# Discontinuous Galerkin methods for first-order PDEs

Alexandre Ern

**Université Paris-Est, CERMICS (ENPC)**

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# Outline

- Friedrichs' systems (steady linear PDEs)
  - design of dG methods
  - convergence analysis for smooth solutions
  - unified view on linear stabilization
  - cf. [AE & Guermond, 06-..], [Di Pietro & AE, 12]
- dG in time (time-dependent linear PDEs)
  - convergence analysis for smooth solutions
  - cf. [AE & Schieweck, 15]
- Weighting linear stabilization (conservation laws)
  - linear stabilization for rough solutions/nonlinear PDEs
  - cf. [AE & Guermond, 13]

# Friedrichs' systems

- Open, bounded, connected, strongly Lipschitz subset $\Omega \subset \mathbb{R}^d$
- $\mathbb{K}^m$-valued functions, $m \geq 1$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$
- $(d+1)$ functions $\mathcal{K}, \{\mathcal{A}^k\}_{1 \leq k \leq d} : \Omega \to \mathbb{K}^{m \times m}$
    - $\mathcal{K}$, $\{\mathcal{A}^k\}_{1 \leq k \leq d}$ and $\mathcal{X} := \sum_{k=1}^{d} \partial_k \mathcal{A}^k$ are **bounded**
    - $\mathcal{A}^k$ is **symmetric** (Hermitian)
    - $\mathcal{K} + \mathcal{K}^{\mathrm{H}} - \mathcal{X}$ is **uniformly positive** ($\geq 2\mu_0 \mathcal{I}$)
- Given $f : \Omega \to \mathbb{K}^m$, find $u : \Omega \to \mathbb{K}^m$ s.t. $Au = f$ in $\Omega$ with

$$Au = \mathcal{K}u + \sum_{k=1}^{d} \mathcal{A}^k \partial_k u$$

- cf. [Friedrichs, 58]

# Examples

- **Advection-reaction** $m = 1$, $\mathbb{K} = \mathbb{R}$
  - $\mu u + \boldsymbol{\beta}\cdot\nabla u = f$
  - $\mu \in L^\infty$, $\boldsymbol{\beta} \in \mathbf{L}^\infty$, $\nabla\cdot\boldsymbol{\beta} \in L^\infty$, $\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \geq \mu_0 > 0$
- **Darcy** (grad-div) $m = d + 1$, $\mathbb{K} = \mathbb{R}$
  - $u = (\boldsymbol{\sigma}, p)$, $\mathbb{d}^{-1}\boldsymbol{\sigma} + \nabla p = \mathbf{f}_1$, $\mu p + \nabla\cdot\boldsymbol{\sigma} = f_2$
  - $\mu \in L^\infty$ and uniformly positive, $\mathbb{d}$ bounded, symmetric, uniformly positive definite
- **Maxwell** (eddy currents, curl-curl) $m = 6$, $\mathbb{K} = \mathbb{C}$
  - $u = (\mathbf{E}, \mathbf{H})$, $\sigma\mathbf{E} - \nabla\times\mathbf{H} = \mathbf{f}_1$, $i\omega\mu\mathbf{H} + \nabla\times\mathbf{E} = \mathbf{0}$
  - $\sigma, \mu \in L^\infty$, uniformly positive (for simplicity)

# Boundary conditions

- Symmetric boundary field $\mathcal{N} : \partial\Omega \to \mathbb{K}^m$ ($\mathbf{n}$ unit outward normal) s.t. $\mathcal{N} = \sum_{k=1}^{d} n_k \mathcal{A}^k$
- Assume there is an additional boundary field $\mathcal{M} : \partial\Omega \to \mathbb{K}^m$ s.t.
  - (real part of) $\mathcal{M}$ is non-negative
  - $\ker(\mathcal{M} - \mathcal{N}) + \ker(\mathcal{M} + \mathcal{N}) = \mathbb{K}^m$
- The boundary condition is $(\mathcal{M} - \mathcal{N})u = 0$ on $\partial\Omega$
- Examples
  - **advection-reaction** $\mathcal{N}u = (\boldsymbol{\beta}\cdot\mathbf{n})u$, $\mathcal{M}u = |\boldsymbol{\beta}\cdot\mathbf{n}|u$
  - **Darcy** $\mathcal{N}(\boldsymbol{\sigma}, p) = (p\mathbf{n}, \boldsymbol{\sigma}\cdot\mathbf{n})$, $\mathcal{M}(\boldsymbol{\sigma}, p) = (\pm p\mathbf{n}, \mp\boldsymbol{\sigma}\cdot\mathbf{n})$
  - **Maxwell** $\mathcal{N}(\mathbf{E}, \mathbf{H}) = (\mathbf{H}\times\mathbf{n}, \mathbf{E}\times\mathbf{n})$, $\mathcal{M}(\mathbf{E}, \mathbf{H}) = (\pm\mathbf{H}\times\mathbf{n}, \mp\mathbf{E}\times\mathbf{n})$
  - note that $\mathcal{M}$ is skew-symmetric for Darcy and Maxwell

# Mathematical theory

- $L^2$-based theory: pivot space $L = L^2(\Omega; \mathbb{K}^m)$
- Graph space $V = \{v \in L \mid Av \in L\}$
  - Friedrichs' operator $Av = \mathcal{K}v + \sum_{k=1}^{d} \mathcal{A}^k \partial_k v$
  - formal adjoint $\tilde{A}v = (\mathcal{K}^{\mathrm{H}} - \mathcal{X})v - \sum_{k=1}^{d} \mathcal{A}^k \partial_k v$
  - $A, \tilde{A} \in \mathcal{L}(V; L)$
- Boundary operators $N, M \in \mathcal{L}(V; V')$
  - $\langle Nv, w \rangle_{V',V} = (Av, w)_L - (v, \tilde{A}w)_L$
  - $\langle Mv, v \rangle_{V',V} \geq 0$ and $\ker(M - N) + \ker(M + N) = V$
  - $L$-dissipativity on $\ker(M - N)$: $(Av, v)_L \geq \mu_0 \|v\|_L^2 + \frac{1}{2}\langle Mv, v \rangle_{V',V}$
- Given $f \in L$, there is a unique $u \in V$ s.t.

$$Au = f \qquad (M - N)u = 0$$

(and there is a unique $\tilde{u} \in V$ s.t. $\tilde{A}\tilde{u} = f$ and $(M^* + N)\tilde{u} = 0$)

# dG setting

- Admissible mesh sequence $\{\mathcal{T}_h\}_{h>0}$
  - matching simplicial meshes: Ciarlet's **shape-regularity**
  - general meshes (non-matching, polyhedral) : **shape- and contact-regularity**, essentially one length scale for mesh faces and cells [Di Pietro & AE, 12]
  - usual FE tools: inverse & discrete trace ineq., polynomial approx.
- Broken polynomial space (of order $r \geq 0$)

$$\mathbb{P}_r(\mathcal{T}_h;\mathbb{R}) = \{v_h \in L^1(\Omega;\mathbb{R}) \mid v_h|_T \in \mathbb{P}_r(T;\mathbb{R})\ \forall T \in \mathcal{T}_h\}$$

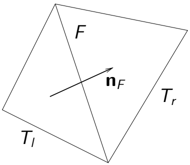
- Jumps and averages at mesh interfaces

$F = \partial T_l \cap \partial T_r$

$\mathbf{n}_F$ points from $T_l$ to $T_r$

$\{\!\{v\}\!\} = \frac{1}{2}(v|_{T_l} + v|_{T_r})$

$[v] = v|_{T_l} - v|_{T_r}$



# dG approximation: centered fluxes

- Standard Galerkin setting with $V_h = \mathbb{P}_r(\mathcal{T}_h; \mathbb{K}^m)$

$$\text{Find } u_h \in V_h \text{ s.t. } a_h^{\mathrm{cf}}(u_h, w_h) = (f, w_h)_L \text{ for all } w_h \in V_h$$

with discrete bilinear form $a_h^{\mathrm{cf}}$ satisfying two key properties
  - **exact consistency** $a_h^{\mathrm{cf}}(u, w_h) = (f, w_h)_L$, $\forall w_h \in V_h$
  - $L$-**dissipativity** $a_h^{\mathrm{cf}}(v_h, v_h) \geq \mu_0 \|v_h\|_L^2 + \frac{1}{2}(\mathcal{M} v_h, v_h)_{L(\partial\Omega)}^2$, $\forall v_h \in V_h$

- Centered fluxes (interfaces $F \in \mathcal{F}_h^{\mathrm{i}}$) and boundary penalty ($F \in \mathcal{F}_h^{\mathrm{b}}$)

$$a_h^{\mathrm{cf}}(v_h, w_h) = \sum_{T \in \mathcal{T}_h} (v_h, \tilde{A} w_h)_{L(T)} + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} (\phi_F^{\mathrm{i}}(v_h), [w_h])_{L(F)} + \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} (\phi_F^{\mathrm{b}}(v_h), w_h)_{L(F)}$$

  - $\phi_F^{\mathrm{i}}(v_h) = \mathcal{N}_F \{\!\{ v_h \}\!\}$ (for AR, $\phi^{\mathrm{i}}(v_h) = (\boldsymbol{\beta} \cdot \mathbf{n}_F) \{\!\{ v_h \}\!\}$)
  - $\phi_F^{\mathrm{b}}(v_h) = \frac{1}{2}(\mathcal{M}_+ - \mathcal{N}) v_h$, $\mathcal{M}_+ = \mathcal{M} = |\boldsymbol{\beta} \cdot \mathbf{n}|$ for AR, $\mathcal{M}_+$ adds least-squares penalty on BC for Darcy and Maxwell

- For smooth solution $u \in H^{r+1}(\Omega; \mathbb{K}^m)$, $\|u - u_h\|_L \lesssim h^r$

# Linear stabilization (upwinding)

- Upwinding amounts to adding a least-squares penalty on interface jumps [Brezzi et al., 04]

$$a_h(v_h, w_h) = a_h^{\mathrm{cf}}(v_h, w_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} (\mathcal{S}_F[\![v_h]\!], [\![w_h]\!])_{L(F)}$$

with $\mathcal{S}_F \sim |\mathcal{N}_F|$, so that

  - $a_h$ is still **exactly consistent**
  - $a_h$ enjoys **strengthened $L$-dissipativity**

$$a_h(v_h, v_h) \geq |\!|\!| v_h |\!|\!|^2 = \mu_0 \|v_h\|_L^2 + \frac{1}{2}(\mathcal{M}v_h, v_h)_{L(\partial\Omega)}^2 + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \|\mathcal{S}_F^{1/2}[\![v_h]\!]\|_{L(F)}^2$$

- Incidence on the flux: $\phi_F^{\mathrm{i}}(v_h) = \mathcal{N}_F\{\!\{v_h\}\!\} + \mathcal{S}_F[\![v_h]\!]$
  - for AR, $\mathcal{S}_F = \frac{1}{2}|\boldsymbol{\beta}\cdot\mathbf{n}_F|$ leads to $\phi_F^{\mathrm{i}}(v_h) = (\boldsymbol{\beta}\cdot\mathbf{n}_F)u_h^{\uparrow}$
  - for Darcy, jumps of both $\boldsymbol{\sigma}_h\cdot\mathbf{n}_F$ and $p_h$ are penalized
  - for Maxwell, jumps of both $\mathbf{H}_h\times\mathbf{n}_F$ and $\mathbf{E}_h\times\mathbf{n}_F$ are penalized

# Error analysis with upwinding

- Assume smooth solution $u \in H^{r+1}(\Omega; \mathbb{K}^m)$
- Strengthened $L$-dissipativity leads to $|||u - u_h||| \lesssim h^{r+1/2} \rightarrow$ quasi-optimal $L$-norm estimate
- Full stability norm and discrete inf-sup stability

$$|||v_h|||_\sharp \lesssim \sup_{w_h \in V_h} \frac{a_h(v_h, w_h)}{|||w_h|||_\sharp} \qquad \forall v_h \in V_h$$

$$|||v_h|||_\sharp^2 = |||v_h|||^2 + \sum_{T \in \mathcal{T}_h} h_T \|Av_h\|_{L(T)}^2$$

- We obtain $|||u - u_h|||_\sharp \lesssim h^{r+1/2} \rightarrow$ optimal graph-norm estimate
- For mixed elliptic PDEs, it is possible to modify the penalty strategy so as to eliminate locally the auxiliary variable

# Unified view on linear stabilization

- Many recent $H^1$-conforming stabilized FEM are analyzed with the same tools and lead to similar error estimates
- Example: **Continuous interior penalty**

$$a_h^{\mathrm{cip}}(v_h, w_h) = (v_h, \tilde{A} w_h)_L + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} (\mathcal{S}_F [\nabla v_h], [\nabla w_h])_{L(F)} + \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} (\phi_F^{\mathrm{b}}(v_h), w_h)_{L(F)}$$

  - penalizes gradient jumps with $\mathcal{S}_F \sim h_F^2$
  - cf. [Burman & Hansbo, 04; Burman, 05; Burman & AE, 07]
- Other examples
  - **Subgrid Viscosity** penalizes gradient of subscale fluctuation, cf. [Guermond, 99]
  - **Local Projection Stabilization** penalizes subscale fluctuation of gradient, cf. [Braack & Burman, 06; Matthies et al., 07]
- Stabilization bilinear form is symmetric (contrast with GaLS/SUPG)

# Time-dependent linear PDEs

- Overview
- Main results
- Some analysis tools
- Error estimates for smooth solutions

# dG in time

- Time semi-discretization of evolution problem by dG method
  - piecewise polynomials in time of order $k \geq 0$
  - time interval $I = (0, T]$ decomposed as $I = \cup_{n=1}^{N} I_n$
  - subintervals $I_n = (t_{n-1}, t_n]$ **(open at left, closed at right endpoint)**
  - discrete times $0 = t_0 < t_1 < \cdots < t_N = T$, time steps $\tau_n = t_n - t_{n-1}$
- For Banach space $B$ (functions in space), let

$$\mathbb{P}_k(I_n, B) = \{w : I_n \to B \ : \ w(t) = \sum_{j=0}^{k} W^j t^j, \ \forall t \in I_n, \ W^j \in B, \ \forall j\}$$

$$X_\tau^k(B) = \{w_\tau : \bar{I} \to B : \ w_\tau|_{I_n} \in \mathbb{P}_k(I_n, B) \quad \forall n\}$$

  - a function $w_\tau \in X_\tau^k(B)$ can be **discontinuous** at discrete times $t_n$ and is **continuous from the left** at all $t_n$
  - jump of $w_\tau$ at $t_n$ is $[w_\tau]_n = w_\tau(t_n^+) - w_\tau(t_n)$

# Evolution problems with coercivity

- Parabolic problems: dG in time (order $k$), dG in space (order $r$) [Thomée, 07]
  - $\ell^\infty(L^2)$ (at discrete time nodes) and $L^2(L^2)$ error estimates of order $(\tau^{2k+1} + h^{r+1})$: **super-convergence in time**
- Nonlinear advection-diffusion, dG in space
  - $\ell^\infty(L^2)$ and $L^2(L^2)$ estimates of order $(\tau^{k+1} + h^r)$ on time-varying meshes (under condition $h^2 \lesssim \tau$) [Feistauer et al., 11-..]
- Linear advection-diffusion, $H^1$-conforming FEM with LPS
  - $\ell^\infty(L^2)$ and $L^2(L^2)$ estimates of order $(\tau^{k+1} + h^{r+1/2} + \varepsilon^{1/2}h^r)$ on static meshes [Ahmed, Matthies, Tobiska & Xie, 11]

# Evolution problems without coercivity

- Linear first-order operator $Av = \mu v + \boldsymbol{\beta}\cdot\nabla v$ in space
  - $\mu : \Omega \to \mathbb{R}$ is a bounded reaction function
  - $\boldsymbol{\beta} : \Omega \to \mathbb{R}^d$ is a given Lipschitz advection field
  - both are **time-independent**
- Mathematical setting of Friedrichs' systems (spaces $V$ and $L$)
- Linear evolution problem
  - data $f \in C^0([0,T],L)$ and $u_0 \in V$
  - find $u \in C^0([0,T],V) \cap C^1([0,T],L)$ s.t.

$$(\partial_t u(t), v)_L + (Au(t), v)_L = (f(t), v)_L \quad \forall v \in L \quad \forall t \in (0,T)$$

    and $u(0) = u_0$
  - well-posedness results from Hille–Yosida Theorem

# dG-in-time semi-discretization

- Time semi-discrete solution $u_\tau$ belongs to $X^k_\tau(V)$
- For all $n = 1\dots N$, $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ s.t. for all $v_\tau \in \mathbb{P}_k(I_n, L)$,

$$\int_{I_n} (\partial_t u_\tau + Au_\tau, v_\tau)_L \, dt + \big([u_\tau]_{n-1}, v_\tau(t^+_{n-1})\big)_L = \int_{I_n} (f, v_\tau)_L \, dt$$

  $(k+1)$ coupled first-order PDEs in space within each time step

- RHS evaluated using the $(k+1)$-point right-sided GR quadrature on each subinterval $I_n$

$$Q_n(g) = \frac{\tau_n}{2} \sum_{\mu=1}^{k+1} \widehat{w}_\mu g(t_{n,\mu}) \approx \int_{I_n} g(t) dt$$

  - weights $\widehat{w}_\mu > 0$, $t_{n,k+1} = t_n$, $Q_n(g)$ exact for all $g \in \mathbb{P}_{2k}(I_n, \mathbb{R})$

- Time semi-discrete problem with quadrature becomes

$$\int_{I_n} (\partial_t u_\tau + Au_\tau, v_\tau)_L \, dt + \big([u_\tau]_{n-1}, v_\tau(t^+_{n-1})\big)_L = Q_n((f, v_\tau)_L)$$

# Full space-time discretization

- Discrete space $V_h^n \subset L$ built from a mesh $\mathcal{T}_h^n$ which can change from one time interval to the next
- FEM with linear stabilization (dG, CIP, ...)
  - $A_h^n : V_h^n \to V_h^n$ s.t. $(A_h^n v_h, w_h)_L = a_h^n(v_h, w_h)$ ($a_h^n$ depends on $\mathcal{T}_h^n$ ...)
- **Fully discrete problem**: $u_{\tau h}|_{I_n} \in \mathbb{P}_k(I_n, V_h^n)$ s.t. for all $v_{\tau h} \in \mathbb{P}_k(I_n, V_h^n)$ and all $n = 1 \ldots N$,

$$\int_{I_n} (\partial_t u_{\tau h} + A_h^n u_{\tau h}, v_{\tau h})_L \, dt + \left([u_{\tau h}]_{n-1}, v_{\tau h}(t_{n-1}^+)\right)_L = Q_n((f, v_{\tau h})_L)$$

# Example: dG(1) in time

- On each time interval $I_n$, we can solve for the two unknowns

$$U_{hn}^j = u_{\tau h}(t_{n,j}) \in V_h^n \qquad j = 1, 2$$

- The coupled (2×2)-block system reads

$$\begin{aligned} \tfrac{3}{4} U_{hn}^1 + \tfrac{\tau_n}{2} A_h^n U_{hn}^1 + \tfrac{1}{4} U_{hn}^2 \qquad\qquad\qquad\qquad &= u_{\tau h}(t_{n-1}) + \tfrac{\tau_n}{2} P_h^n f(t_{n,1}) \\ -\tfrac{9}{4} U_{hn}^1 \qquad\qquad + \tfrac{5}{4} U_{hn}^2 + \tfrac{\tau_n}{2} A_h^n U_{hn}^2 &= -u_{\tau h}(t_{n-1}) + \tfrac{\tau_n}{2} P_h^n f(t_{n,2}) \end{aligned}$$

where $P_h^n$ is the $L$-orthogonal projector onto $V_h^n$

# Main results

- Improved and new error estimates for smooth solutions
  - polynomial order $k \geq 1$ in time
  - unified analysis for FEM with linear stabilization in space
- Two main analysis tools in time
  - post-processed, time-continuous discrete solution $\mathcal{L}_\tau u_{\tau h}$
  - special time-interoplate $R_\tau^{k+1} u$ of order $(k+1)$
- $\ell^\infty(L^2)$ and $L^2(L^2)$ estimates for $(u - \mathcal{L}_\tau u_{\tau h})$
  - **super-convergent bound** of order $(\tau^{k+2} + h^{r+1/2})$ on static meshes
  - **novel estimate on projection error** for time-varying meshes
- Estimates on error derivatives (on static meshes)
  - bound on $(\partial_t u - \mathcal{L}_\tau \partial_t \mathcal{L}_\tau u_{\tau h})$ of order $(\tau^{k+1} + h^{r+1/2})$ in $\ell^\infty(L^2)$ and in $L^2(L^2)$
  - optimal bound on the discrete graph norm of $(u - \mathcal{L}_\tau u_{\tau h})$

# Comparison with RK methods (1)

- Explicit RK methods in time combined with **dG in space** (and suitable limiters) [Cockburn, Shu et al., 89-..]

- Explicit time-marching schemes are conditionally stable
  - error bounds require **Gronwall's argument**
  - error constant blows up **exponentially** in $T$

- Analysis of explicit RK2 and RK3 schemes: $\ell^\infty(L^2)$ estimates
  - nonlinear conservation laws and dG in space [Zhang & Shu, 04, 10]
  - Friedrichs' systems, stabilized FEM [Burman, AE & Fernández, 10]
  - $O(\tau^2 + h^{r+1/2})$ for RK2 under tightened CFL condition $\tau = O(h^{4/3})$
  - for RK2 with $r = 1$, usual CFL suffices ($\tau = O(h)$)
  - $O(\tau^3 + h^{r+1/2})$ for RK3 under usual CFL
  - no unified analysis available for arbitrary order in time

# Comparison with RK methods (2)

- Advantages of time-dG schemes are
  - unconditional stability
  - super-convergent error estimates
  - error constants behave as $T^{1/2}$
  - unified analysis for all polynomial orders $k \geq 1$ (implicit Euler corresponding to $k = 0$ being slightly different)
- The prize to pay is **increased computational cost**
  - can be tamed by efficient multigrid solvers
  - heat, Stokes and NS equations [Hussain, Schieweck & Turek, 11, 12]
- Implicit RK schemes share various advantages with dG in time
  - recent analysis for linear Maxwell equations [Hochbruck & Pažur, 15]

# Analysis tools

- Recall $X_\tau^k(B) = \{w_\tau : \bar{I} \to B : \; w_\tau\big|_{I_n} \in \mathbb{P}_k(I_n, B), \, \forall n = 1 \dots N\}$
- **Lifting operator**

$$\mathcal{L}_\tau : X_\tau^k(B) \to X_\tau^{k+1}(B) \cap C^0(\bar{I}, B)$$

such that $\mathcal{L}_\tau w_\tau(0) = w_\tau(0)$ and, for all $n = 1 \dots N$,

$$\mathcal{L}_\tau w_\tau(t) = w_\tau(t) - [w_\tau]_{n-1} \vartheta_n(t) \quad \forall t \in I_n = (t_{n-1}, t_n]$$

where $\vartheta_n \in \mathbb{P}_{k+1}(I_n, \mathbb{R})$, $\vartheta_n(t_{n-1}) = 1$ and vanishes at the $(k+1)$ RS GR points, so that $\mathcal{L}_\tau w_\tau(t_{n,\mu}) = w_\tau(t_{n,\mu})$ for all $\mu = 1 \dots (k+1)$

- The fully discrete problem can be rewritten as

$$\int_{I_n} (\partial_t \mathcal{L}_\tau u_{\tau h} + A_h^n u_{\tau h}, v_{\tau h})_L \, dt = Q_n((f, v_{\tau h})_L)$$

# A higher-order time interpolate (1)

- Let $u \in C^1(\overline{I}, B)$
- **Step 1.** Choose a Lagrange/Hermite interpolate $I_\tau^{k+2} u \in C^1(\overline{I}, B)$ such that, for all $n = 1 \dots N$, $I_\tau^{k+2} u|_{I_n} \in \mathbb{P}_{k+2}(I_n, B)$ and

$$I_\tau^{k+2} u(t_n) = u(t_n) \qquad \text{and} \qquad \partial_t I_\tau^{k+2} u(t_n) = \partial_t u(t_n)$$

  - for $k = 1$, these conditions fully determine $I_\tau^{k+2} u$ in $I_n$
  - for $k \geq 2$, values at additional Lagrange nodes in $I_n$ are prescribed
  - for $k = 0$, this construction is not possible

- **Step 2.** Define $R_\tau^{k+1} u|_{I_n} \in \mathbb{P}_{k+1}(I_n, B)$ by the $(k+2)$ conditions

$$\partial_t R_\tau^{k+1} u(t_{n,\mu}) = \partial_t I_\tau^{k+2} u(t_{n,\mu}) \qquad \forall \mu = 1 \dots (k+1)$$
$$R_\tau^{k+1} u(t_{n-1}^+) = I_\tau^{k+2} u(t_{n-1})$$

  and set $R_\tau^{k+1} u(0) = u(0)$

# A higher-order time interpolate (2)

- **Continuity**: $R_\tau^{k+1} u \in C^0(\bar{I}, B)$ and $R_\tau^{k+1} u(t_n) = u(t_n)$ for all $n = 0 \dots N$
- **Approximation of smooth functions**

$$\|u - R_\tau^{k+1} u\|_{C^0(\bar{I}_n, B)} \lesssim \tau_n^{k+2} |u|_{C^{k+2}(\bar{I}_n, B)}$$
$$\|\partial_t u - \partial_t R_\tau^{k+1} u\|_{C^0(\bar{I}_n, B)} \lesssim \tau_n^{k+1} |u|_{C^{k+2}(\bar{I}_n, B)}$$

- **Stability**: $\|R_\tau^{k+1} u\|_{C^1(\bar{I}_n, B)} \lesssim \|u\|_{C^1(\bar{I}_n, B)}$ for all $u \in C^1(\bar{I}_n, B)$

# $\ell^\infty(L^2)$ and $L^2(L^2)$ error estimates

- Static meshes
- Post-processed error $\tilde{e} = u - \mathcal{L}_\tau u_{\tau h}$: For all $m = 1 \dots N$,

$$\|\tilde{e}(t_m)\|_L^2 \lesssim (E_0)^2 + t_m \max_{1\le n\le m} \left\{ C_n^{\mathrm{T}}(u)\tau_n^{2(k+2)} + C_n^{\mathrm{S}}(u)h^{2r+1} \right\} + \mathrm{hot}$$

and under the mild assumption $\tau_n \lesssim \tau_{n-1}$,

$$\|\tilde{e}\|_{L^2(I,L)}^2 \lesssim (E_0)^2 + T \max_{1\le n\le N} \left\{ C_n^{\mathrm{T}}(u)\tau_n^{2(k+2)} + C_n^{\mathrm{S}}(u)h^{2r+1} \right\}$$

- For the error $(u - u_{\tau h})$, same super-convergent bound in $\ell^\infty(L^2)$, but only optimal $(\tau^{k+1} + h^{r+1/2})$ bound in $L^2(L^2)$
- Super-convergence does not hold for implicit Euler $(k = 0)$

# Time-varying meshes

- Time-varying meshes lead to an additional projection error
- Assume that $\mathcal{T}_h^n$ is created from $\mathcal{T}_h^{n-1}$ by **local refinements and coarsenings** (using a common finest mesh)
- The **local (in time)** projection error is defined as

$$E_n^{\mathrm{P}}(u) = \sup_{v_h \in V_h^n} \frac{\left(u(t_{n-1}) - P_h^{n-1} u(t_{n-1}), v_h - \Pi_h^{n-1} v_h\right)_L}{\|v_h - \Pi_h^{n-1} v_h\|_L}$$

  - $\Pi_h^{n-1} : V_h^{n-1} + V_h^n \to V_h^{n-1}$ denotes an $L^2$-stable, linear quasi-interpolation operator
  - Lagrange interpolate for $H^1$-conf. FEM, $L^2$-projection for dG
  - local projection error vanishes if there is only mesh coarsening
- The **global** projection error entering the $\ell^\infty(L^2)$ and $L^2(L^2)$ error estimates is $(E_{P,m}(u))^2 = \sum_{n=1}^m (E_n^{\mathrm{P}}(u))^2$

# Bound on projection error

- Decompose mesh as $\mathcal{T}_h^n = \mathcal{T}_h^{n,\mathrm{ref}} \cup \mathcal{T}_h^{n,\mathrm{coa}}$ where $\mathcal{T}_h^{n,\mathrm{coa}}$ collects mesh cells in $\mathcal{T}_h^n$ that can be decomposed into one or more cells of $\mathcal{T}_h^{n-1}$
- Quasi-interpolation operator satisfies $\left(v_h - \Pi_h^{n-1} v_h\right)\big|_K = 0$, $\forall v_h \in V_h^n$, $\forall K \in \mathcal{T}_h^{n,\mathrm{coa}}$
- On dG spaces, the local projection error is bounded as

$$E_n^{\mathrm{P}}(u) \lesssim |\Omega_n^{\mathrm{ref}}|^{1/2}(h_n^{\mathrm{ref}})^{1/2}\left\{(h_n^{\mathrm{ref}})^{r+1/2}|u(t_{n-1})|_{W^{r+1,\infty}(\Omega_n^{\mathrm{ref}})}\right\}$$

  and on $H^1$-conforming spaces, it is bounded as

$$E_n^{\mathrm{P}}(u) \lesssim (h_n^{\mathrm{ref}})^{1/2}\left\{(h_n)^{r+1/2}|u(t_{n-1})|_{H^{r+1}(\Omega)}\right\}$$

- The bound on dG spaces can exploit that, often in practice, $|\Omega_n^{\mathrm{ref}}| \lesssim h_n^{\mathrm{ref}}$ (up to a slightly stronger regularity on $u$)

# Estimates on error derivatives

- Bounds on error derivatives are rarely explored in the literature
- Assume **static meshes**
- General methodology
  - derive super-convergent (in time) $\ell^\infty(L^2)$ and $L^2(L^2)$ error bounds on time-derivative
  - infer optimal (in time) discrete graph-norm error estimate using discrete inf-sup stability

# Estimate on time derivative

- Key idea: error on time-derivative is defined as

$$\widehat{e} = \partial_t u - \mathcal{L}_\tau \partial_t \mathcal{L}_\tau u_{\tau h}$$

- For all $m = 1 \dots N$,

$$\|\widehat{e}(t_m)\|_L^2 \lesssim (\widehat{E}_0)^2 + t_m \max_{1\le n\le m} \left\{ \widehat{C}_n^{\mathrm{T}}(u,f)\tau_n^{2(k+1)} + C_n^{\mathrm{S}}(u)h^{2r+1} \right\} + \text{hot}$$

and under the mild assumption $\tau_n \lesssim \tau_{n-1}$,

$$\|\widehat{e}\|_{L^2(I,L)}^2 \lesssim (\widehat{E}_0)^2 + T \max_{1\le n\le N} \left\{ \widehat{C}_n^{\mathrm{T}}(u,f)\tau_n^{2(k+1)} + C_n^{\mathrm{S}}(u)h^{2r+1} \right\}$$

# Discrete graph norm error estimate

- Recall discrete inf-sup stability with stability norm

$$\|\!|v_h|\!\|_\sharp^2 = \|\!|v_h|\!\|^2 + \sum_{T\in\mathcal{T}_h} h_T \|\beta\cdot\nabla v_h\|_{L,T}^2$$

- $\ell^2(V)$-estimate on $\tilde{e} = u - \mathcal{L}_\tau u_{\tau h}$: For all $m = 1\dots N$,

$$\sum_{n=1}^{m} Q_n(\|\!|\tilde{e}|\!\|_\sharp^2) \lesssim (\widehat{E}_0)^2 + t_m \max_{1\le n\le m}\left\{\widetilde{C}_n^{\mathrm{T}}(u,f)\tau_n^{2(k+1)} + C_n^{\mathrm{S}}(u)h^{2r+1}\right\}$$

- This bound is optimal in time and exhibits the usual (quasi-)optimal behavior in space for steady problems

# Weighting linear stabilization

- Motivations
- Weighting LS: theory
- Weighting LS: numerics
- We focus on Continuous Interior Penalty, but conjecture most conclusions extend to other LS

# Motivations

- LS adds least-squares penalty to standard Galerkin FEM
  - acts as a high-order dissipation (in contrast to first-order viscosity)
  - LS is very effective for linear first-order PDEs with smooth data
- The situation is not so bright when it comes to solving
  - linear problems with non-smooth data
  - nonlinear problems with non-unique weak solutions
- LS promotes the Gibbs phenomenon, leading to
  - **spurious oscillations** in the vicinity of shocks
  - **failure to satisfy a maximum principle**
  - convergence to **non-entropic weak solutions**

# Nonlinear viscosity

- LS is often supplemented by some nonlinear viscosity technique
  - shock-capturing [Hughes & Mallet, 86; Johnson & Szepessy, 87]
  - crosswind diffusion [Codina, 93; Burman & AE, 02; Burman, 07]
  - entropy viscosity [Guermond, 08; G. & Pasquetti, 08; G., Pasquetti & Popov, 11]
- It is not clear that LS and nonlinear viscosity work hand in hand
- Numerical tests indicate they can antagonize each other

# Some illustrations

- Nonlinear conservation law

$$\begin{cases} \partial_t u + \nabla\cdot\mathbf{f}(u) = 0 & (x,t) \in \Omega \times (0,T) \\ u|_{t=0} = u_0 & x \in \Omega \end{cases} \tag{1}$$

  - $\Omega$ open polyhedral domain in $\mathbb{R}^d$; $f \in C^1(\mathbb{R};\mathbb{R}^d)$
  - **no issues with BCs** (either periodic or compactly supported $u_0$)
  - we assume that (1) admits a unique weak entropic solution
  - we consider space semi-discretization

- Galerkin solution $u_h \in C^1([0,T];V_h)$ s.t. $u_h|_{t=0} = u_{0,h}$ and

$$\int_\Omega w_h \partial_t u_h \,\mathrm{d}\Omega + \int_\Omega w_h \nabla\cdot\mathbf{f}(u_h)\,\mathrm{d}\Omega = 0 \qquad \forall w_h \in V_h \quad \forall t \in (0,T)$$

with $H^1$-conforming FE space $V_h$ (of order $r$)
... **globally polluted by spurious oscillations**

# Viscous solution

- Viscous solution

$$\int_\Omega w_h \partial_t u_h \,\mathrm{d}\Omega + \int_\Omega w_h \nabla\cdot \mathbf{f}(u_h)\,\mathrm{d}\Omega + n_{\mathrm{visc}}(u_h; w_h) = 0$$

with

$$n_{\mathrm{visc}}(v_h; w_h) = c_{\max} \sum_{T\in\mathcal{T}_h} h_T \|\mathbf{f}'(v_h)\|_{L^\infty(T)} \int_T \nabla v_h \cdot \nabla w_h \,\mathrm{d}T$$

  - typically $c_{\max} = \frac{1}{2r}$ in 1D and $c_{\max} = \frac{1}{4r}$ in 2D
  - for linear transport, $\mathbf{f}(v_h) = \boldsymbol{\beta} v_h$ so that $\|\mathbf{f}'(v_h)\|_{L^\infty(T)} = \|\boldsymbol{\beta}\|_{L^\infty(T)}$

... **only first-order accurate**

# CIP stabilized solution

- CIP stabilized solution

$$\int_\Omega w_h \partial_t u_h \,\mathrm{d}\Omega + \int_\Omega w_h \nabla\cdot \mathbf{f}(u_h)\,\mathrm{d}\Omega + n_{\mathrm{CIP}}(u_h; w_h) = 0$$

with

$$n_{\mathrm{CIP}}(v_h; w_h) = c_{\mathrm{CIP}} \sum_{F\in\mathcal{F}_h^i} h_F^2 \|\mathbf{f}'(v_h)\|_{L^\infty(F)} \int_F [\nabla v_h]\cdot[\nabla w_h]\,\mathrm{d}F$$

  - typically, $c_{\mathrm{CIP}} = 0.05$

... $O(h^{r+1/2})$ $L^2$**-estimates** for linear transport and smooth solutions

# Entropy-viscosity solution

- Entropy-viscosity solution (nonlinear stabilization)

$$\int_\Omega w_h \partial_t u_h \,\mathrm{d}\Omega + \int_\Omega w_h \nabla\cdot \mathbf{f}(u_h)\,\mathrm{d}\Omega + n_{\mathrm{entr}}(u_h; u_h, w_h) = 0$$

with

$$n_{\mathrm{entr}}(z_h; v_h, w_h) = \sum_{T\in\mathcal{T}_h} \nu_T(z_h) \int_T \nabla v_h \cdot \nabla w_h \,\mathrm{d}T$$

and $\nu_T(z_h)$ is designed s.t.

$$\nu_T(z_h) = \min(c_{\max}\beta_T(z_h)h_T, c_{\mathrm{ev}} D_T(z_h)h_T^2)$$

and $\beta_T(z_h) = \|\mathbf{f}'(z_h)\|_{L^\infty(T)}$, $D_T(z_h)$ is the local residual for a chosen entropy (e.g., the quadratic one)

... **weak maximum principle** (proof in 1D)

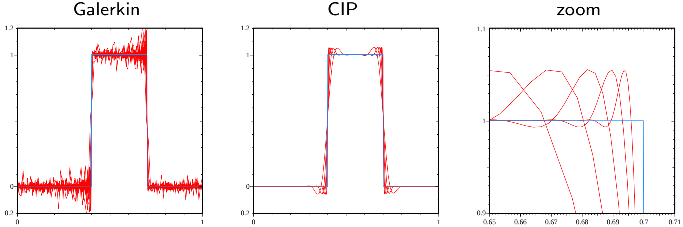
$$\|u_h(t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + ch^\alpha$$

# Illustration of difficulties

- Numerical tests in 1D
  - linear transport with non-smooth data
  - nonlinear transport with composite wave (non-convex flux)
  - CIP stabilization and first-order viscosity
- Time discretization is performed using SSP RK3
  - (very) small time steps to avoid time discretization errors
- The mass matrix is never lumped

# Linear transport with non-smooth data I

- $\partial_t u + \partial_x u = 0$, $u(x,0) = 1_{(0.4,0.7)}$, periodic BCs, and $T = 1$




- Stabilizing capability of CIP stabilization, **but inability to counter Gibbs phenomenon**

- Maximum principle indicators at final time

$$e_{\mathrm{Max}} = \max_{x\in\Omega} u_h(x,T) - 1 \qquad e_{\mathrm{Min}} = -\min_{x\in\Omega} u_h(x,T)$$

remain bounded away from zero for CIP

# Linear transport with non-smooth data II

| $h$ | entropy | | entropy + CIP | |
|---|---|---|---|---|
| | $e_{\mathrm{Max}}$ | rate | $e_{\mathrm{Max}}$ | rate |
| 2.500E-03 | 6.715E-03 | – | 1.597E-02 | – |
| 1.250E-03 | 5.434E-03 | 0.305 | 1.600E-02 | -0.003 |
| 6.250E-04 | 2.854E-03 | 0.929 | 1.633E-02 | -0.030 |
| 3.125E-04 | 2.235E-03 | 0.353 | 1.626E-02 | 0.006 |
| 1.563E-04 | 1.785E-03 | 0.324 | 1.646E-02 | -0.017 |

- Entropy-viscosity solution satisfies a weak maximum principle
- Adding CIP to entropy-viscosity, the WMP is lost!

# Nonlinear transport with composite wave I

- Riemann problem with non-convex flux (S-shaped)

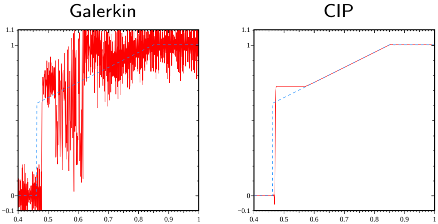
$$f(u) = \begin{cases} \frac{1}{4}u(1-u) & \text{if } u < \frac{1}{2} \\ \frac{1}{2}u(u-1) + \frac{3}{16} & \text{if } \frac{1}{2} \le u \end{cases} \qquad u_0(x) = \begin{cases} 0 & x \in [0, 0.35] \\ 1 & x \in (0.35, 1] \end{cases}$$

- Entropy solution at $T = 1$ is composed of a **shock wave followed by a rarefaction wave**

- Many second-order central schemes with limiters converge to a non-entropic (weak) solution
  - e.g., central upwind with second-order reconstruction and either superbee or minmod2 limiters [Kurganov, Petrova & Popov, 07]

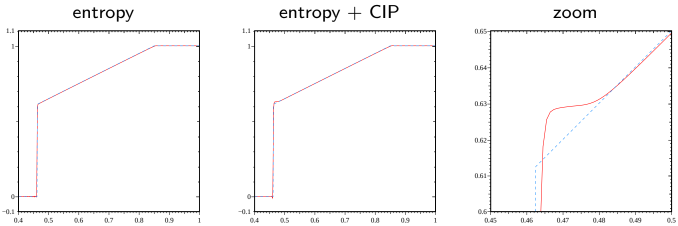
# Nonlinear transport with composite wave II

- Uniform mesh with 1000 cells, SSP RK3 with $\mathrm{CFL} = 0.01$




- The CIP-stabilized solution converges to a non-entropic solution

# Nonlinear transport with composite wave III

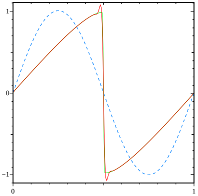



- Entropy-viscosity solution converges to (correct) entropic solution
- Adding CIP destroys this property!

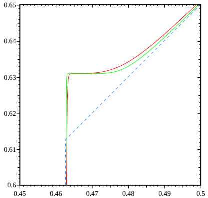
# CIP stabilization and first-order viscosity

- CIP can have adverse effects even on first-order viscosity
- **(Inviscid) Burgers equation** with $u(x,0) = \sin(2\pi x)$, 200 mesh cells, $r = 1$, $\mathrm{CFL} = 0.025$
  - adding CIP to 1st-order visc. leads to over/under-shoots
  - $c_{\max} = 2$ makes 1st-order visc. overcome Gibbs phenomenon triggered by CIP
- **Riemann problem with non-convex flux**, 4,000 and 10,000 cells
  - viscous solution converges to entropic solution (as expected!)
  - adding CIP stabilization destroys this property

Burgers




non-convex flux



# Conclusions from numerical tests

- CIP does a great job at suppressing oscillations in smooth regions
- It promotes the Gibbs phenomenon
  - failure to satisfy a (weak) **maximum principle**
  - convergence to **non-entropic weak solutions**
- These effects can even overcome convergent viscosity methods (both nonlinear and first-order)

# Key idea

- Temper the amount of LS in the vicinity of shocks
  - nonlinear weights depending on the **local gradient** of discrete solution
  - may seem counter-intuitive at first glance since LS is often motivated to counter spurious oscillations near large gradients ...
- We show that CIP stabilization can be tempered in such a way that
  - $O(h^{r+1/2})$ $L^2$-error estimates are preserved for smooth solutions in linear problems [proof]
  - LS no longer antagonizes nonlinear viscosity methods [numerics]
- This is a win-win situation
  - nonlinear viscosity alone does not deliver full-order accuracy in smooth regions

# Theoretical insight

- Weighted CIP-stabilized solution

$$\int_\Omega w_h \partial_t u_h \,\mathrm{d}\Omega + \int_\Omega w_h \nabla\cdot\mathbf{f}(u_h)\,\mathrm{d}\Omega + n_{\text{wei,ed}}(u_h; u_h, w_h) = 0$$

with

$$n_{\text{wei,ed}}(z_h; v_h, w_h) = c_{\mathrm{CIP}} \sum_{F\in\mathcal{F}_h^i} \alpha(g_F(z_h)) h_F^2 \|\mathbf{f}'(v_h)\|_{L^\infty} \int_F [\nabla v_h]\cdot[\nabla w_h]\,\mathrm{d}F$$

where $g_F(z_h) = |\langle \nabla z_h\rangle_{\Delta_F}|/\ell(u_0)$ is a local measure of $\nabla z_h$ around $F$

- The weighting function $\alpha$ is non-increasing and

$$\exists\lambda > 0, \quad (r \geq r_0) \Rightarrow (\alpha(r) \geq r^{-\lambda})$$

$\alpha$ **cannot decrease too fast** (typically $\alpha(0) = 1$ and $\alpha(\infty) = 0$)

# Convergence analysis

- **Linear transport, smooth solutions**
- For all $t \in [0, T]$, with $e = u - u_h$,

$$\|e(t)\|^2_{L^2(\Omega)} + \int_0^t n_{\mathrm{wei,ed}}(u_h; e, e)\,\mathrm{d}\tau \lesssim h^{2r+1}$$

with

  - for all $\lambda > 0$, if $d = 2$ or if $d = 3$ and $r \geq 3$
  - for $d = 3$ and $r \in \{1, 2\}$, upper bound is $h^{r+\epsilon_\lambda}$ with $\epsilon_\lambda \in (0, \frac{1}{2})$ and $\lambda \in (0, 2)$ for $r = 2$ and $\lambda \in (0, \frac{2}{3})$ if $r = 1$

- Proof on quasi-uniform meshes

# Principle of proof I

- Classical techniques lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{L^2(\Omega)} + n_{\mathrm{wei,ed}}(u_h; e, e) \leq \mathrm{RHS}(\Omega) \lesssim h^r \|e\|_{L^2(\Omega)}$$

where control on $n_{\mathrm{wei,ed}}(u_h; e, e)$ is not yet used

- Let $\epsilon \geq 0$ and consider the sets collecting "bad" and "good" cells

$$\Omega^\sharp = \{g_F(u_h) \geq h^{-\epsilon}\}$$
$$\Omega^\flat = \Omega \setminus \Omega^\sharp$$

$\Omega^\sharp$ collects mesh cells where the gradient of $u_h$ is locally high

# Principle of proof II

- On $\Omega^\flat$, owing to the behavior of weighting function $\alpha$, there is enough CIP stabilization to infer that

$$\mathrm{RHS}(\Omega^\flat) \lesssim h^{r+\frac{1}{2}-\frac{1}{2}\lambda\epsilon} n_{\mathrm{wei,ed}}(u_h; e, e)^{\frac{1}{2}}$$

- On $\Omega^\sharp$, the following holds:

$$\mathrm{RHS}(\Omega^\sharp) \lesssim h^{2r}|\Omega^\sharp|^{\frac{1}{2}} \quad \text{and} \quad |\Omega^\sharp| \lesssim h^{2(r-1+\epsilon)}$$

since $\|\nabla u\|_{L^2(\Omega^\sharp)}$ and $\|\nabla e\|_{L^2(\Omega^\sharp)}$ are bounded

- This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{L^2(\Omega)} + n_{\mathrm{wei,ed}}(u_h; e, e) \lesssim h^{3r-1+\epsilon} + h^{2r+1-\lambda\epsilon}$$

Choose $\epsilon$ to equilibrate both terms and derive an **improved error estimate** $O(h^{r+\rho})$, and then use a **bootstrap argument**

# Numerical examples

- We study the effectiveness of the weighted CIP-stabilization on
  - linear transport with smooth data
  - linear transport with non-smooth data
  - nonlinear transport with composite wave
- 1D and 2D tests are considered

# 1D tests I

- $\Omega = (0,1)$ with periodic BCs, $r = 1$, SSP RK3 with $\mathrm{CFL} = 0.2$
  - stab. parameters $c_{\mathrm{CIP}} = 0.05$, $c_{\max} = 0.5$, and $c_{\mathrm{ev}} = 0.5$
- **Linear transport with smooth data**, CIP stabilization with and without weighting

$$\|e\|_{L^1(\Omega)} \sim h^2 \qquad \|e\|_{L^2(\Omega)} \sim h^2$$

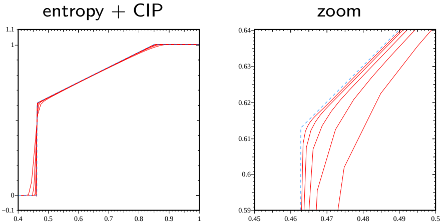
- **Linear transport with non-smooth data**, entropy viscosity plus CIP stabilization, uniform and non-uniform meshes

$$\|e\|_{L^1(\Omega)} \sim h^{0.75} \qquad \|e\|_{L^2(\Omega)} \sim h^{0.37}$$

and weak maximum principle is satisfied (with rate $h^{0.5}$)

# 1D tests II

- **Riemann problem with non-convex flux**
  - five uniform meshes from 100 up to 1,600 cells
  - entropy viscosity plus CIP stabilization




- Convergence to the (correct) entropic solution

# 2D tests I

- **Linear transport** (rotating velocity field in unit disk)
  - $r \in \{1, 2\}$, RK4, $\mathrm{CFL} = 0.25$
  - stab. parameters $c_{\mathrm{CIP}} = 0.025$, $c_{\max} = \frac{1}{4r}$, and $c_{\mathrm{ev}} = 0.1$
- CIP stabilization with and without weighting leads to optimal convergence on smooth solutions
- Entropy viscosity plus weighted CIP stabilization
  - $r = 1$: entropy viscosity alone and with CIP is second-order
  - $r = 2$: entropy viscosity alone is $h^{2+\epsilon}$, while adding CIP improves CV at least to $h^{2.5}$ $\rightarrow$ win-win situation

# 2D tests II

- Linear transport, **non-smooth data**, entropy visc. + CIP, $r \in \{1,2\}$
- CV rates (in $L^1$-norm, rates are $h^{0.75}$ for $r=1$ and $h^{0.8}$ for $r=2$)

| $h$ | $r=1$ | | $r=2$ | |
|---|---|---|---|---|
| | $L^2$-norm | rate | $L^2$-norm | rate |
| 5.00E-02 | 4.172E-01 | – | 2.794E-01 | – |
| 2.50E-02 | 3.158E-01 | 0.402 | 2.114E-01 | 0.402 |
| 1.25E-02 | 2.411E-01 | 0.389 | 1.601E-01 | 0.401 |
| 1.00E-02 | 2.214E-01 | 0.383 | 1.466E-01 | 0.394 |

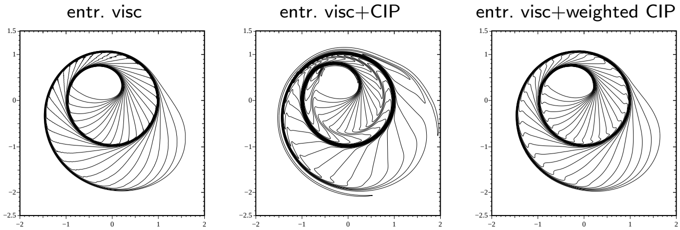
- Weak maximum principle for $e_{\mathrm{Max}}$ (similar results for $e_{\mathrm{Min}}$)

| $h$ | $r=1$ | | $r=2$ | |
|---|---|---|---|---|
| | $e_{\mathrm{Max}}$ | rate | $e_{\mathrm{Max}}$ | rate |
| 5.00E-02 | 3.546E-02 | – | 7.904E-03 | – |
| 2.50E-02 | 1.283E-02 | 1.467 | 6.943E-03 | 0.187 |
| 1.25E-02 | 7.776E-02 | 0.722 | 5.953E-03 | 0.222 |
| 1.00E-02 | 6.798E-02 | 0.603 | 5.211E-03 | 0.596 |

# 2D tests III

- **Cauchy problem in $\mathbb{R}^2$ with non-convex flux**

$$\mathbf{f}(u) = (\sin u, \cos u) \qquad u(x,y,0) = \begin{cases} 3.5\pi & x^2+y^2<1 \\ 0.25\pi & \text{otherwise} \end{cases}$$




- entropy viscosity ($c_{\max} = \frac{1}{2}$, $c_{\mathrm{ev}} = 1$) predicts correct rotating composite wave structure
- adding CIP ($c_{\mathrm{CIP}} = 1$) leads to non-physical layers
- weighting CIP **pushes spurious layer back to the shock**

# Conclusions

- In the literature, much efforts are devoted to constructing LS techniques in various flavors
- It is often believed that LS is the workhorse, whereas shock-capturing is only meant to remove remaining oscillations
- We believe that
  - nonlinear viscosities should be the workhorses killing the Gibbs phenomenon and ensuring convergence to the entropic solution
  - LS plays the role of an **auxiliary tool** whose job is to improve convergence in **smooth regions**