid

Space-filling advertising: hardware computing just right
To sum up,

- Doug Burger says “we should specialize our hardware”
- Kaul et al say “we should design hardware that computes just right”
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- Doug Burger says “we should specialize our hardware”
- Kaul et al say “we should design hardware that computes just right”

We’ve been doing both since 2003.

The FloPoCo project

http://flopoco.gforge.inria.fr/
Two different ways of wasting silicon

Here are two universally programmable chips.

Who’s best for (insert your computation here) ?
Are FPGAs any good at floating-point?

Long ago (1995), people ported the basic operations: $+, -, \times$

- Versus the highly optimized FPU in the processor,
- each operator 10x slower in an FPGA

This is the unavoidable overhead of programmability.
Are FPGAs any good at floating-point?

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This is the inevitable overhead of programmability.

If you lose according to a metric, change the metric.

Peak figures for double-precision floating-point exponential

- Pentium core: 20 cycles / DPExp @ 4GHz: $200\; \text{MDPExp/s}$
- FPExp in FPGA: 1 DPExp/cycle @ 400MHz: $400\; \text{MDPExp/s}$
- Chip vs chip: 6 Pentium cores vs 150 FPExp/FPGA
- Power consumption also better
- Single precision data better

(Intel MKL vector libm, vs FPExp in FloPoCo version 2.0.0)
### Table 2. Verilog-AMS Compiler Output Instruction Distribution

<table>
<thead>
<tr>
<th></th>
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<th></th>
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<td>2</td>
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<td>0</td>
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<td>43</td>
<td>18</td>
<td>1</td>
<td>10</td>
<td>4</td>
</tr>
</tbody>
</table>
Custom arithmetic (not your Pentium’s)

Florent de Dinechin, projet AriC (ex-Arénaire)
Custom arithmetic (not your Pentium’s)

Florent de Dinechin, projet AriC (ex-Arénaire)

De calculer juste à calculer au plus juste
Custom arithmetic (not your Pentium’s)

Constant multipliers

precomputed ROM

Need a generator

generic polynomial evaluator

truncated multiplier

Never compute 1 bit more accurately than needed!

Florent de Dinechin, projet AriC (ex-Arénaire)
Useful operators that make sense in a processor

- Should a processor include elementary functions?  
  Yes (Paul & Wilson, 1976), No since the transition to RISC

- Should a processor include a divider and square root?  
  Yes (Oberman et al, Arith, 1997), No since the transition to FMA (IBM then HP then Intel)

- Should a processor include decimal hardware?  
  Yes say IBM, No say Intel

- Should a processor include a multiplier by log(2)?  
  No of course.
Useful operators that make sense in an FPGA or ASIC

- Elementary functions?  
  Yes iff your application needs it
- Divider or square root?  
  Yes iff your application needs it
- Decimal hardware?  
  Yes iff your application needs it
- A multiplier by \log(2)?  
  Yes iff your application needs it

In FPGAs, useful means: useful to one application.

Florent de Dinechin, projet AriC (ex-Arénaire)
Arithmetic operators useful to at least one application:

- Elementary functions (sine, exponential, logarithm...)
- Algebraic functions \( \frac{x}{\sqrt{x^2 + y^2}} \), polynomials, ...
- Compound functions \( \log_2(1 \pm 2^x), e^{-Kt^2}, ... \)
- Floating-point sums, dot products, sums of squares
- Specialized operators: constant multipliers, squarers, ...
- Complex arithmetic
- LNS arithmetic
- Decimal arithmetic
- Interval arithmetic
- ...

Oh yes, basic operations, too.

Florent de Dinechin, projet AriC (ex-Arénaire)
Enough work to keep me busy to retirement

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An arithmetic **operation** is a *function* (in the mathematical sense)
- few well-typed inputs and outputs
- no memory or side effect (usually)
What do we call arithmetic operators?

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- An **operator** is the *implementation* of such a function
  - IEEE-754 FP standard: \( \text{operator}(x) = \text{rounding}(\text{operation}(x)) \)

→ Clean mathematical definition (even for floating-point arithmetic)
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→ Clean mathematical definition (even for floating-point arithmetic)

The operator as a circuit...

... is a direct acyclic graph (DAG):
- easy to build and pipeline
- easy to test against its mathematical specification
The benefits of custom computing

Example: a floating-point sum of squares

\[ x^2 + y^2 + z^2 \]

(not a toy example but a useful building block)
The benefits of custom computing

Example: a floating-point sum of squares

\[ x^2 + y^2 + z^2 \]

(not a toy example but a useful building block)

- A square is simpler than a multiplication
  - half the hardware required

Accuracy can be improved:

\[(x^2 \times y^2 + z^2)\]: asymmetrical
The benefits of custom computing

Example: a floating-point sum of squares

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(not a toy example but a useful building block)

- A square is simpler than a multiplication
  - half the hardware required
- \( x^2, y^2, \) and \( z^2 \) are positive:
  - one half of your FP adder is useless

Accuracy can be improved:

5 rounding errors in the floating-point version

\((x^2 + y^2) + z^2\): asymmetrical
The benefits of custom computing

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  - \((x*x+y*y)+z*z\) : asymmetrical
The benefits of custom computing

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  - \((x^2+y^2)+z^2\) : asymmetrical

The FloPoCo Recipe

- Floating-point interface for convenience
- Clear accuracy specification for computing just right
- Fixed-point internal architecture for efficiency
A floating-point adder

Florent de Dinechin, projet AriC (ex-Arénaire)
A fixed-point architecture

Florent de Dinechin, projet AriC (ex-Arénaire)
De calculer juste à calculer au plus juste
### The benefits of custom computing

A few results for floating-point sum-of-squares on Virtex4:

<table>
<thead>
<tr>
<th>Simple Precision</th>
<th>area</th>
<th>performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>LogiCore classic</td>
<td>1282 slices, 20 DSP</td>
<td>43 cycles @ 353 MHz</td>
</tr>
<tr>
<td>FloPoCo classic</td>
<td>1188 slices, 12 DSP</td>
<td>29 cycles @ 289 MHz</td>
</tr>
<tr>
<td>FloPoCo custom</td>
<td>453 slices, 9 DSP</td>
<td>11 cycles @ 368 MHz</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Double Precision</th>
<th>area</th>
<th>performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>FloPoCo classic</td>
<td>4480 slices, 27 DSP</td>
<td>46 cycles @ 276 MHz</td>
</tr>
<tr>
<td>FloPoCo custom</td>
<td>1845 slices, 18 DSP</td>
<td>16 cycles @ 362 MHz</td>
</tr>
</tbody>
</table>

- all performance metrics improved, FLOP/s/area more than doubled
- Plus: custom operator more accurate, and symmetrical
Custom also means: custom pipeline

One operator does not fit all
- Low frequency, low resource consumption
Custom also means: custom pipeline

\[
\begin{align*}
X & \rightarrow \text{unpack} \\
E_X & \rightarrow M_{\text{X1}} + w_F \\
E_Y & \rightarrow M_{\text{Y1}} + w_F \\
E_Z & \rightarrow M_{\text{Z1}} + w_F \\
\end{align*}
\]

\[
\begin{align*}
\text{sort} \rightarrow \text{squarer} \\
E_B & \rightarrow M_{\text{A2}} + 2 + w_F + g \\
\end{align*}
\]

\[
\begin{align*}
\text{add} \rightarrow \text{normalize/pack} \\
4 + w_F + g & \rightarrow w_E + w_F + g \\
R & \\
\end{align*}
\]

One operator does not fit all

- Low frequency, low resource consumption
- Faster but larger (more registers)

Florent de Dinechin, projet AriC (ex-Arénaire)
Custom also means: custom pipeline

One operator does not fit all
- Low frequency, low resource consumption
- Faster but larger (more registers)
- Combinatorial

Florent de Dinechin, projet AriC (ex-Arénaire)
- All you ever wanted to know about division by 3
- Application-specific floating-point accumulation
- Architectures computing the floating-point exponential
- ...