Discontinuous Galerkin methods for aerodynamic flows

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Aerodynamic

Navier-Stokes equations

- viscous compressible Navier-Stokes equations

$$\left\{\begin{array}{r}\partial_t\rho+\mathrm{div}_{\mathbf{x}}(\rho\mathbf{u})=0\\ \partial_t(\rho\mathbf{u})+\mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}\otimes\mathbf{u}+P\mathbf{I})=\mathrm{div}_{\mathbf{x}}\tau\\ \partial_t(\rho E)+\mathrm{div}_{\mathbf{x}}((\rho E+P)\mathbf{u})=\mathrm{div}_{\mathbf{x}}\left(\tau\cdot\mathbf{u}\right)+\mathrm{div}_{\mathbf{x}}\left(\lambda\nabla T\right)\end{array}\right.$$

$$\tau=\mu\left(\nabla\otimes\mathbf{u}+(\nabla\otimes\mathbf{u})^T\right)-\frac{2\mu}{3}\left(\mathrm{div}_{\mathbf{x}}\mathbf{u}\right)\mathbf{I}$$

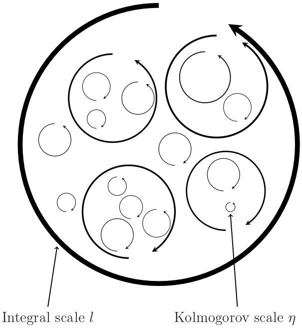
- Closure conditions

$$E=\frac{|\mathbf{u}|^2}{2}+\varepsilon \qquad P=P(\varepsilon,\rho) \qquad T=T(\varepsilon,\rho)$$

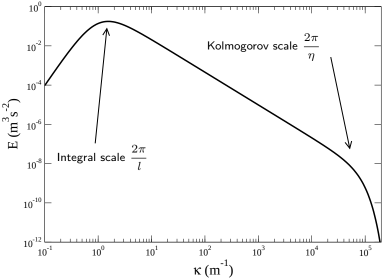
- Difficulties
  - Nonlinear system
  - hyberbolic terms
  - parabolic terms

Aerodynamic

Turbulent flows



Physical space



Fourier space

- Integral scale l
- Kolmogorov scale η

Aerodynamic

Turbulent flows

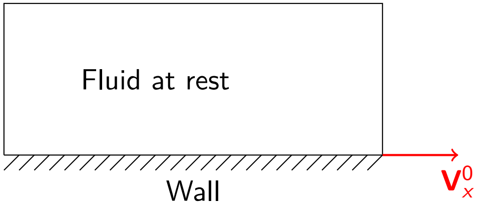
- Different flow configurations

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{Ma} \nabla P = \frac{1}{Re} \mathrm{div}_{\mathbf{x}} \tau$$

  - Re Reynolds number
  - Ma Mach number
- Computational cost
  - Resolution scale (size of the cells) is driven by the smallest scales η.
  - $\frac{l}{\eta} \sim Re^{3/4}$
  - Mesh size increases as $Re^{9/4}$.

Aerodynamic

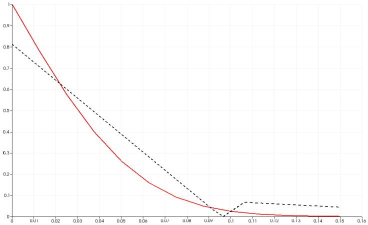
Boundary layer

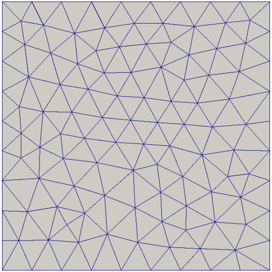


$$\mathbf{V}_x(t,x) = \mathbf{V}_x^0 \left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)$$

Aerodynamic

Boundary layer



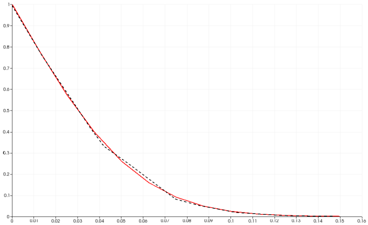


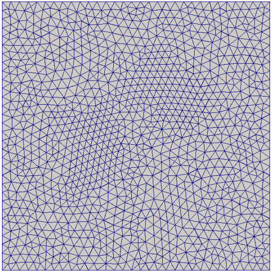
$N = 200$

$$\mathbf{V}_x(t,x) = \mathbf{V}_x^0 \left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)$$

Aerodynamic

Boundary layer



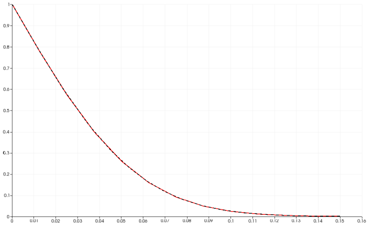


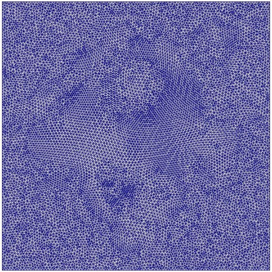
$$N = 2326$$

$$\mathbf{V}_x(t,x) = \mathbf{V}_x^0\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)$$

Aerodynamic

Boundary layer



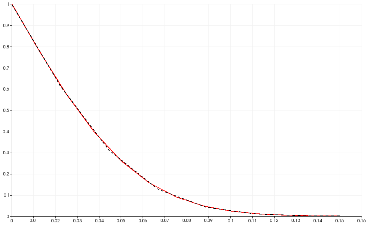


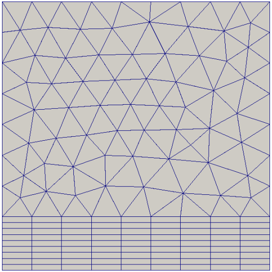
$N = 22804$

$$\mathbf{V}_x(t,x) = \mathbf{V}_x^0 \left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)$$

Aerodynamic

Boundary layer





$$N = 223$$

$$\mathbf{V}_x(t,x) = \mathbf{V}_x^0\left(1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right)$$

Use hybrid meshes

Aerodynamic

Turbulence modelling

- Direct Numerical Simulation
  - mesh size scales as $Re^{9/4}$; Re^3 in boundary layers
  - timestep scales as $Re^{-3/4}$
  - might be available in 2080 (Spalart, Boeing).
- Use turbulence models
  - Reynolds Averaged Navier Stokes (RANS)
    - fully time filtered (stationary)
    - Mesh loop adaptation
    - Standard in industry
  - Large Eddy Simulation: partially time filtered
    - Reliable if the mesh is fine enough
    - Mesh adaptation at each time step?
    - Beware of diffusion ▸ Example Vortex
  - Hybrid RANS/LES

Aerodynamic

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Need for high order schemes

Aerodynamic

Which scheme?

We want

- Work on unstructured hybrid meshes.
- Discretize convection problems (upwinding of the fluxes).
- Have a compact scheme.

- Punctual approximations
  - Finite differences: structured meshes, filters for stability
- Weak approximations
  - High order Finite volumes: uncompact stencil
  - Continuous Galerkin: difficulties for upwinding, non diagonal mass matrix.

$\Longrightarrow$ Discontinuous Galerkin

Outline

Numerical scheme for compressible Navier-Stokes equations

- Advection terms
- Diffusion terms
- Efficient implementation

Discontinuous Galerkin methods for low (but not zero) Mach flow

- Review of the steady case
- Problem in the unsteady case
- A new scheme stable for both steady and unsteady flow

Application

Outline

Numerical scheme for compressible Navier-Stokes equations

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- Diffusion terms
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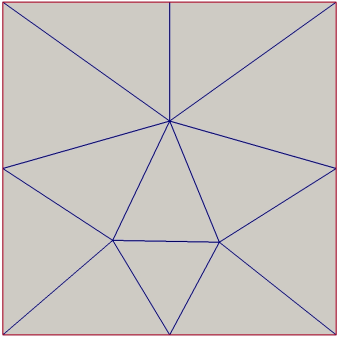
Discontinuous Galerkin methods for low (but not zero) Mach flow

Application

Notations

- We consider a mesh $\mathcal{T}_h$



- S_b boundary faces
- S_i interior faces
- $\mathbf{n}^S$ outward normal of the face S
- $\mathbf{n}^{\mathrm{out}}$ outward normal from an element

$$\varphi^R(\mathbf{x}) = \lim_{\varepsilon\to 0, \varepsilon>0} \varphi(\mathbf{x}+\varepsilon \mathbf{n}^S) \quad \text{et} \quad \varphi^L(\mathbf{x}) = \lim_{\varepsilon\to 0, \varepsilon>0} \varphi(\mathbf{x}-\varepsilon \mathbf{n}^S)$$

$$[\![\varphi]\!](\mathbf{x}) = \varphi^R(\mathbf{x}) - \varphi^L(\mathbf{x}) \qquad \text{and} \qquad \{\!\{ \varphi \}\!\}(\mathbf{x}) = \frac{\varphi^R(\mathbf{x}) + \varphi^L(\mathbf{x})}{2}.$$

- Integrate the equation $\varphi \mathrm{div}_{\mathbf{x}} \mathbf{f}(\mathbf{u}) = 0$
- $\mathbf{u}$ and φ are continuous in the cells, discontinuous on the faces

$$\int_{\Omega} \varphi \mathrm{div}_{\mathbf{x}} \mathbf{f}(u) = \sum_{T \in \mathcal{T}_h} \int_{T} \varphi \mathrm{div}_{\mathbf{x}} \mathbf{f}(u) + \sum_{S \in S_i} \int_{S} \hat{\varphi} [\![\mathbf{f}(u)]\!] \cdot \mathbf{n}^S$$

where $\hat{\varphi}$ is the test function numerical flux

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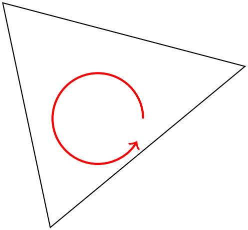
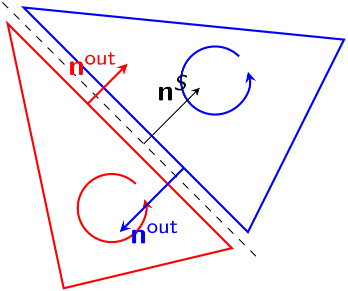


- Stokes formula

$$\begin{aligned}
&= \int_T \mathrm{div}_{\mathbf{x}} \left(\varphi \mathbf{f}(u) \right) - \int_T \mathbf{f}(u) \nabla \varphi \\
&= \int_{\partial T} \varphi \mathbf{f}(u) \cdot \mathbf{n}^{\mathrm{out}} - \int_T \mathbf{f}(u) \nabla \varphi \\
&= \sum_{S \in \partial T} \int_S \varphi \mathbf{f}(u) \cdot \mathbf{n}^{\mathrm{out}} - \int_T \mathbf{f}(u) \nabla \varphi
\end{aligned}$$

Face integrals and numerical flux $\hat{\varphi}$



$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\sum_{S\in\partial T}\int_S \varphi \mathbf{f}(u)\cdot\mathbf{n}^{\text{out}} \\
=&-\sum_{S\in S_i}\int_S [\![\varphi \mathbf{f}(u)]\!]\cdot\mathbf{n}^S \\
&+\sum_{S\in S_b}\int_S \varphi \mathbf{f}(u)\cdot\mathbf{n}^{\text{out}}
\end{aligned}
$$

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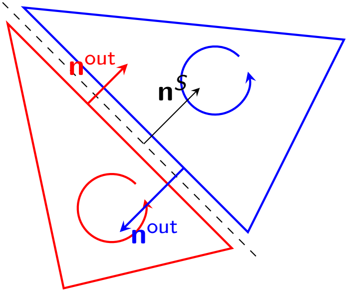

$$\sum_{T\in\mathcal{T}_h}\sum_{S\in\partial T}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{\text{out}}$$
$$= -\sum_{S\in S_i}\int_S [\![\varphi\mathbf{f}(u)]\!]\cdot\mathbf{n}^S + \sum_{S\in S_b}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{\text{out}}$$

$$[\![ab]\!] = \{\!\{a\}\!\}[\![b]\!] + [\![a]\!]\{\!\{b\}\!\}$$

$$\int_\Omega \varphi\,\mathrm{div}_{\mathbf{x}}\mathbf{f}(u) = -\sum_{T\in\mathcal{T}_h}\int_T \mathbf{f}(u)\nabla\varphi - \sum_{S\in S_i}\int_S [\![\varphi]\!]\{\!\{\mathbf{f}(u)\}\!\}\cdot\mathbf{n}^S$$
$$-\sum_{S\in S_i}\int_S \{\!\{\varphi\}\!\}[\![\mathbf{f}(u)]\!]\cdot\mathbf{n}^S + \sum_{S\in S_b}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{\text{out}}$$
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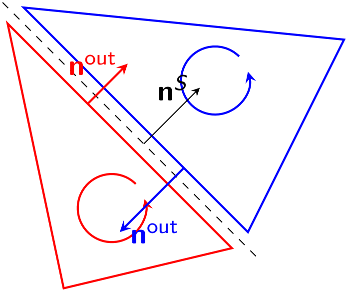

$$\sum_{T\in\mathcal{T}_h}\sum_{S\in\partial T}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{\text{out}} = -\sum_{S\in S_i}\int_S [\![\varphi\mathbf{f}(u)]\!]\cdot\mathbf{n}^S + \sum_{S\in S_b}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{\text{out}}$$

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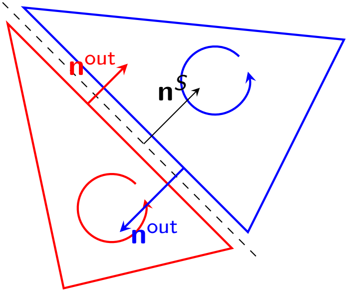

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$$\int_\Omega \varphi \mathrm{div}_{\mathbf{x}}\mathbf{f}(u) = -\sum_{T\in\mathcal{T}_h}\int_T \mathbf{f}(u)\nabla\varphi - \sum_{S\in S_i}\int_S [\![\varphi]\!]\{\!\{\mathbf{f}(u)\}\!\}\cdot\mathbf{n}^S$$
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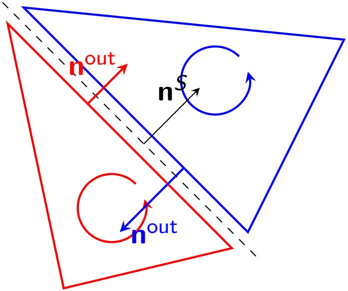

$$\sum_{T\in\mathcal{T}_h}\sum_{S\in\partial T}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{out}$$
$$=-\sum_{S\in S_i}\int_S [\![\varphi\mathbf{f}(u)]\!]\cdot\mathbf{n}^S$$
$$+\sum_{S\in S_b}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{out}$$

$$[\![ab]\!]=\{\!\{a\}\!\}[\![b]\!]+[\![a]\!]\{\!\{b\}\!\}$$

$$\int_\Omega \varphi\,\mathrm{div}_{\mathbf{x}}\mathbf{f}(u)=-\sum_{T\in\mathcal{T}_h}\int_T \mathbf{f}(u)\nabla\varphi-\sum_{S\in S_i}\int_S [\![\varphi]\!]\{\!\{\mathbf{f}(u)\}\!\}\cdot\mathbf{n}^S$$
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Numerical flux

$$\int_\Omega \varphi \mathrm{div}_{\mathbf{x}} \mathbf{f}(u) = -\sum_{T\in\mathcal{T}_h}\int_T \mathbf{f}(u)\nabla\varphi - \sum_{S\in S_i}\int_S [\![\varphi]\!]\{\!\{\mathbf{f}(u)\}\!\}\cdot\mathbf{n}^S + \sum_{S\in S_b}\int_S \varphi\mathbf{f}(u)\cdot\mathbf{n}^{\mathrm{out}}$$

- For stabilizing, use a numerical flux ▸ Riemann solver example

$$\{\!\{\mathbf{f}(\mathbf{u})\}\!\} \approx \tilde{\mathbf{f}}(\mathbf{u}^L, \mathbf{u}^R)$$

- Lax-Friedrich scheme

$$\tilde{\mathbf{f}}(\mathbf{u}^L, \mathbf{u}^R) = \frac{\mathbf{f}(\mathbf{u}^L) + \mathbf{f}(\mathbf{u}^R)}{2} - \frac{\lambda}{2}\left(\mathbf{u}^R - \mathbf{u}^L\right)$$

- Similar to penalization terms, right physical scaling

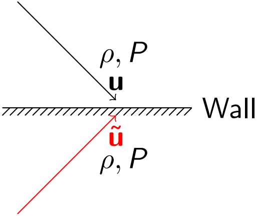
Boundary conditions

$$\sum_{S \in S_b} \int_S \varphi \mathbf{f}(u) \cdot \mathbf{n}^{\text{out}}$$

- Inlet/outlet boundary conditions: use characteristic decomposition
- Slipping wall boundary condition: normal velociy vanishes, no flux on mass nor energy

- $\tilde{\mathbf{u}} = \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$
- Use the Lax Friedrich scheme

$$\begin{pmatrix} 0 \\ [P + (\lambda + \mathbf{u} \cdot \mathbf{n})\,(\mathbf{u} \cdot n - 0)] \cdot \mathbf{n} \\ 0 \end{pmatrix}$$



Diffusive part

- Original system

$$\begin{cases} \mathrm{div}_{\mathbf{x}}\left(A\nabla\mathbf{U}\right) = 0 & \text{on} \quad \Omega \\ A\dfrac{\partial \mathbf{U}}{\partial \mathbf{n}} = f_n & \text{on} \quad \Gamma_n \\ \mathbf{U} = \mathbf{U}^b & \text{on} \quad \Gamma_d \end{cases}$$

- Mixed formulation

$$\begin{cases} \mathrm{div}_{\mathbf{x}}\left(A\mathbf{z}\right) = 0 & \text{on} \quad \Omega \\ \nabla\mathbf{U} = \mathbf{z} & \text{on} \quad \Omega \\ A\mathbf{z}\cdot\mathbf{n} = f_n & \text{on} \quad \Gamma_n \\ \mathbf{U} = \mathbf{U}^b & \text{on} \quad \Gamma_d \end{cases}$$

- Perform advection scheme on both equations.

Diffusive part

- Equivalent to lifting the gradient: the gradient is the sum of a cell contribution and face jumps contribution

$$z = \nabla U + R(\llbracket \mathbf{U} \rrbracket)$$

- BR1 formulation: global lifting

$$\forall \mathbf{g} \qquad \int_\Omega \mathbf{g} \cdot R(\llbracket \mathbf{U} \rrbracket) = -\sum_{S \in S_i} \int_{S_i} \{\!\{ \mathbf{g} \}\!\} \cdot \llbracket \mathbf{U} \rrbracket$$

- BR2 formulation: local lifting

$$\forall S_i \quad \forall \mathbf{g} \qquad \int_{T \ni S_i} \mathbf{g} \cdot R^{S_i}(\llbracket \mathbf{U} \rrbracket) = -\int_{S_i} \{\!\{ \mathbf{g} \}\!\} \cdot \llbracket \mathbf{U} \rrbracket$$

Diffusive part

Final scheme

$$
\begin{aligned}
&-\sum_{T\in\mathcal{T}_h}\int_T \nabla\varphi\cdot A\nabla\mathbf{U}\\
&\quad+\sum_{S\in S_i}\int_{S_i}\left[\{\!\{\, A^T\nabla\varphi \,\}\!\}\cdot[\![\,\mathbf{U}\,]\!]+\{\!\{\, A\nabla\mathbf{U}\,\}\!\}\cdot[\![\,\varphi\,]\!]\right]\\
&\qquad+\sum_{S\in S_i}\int_{S_i}\eta\,\{\!\{\, AR^{S_i}([\![\,\mathbf{U}\,]\!])\,\}\!\}\,[\![\,\varphi\,]\!]\\
&\qquad\quad+\sum_{S\in S_b}\cdots=0
\end{aligned}
$$

- η = the number of faces

Practical implementation of the BR2 scheme

Symmetry term

- Compute

$$\int_{S_i} \left\{\!\!\left\{ A^T \nabla\varphi \right\}\!\!\right\} \cdot [\![\mathbf{U}]\!]$$

- Transformed as

$$\frac{1}{2}\left(\nabla\varphi_L A(\mathbf{U}_L)[\![\mathbf{U}]\!] \otimes \mathbf{n} + \nabla\varphi_R A(\mathbf{U}_R)[\![\mathbf{U}]\!] \otimes \mathbf{n}\right)$$

Practical implementation of the BR2 scheme

Gradient lifting

- Compute

$$\int_{T\ni S} \mathbf{g}\cdot R^S(\llbracket \mathbf{U} \rrbracket) = -\int_{S_i} \{\!\{\mathbf{g}\}\!\}\cdot \llbracket \mathbf{U} \rrbracket$$

- Project R^{S_i} on the finite element basis

$$R^S = \sum_{\varphi_{i_L}\in T_L} R^S_{i_L}\varphi^L_{i_L} + \sum_{\varphi_{i_R}\in T_R} R^S_{i_R}\varphi^R_{i_R}$$

- Test with $g = \varphi_i^R$ and $g = \varphi_i^L$

for $K = L$ or R $\qquad M_K R^{S,K} = M_S^{K,R}\mathbf{U}^R - M_S^{K,L}\mathbf{U}^L$

with $\qquad M_K = \left(\int_K \varphi_i^K \varphi_j^K\right)$ and $M_S^{K,R} = \frac{1}{2}\left(\int_S \varphi_i^R \varphi_j^K\right)$

Practical implementation of the BR2 scheme

Factorization

- We want to compute

$$\{\!\{ A\nabla \mathbf{U} \}\!\} \cdot [\![\varphi]\!] + \eta \left\{\!\!\left\{ AR^{S_i}([\![\mathbf{U}]\!]) \right\}\!\!\right\} [\![\varphi]\!]$$

- Develop

$$\begin{aligned} &\frac{1}{2} A(\mathbf{U}_L)\nabla\mathbf{U}_L + \frac{1}{2} A(\mathbf{U}_R)\nabla\mathbf{U}_R + \frac{\eta}{2} \left(A(\mathbf{U}_L)R_L^S([\![\mathbf{U}]\!]) + A(\mathbf{U}_R)R_L^S([\![\mathbf{U}]\!])\right) \\ = &\frac{1}{2} A(\mathbf{U}_L)\nabla\mathbf{U}_L + \frac{1}{2} A(\mathbf{U}_R)\nabla\mathbf{U}_R + \frac{\eta}{2} \Big(A(\mathbf{U}_L)\left(M_L^{-1}M_S^{L,R}\mathbf{U}^R - M_L^{-1}M_S^{L,L}\mathbf{U}_L\right) \\ &\qquad + A(\mathbf{U}_R)\left(M_R^{-1}M_S^{R,R}\mathbf{U}^R - M_R^{-1}M_S^{R,L}\mathbf{U}_L\right)\Big) \end{aligned}$$

- Factorize

$$\begin{aligned} &\frac{1}{2} A(\mathbf{U}_L)\left(\nabla\color{red}{\mathbf{U}_L}\color{black} - \eta M_L^{-1}M_S^{L,L}\color{red}{\mathbf{U}_L}\color{black} + \eta M_L^{-1}M_S^{L,R}\mathbf{U}^R\right) \\ &\qquad + \frac{1}{2} A(\mathbf{U}_R)\left(\nabla\color{blue}{\mathbf{U}_R}\color{black} + \eta M_R^{-1}M_S^{R,R}\color{blue}{\mathbf{U}^R}\color{black} - \eta M_R^{-1}M_S^{R,L}\mathbf{U}_L\right) \end{aligned}$$

Boundary conditions

- Aim: isothermal and adiabatic wall
  - Velocity: Dirichlet boundary condition
  - Temperature: Neumann (adiabatic) or Dirichlet (isothermal)
- Advection term ≈ nearly as slipping wall
- Diffusion terms:
  - Neumann is directly replaced by the imposed flux,
  - Dirichlet: compute jumps based on imposed values
- The system is expressed in conservative variables $(\rho, \rho\mathbf{u}, \rho E)$
  - Fourier contribution

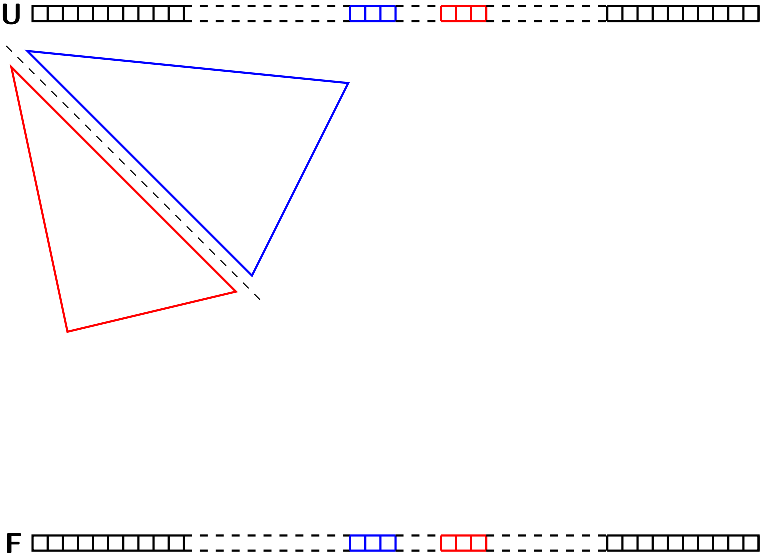
$$\nabla T = \frac{1}{\rho C_v}\left(\left(\frac{|\mathbf{u}|^2}{2} - \varepsilon + \frac{\ell}{\rho}\right)\nabla\rho - \sum u_i \nabla(\rho u_i) + \nabla\rho E\right)$$

- How to define ρ and ρE on the wall?
  - locally switch back to primitive variables (nonlinearity within the lifting operator)

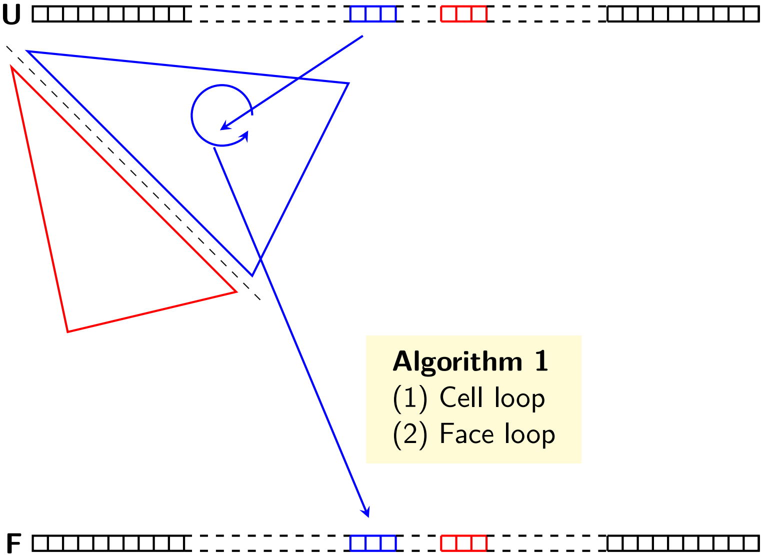
Implementation

- Discontinuous Galerkin methods are linear methods for linear problems
- Three steps:
  - Interpolate on quadrature points (values, gradient, lifting gradient)
  - Compute (numerical) flux
  - Project on degrees of freedom
- Three loops
  - Cells
  - Interior sides
  - Boundary sides
- One nonlinear step nested in two linear steps

Implementation

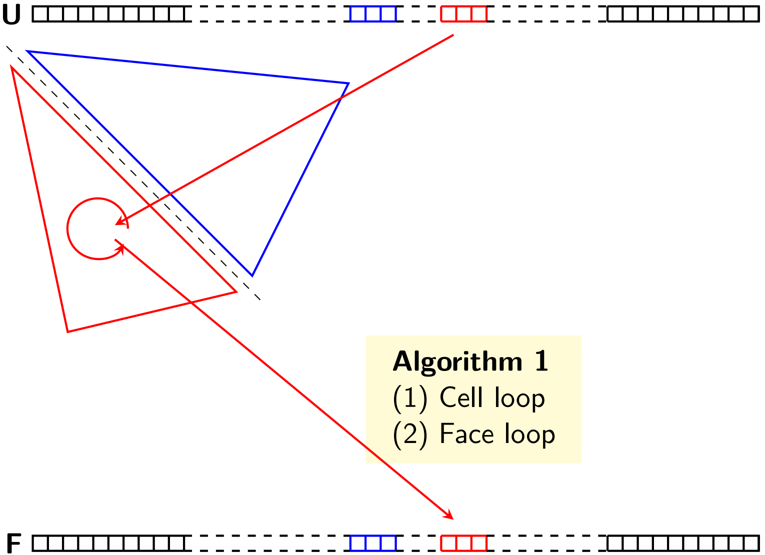


Implementation

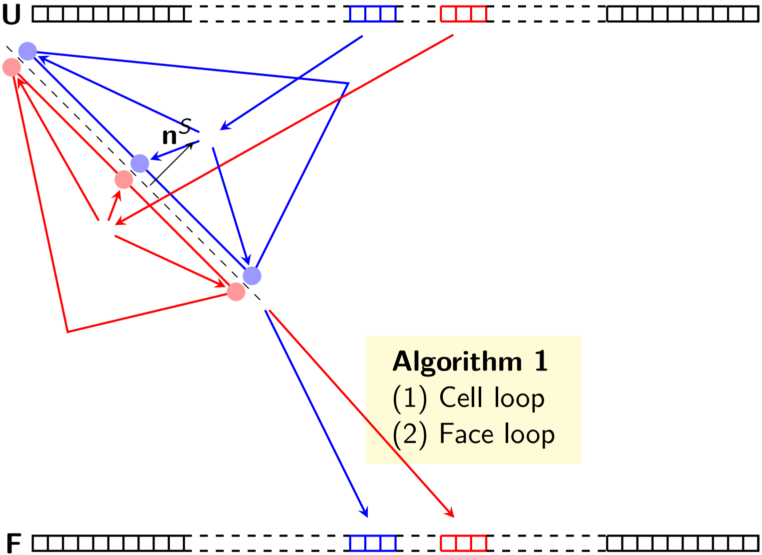


Algorithm 1
(1) Cell loop
(2) Face loop

Implementation

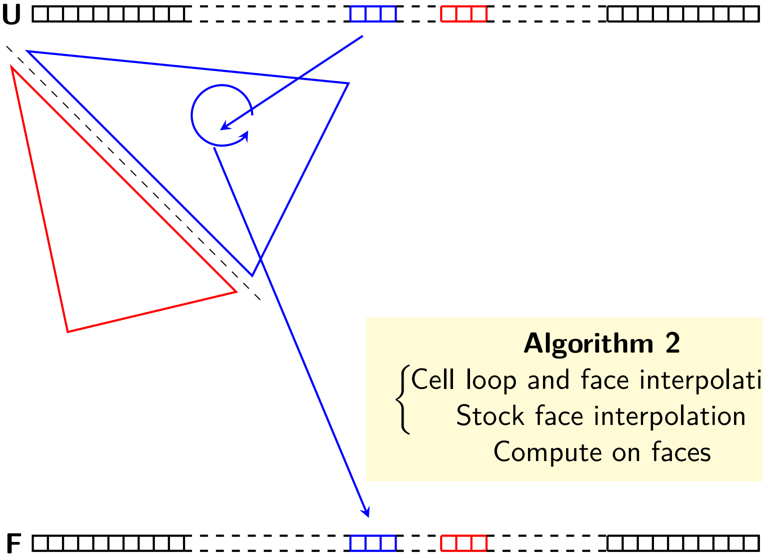


Implementation



Algorithm 1
(1) Cell loop
(2) Face loop

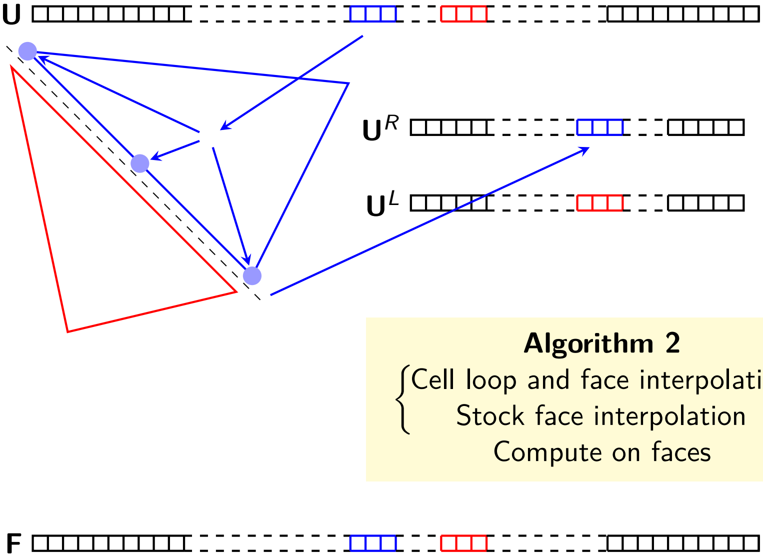
Implementation



Algorithm 2

Cell loop and face interpolation
Stock face interpolation
Compute on faces

Implementation

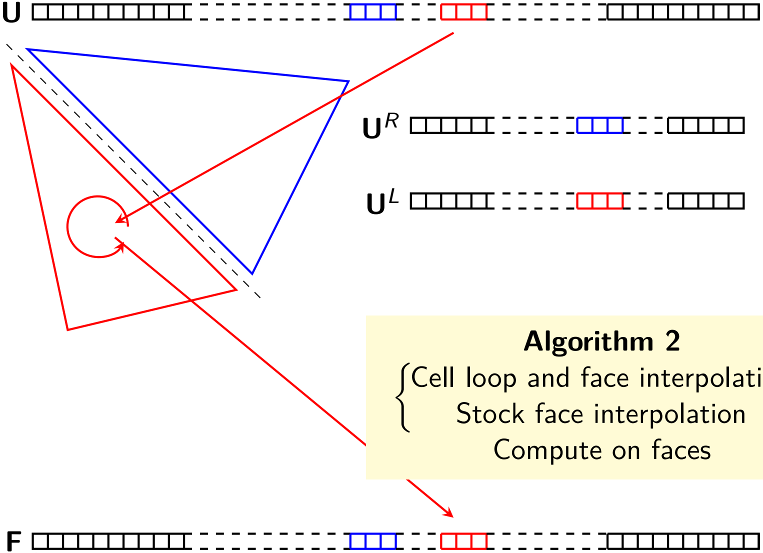


Algorithm 2

$\left\{\begin{array}{l}\text{Cell loop and face interpolation}\\ \text{Stock face interpolation}\end{array}\right.$

Compute on faces

Implementation

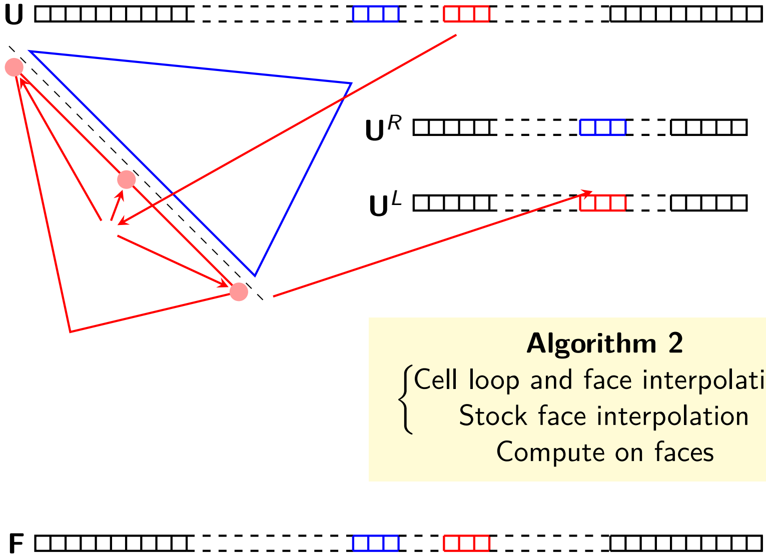


Algorithm 2

$\begin{cases}\text{Cell loop and face interpolation} \\ \text{Stock face interpolation}\end{cases}$

Compute on faces

Implementation

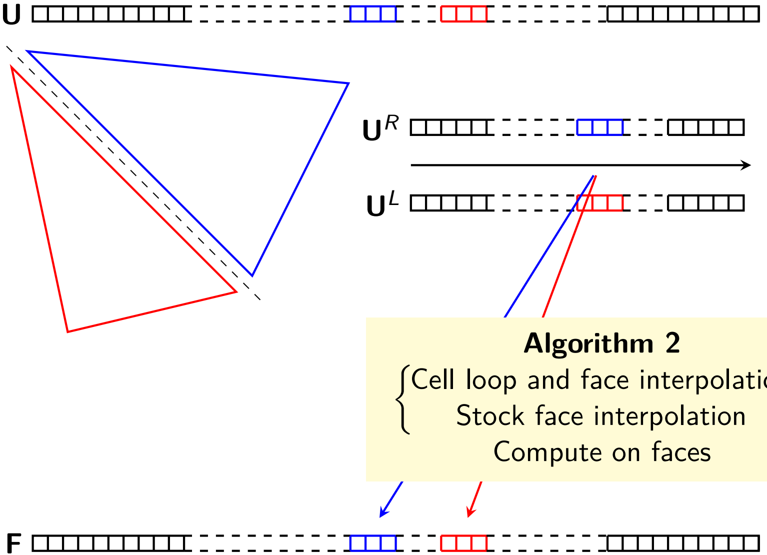


Algorithm 2

$\begin{cases} \text{Cell loop and face interpolation} \\ \text{Stock face interpolation} \end{cases}$

Compute on faces

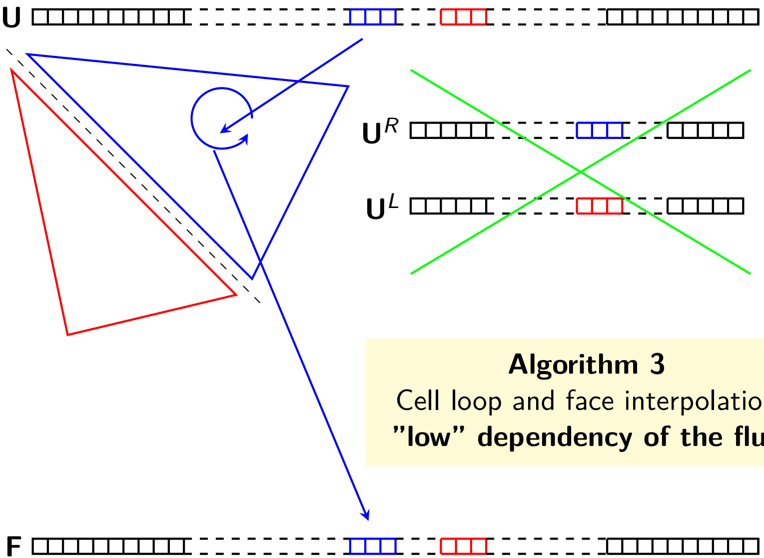
Implementation



Algorithm 2

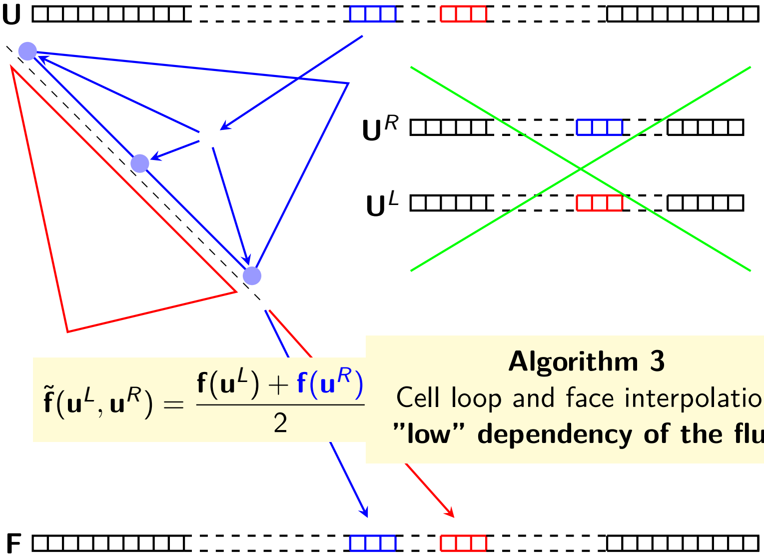
$\{$ Cell loop and face interpolation
Stock face interpolation
Compute on faces

Implementation



Algorithm 3
Cell loop and face interpolation
"low" dependency of the flux

Implementation



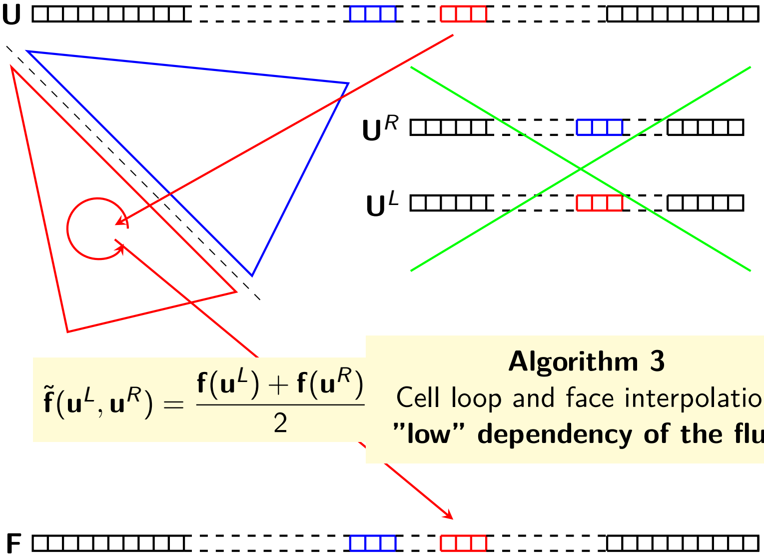
$$\tilde{\mathbf{f}}(\mathbf{u}^L, \mathbf{u}^R) = \frac{\mathbf{f}(\mathbf{u}^L) + \mathbf{f}(\mathbf{u}^R)}{2}$$

Algorithm 3

Cell loop and face interpolation

"low" dependency of the flux

Implementation



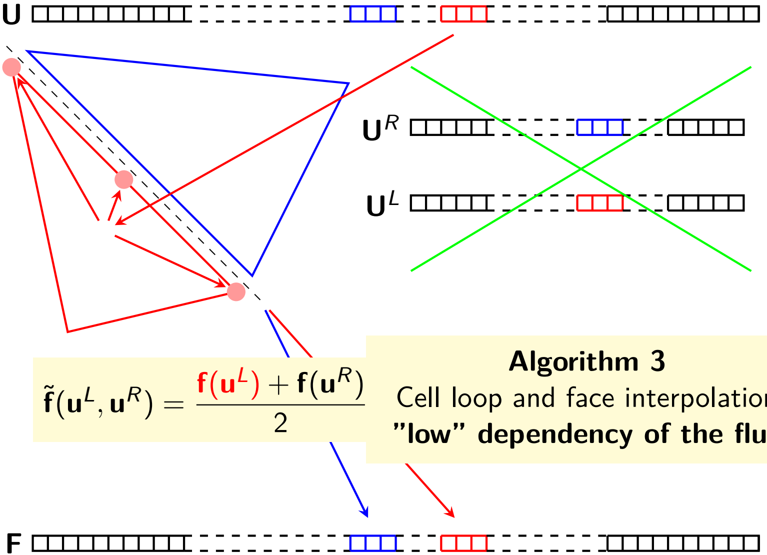
$$\tilde{\mathbf{f}}(\mathbf{u}^L, \mathbf{u}^R) = \frac{\mathbf{f}(\mathbf{u}^L) + \mathbf{f}(\mathbf{u}^R)}{2}$$

Algorithm 3

Cell loop and face interpolation

"low" dependency of the flux

Implementation



$$\tilde{\mathbf{f}}(\mathbf{u}^L, \mathbf{u}^R) = \frac{\mathbf{f}(\mathbf{u}^L) + \mathbf{f}(\mathbf{u}^R)}{2}$$

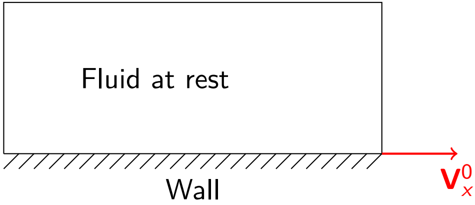
Algorithm 3

Cell loop and face interpolation

"low" dependency of the flux

Validation test

Second Stokes problem



$$\mathbf{V}_x^0 = \sin(\omega t)$$

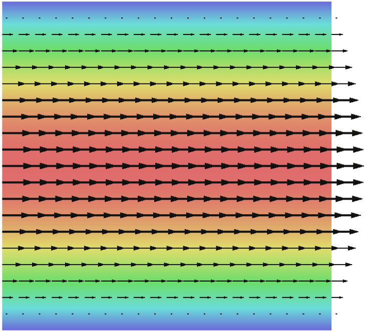
Movie

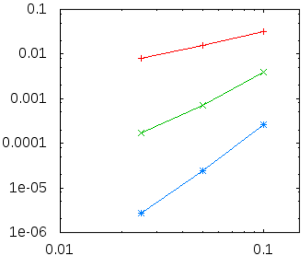
$$-\sum_{T\in\mathcal{T}_h}\int_T \nabla\varphi\cdot A\nabla\mathbf{U} + \sum_{S\in S_i}\int_{S_i}\left[\{\!\{ A^T\nabla\varphi \}\!\}\cdot[\![\mathbf{U}]\!] + \{\!\{ A\nabla\mathbf{U} \}\!\}\cdot[\![\varphi]\!]\right] + \sum_{S\in S_i}\int_{S_i}\eta\,\{\!\{ AR^{S_i}([\![\mathbf{U}]\!]) \}\!\}\,[\![\varphi]\!] = 0$$

Validation test

Poiseuille flows

- Poiseuille flow

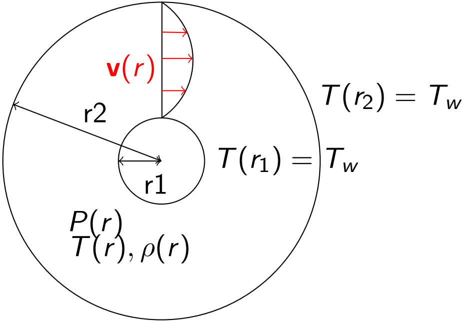




Validation test

Poiseuille flows

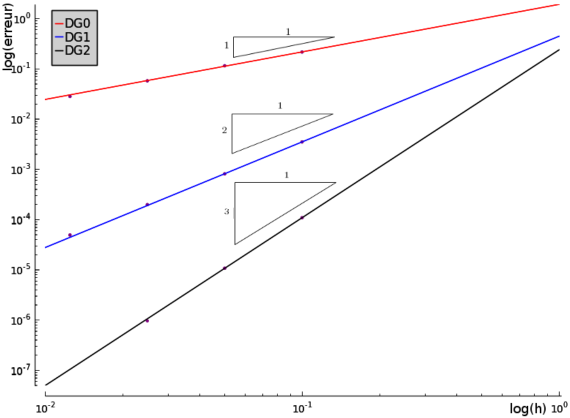
- Circular Poiseuille flow



Validation test

Poiseuille flows

- Circular Poiseuille flow



Validation test

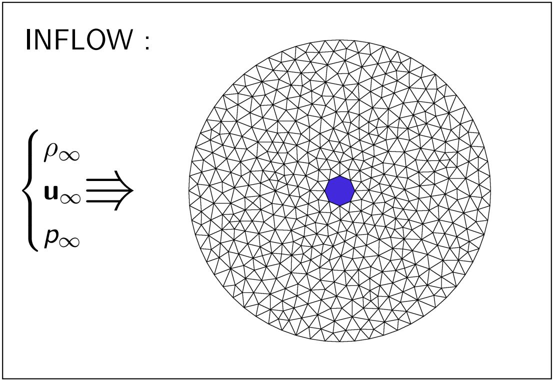
Taylor-Green Vortex

- Three dimensional test
- Cube with periodic boundary conditions
- Low Mach flow with z-anisotropy
- No mechanism for maintaining turbulence

Movie

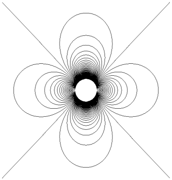
Outline

A steady test case

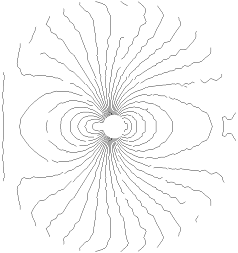


- Discontinuous Galerkin.
- High order discretisation.
- Unstructured Mesh.
- Roe Riemann Solver.

A steady test case



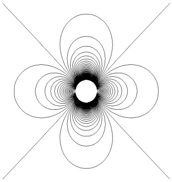
Incompressible exact



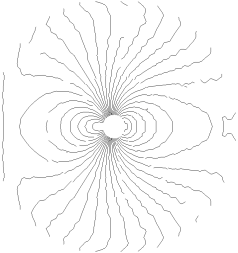
Roe, Quad

- The **compressible discrete** solution does not converge toward the **incompressible** one as the Mach number tends to zero on quadrangular meshes. (Here at $M = 10^{-3}$)

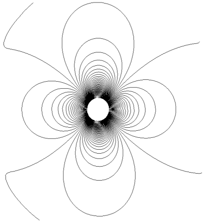
A steady test case



Incompressible exact



Roe, Quad



Roe, Tri

- The **compressible discrete** solution does not converge toward the **incompressible** one as the Mach number tends to zero on quadrangular meshes. (Here at $M = 10^{-3}$)
- H. Guillard, On the behavior of upwind schemes in the low Mach number limit. IV: *P*0 approximation on triangular and tetrahedral cells, Computers & Fluids, 2009, 38 (10).

Euler equations

Non-dimensional Euler equations

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0$$
$$\frac{\partial \rho \mathbf{u}}{\partial t} + div(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2}\nabla p = 0$$
$$\frac{\partial \rho e}{\partial t} + div(\rho e \mathbf{u} + p\mathbf{u}) = 0$$
$$p = (\gamma - 1)(\rho e - \frac{M^2}{2}\rho\|\mathbf{u}\|^2)$$

Where the dimensionless variables are :

- ρ density.
- $\mathbf{u}$ velocity vector.
- e total energy.
- p the pressure.
- $M = \frac{u_\infty}{a_\infty}$ Mach number.

One scale asymptotic expansion

Let's consider the system :

$$\partial_t \mathbf{u} + A(\mathbf{u})\nabla \mathbf{u} + \frac{B(\mathbf{u})}{M^2}\nabla \mathbf{u} = 0$$

And expand the variables using a one scale asymptotic development in power of the Mach number :

$$\mathbf{u}(x,t;M) = \sum_{n=0}^{N} M^n \mathbf{u}^{(n)}(x,t) + O(M^N)$$

We want to compare the behaviour of the **continuous** and **discrete** system.

- B. Müller (1999). Low Mach number asymptotics of the Navier-Stokes equations and numerical implications.
- H. Guillard, & C. Viozat (1999). On the behaviour of upwind schemes in the low Mach number limit. Computers & Fluids

Comparison between continuous and discrete cases

- Continuous :

Order $\mathbf{M}^{-2}$:

$$\nabla p^{(0)} = 0$$

Order $\mathbf{M}^{-1}$:

$$\nabla p^{(1)} = 0$$

Order $\mathbf{M}^{0}$:

$$\nabla.(\rho^{(0)} u^{(0)}) = 0$$
$$\nabla.(\rho^{(0)} u^{(0)} u^{(0)}) + \nabla p^{(2)} = 0$$
$$\nabla.(\rho^{(0)} H^{(0)} u^{(0)}) = 0$$
$$p^{(0)} = (\gamma - 1)\rho^{(0)} e^{(0)}$$

- Discrete using Roe scheme :

Order $\mathbf{M}^{-2}$:

$$\sum p^{(0)} \mathbf{n} = 0$$

Order $\mathbf{M}^{-1}$:

$$\frac{1}{2} \sum \frac{\Delta \bar{p}^{(0)}}{\bar{c}^{(0)}} = 0$$
$$\frac{1}{2} \sum \bar{p}^{(0)} \bar{c}^{(0)} \mathbf{n} \Delta \bar{U}^{(0)} + \sum p^{(1)} \mathbf{n} = 0$$

Order $\mathbf{M}^{0}$:

$$d_t \rho^{(0)} + \frac{1}{2} \sum \frac{\Delta p^{(1)}}{\bar{a}^{(0)}} = 0$$

Comparison between continuous and discrete cases

- Continuous :

Order $\mathbf{M}^{-2}$:

$$\nabla p^{(0)} = 0$$

Order $\mathbf{M}^{-1}$:

$$\nabla p^{(1)} = 0$$

Order $\mathbf{M}^{0}$:

$$\nabla.(\rho^{(0)} u^{(0)}) = 0$$

$$\nabla.(\rho^{(0)} u^{(0)} u^{(0)}) + \nabla p^{(2)} = 0$$

$$\nabla.(\rho^{(0)} H^{(0)} u^{(0)}) = 0$$

$$p^{(0)} = (\gamma - 1)\rho^{(0)} e^{(0)}$$

- Discrete using Roe scheme :

Order $\mathbf{M}^{-2}$:

$$\sum p^{(0)}\mathbf{n} = 0$$

Order $\mathbf{M}^{-1}$:

$$\frac{1}{2}\sum \frac{\Delta \bar{p}^{(0)}}{\bar{c}^{(0)}} = 0$$

$$\frac{1}{2}\sum \bar{p}^{(0)}\bar{c}^{(0)}\mathbf{n}\Delta \bar{U}^{(0)} + \sum p^{(1)}\mathbf{n} = 0$$

Order $\mathbf{M}^{0}$:

$$d_t \rho^{(0)} + \frac{1}{2}\sum \frac{\Delta p^{(1)}}{\bar{a}^{(0)}} = 0$$

Comparison between continuous and discrete cases

- Continuous :

Order $\mathbf{M}^{-2}$:

$$\nabla p^{(0)} = 0$$

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$$\nabla.(\rho^{(0)} u^{(0)}) = 0$$
$$\nabla.(\rho^{(0)} u^{(0)} u^{(0)}) + \nabla p^{(2)} = 0$$
$$\nabla.(\rho^{(0)} H^{(0)} u^{(0)}) = 0$$
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- Discrete using Roe scheme :

Order $\mathbf{M}^{-2}$:

$$\sum p^{(0)} \mathbf{n} = 0$$

Order $\mathbf{M}^{-1}$:

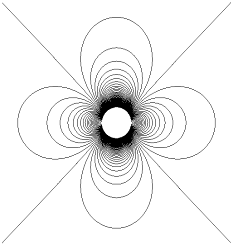
$$\frac{1}{2} \sum \frac{\Delta \bar{p}^{(0)}}{\bar{c}^{(0)}} = 0$$

$$\boxed{\frac{1}{2} \sum \bar{p}^{(0)} \bar{c}^{(0)} \mathbf{n} \Delta \bar{U}^{(0)} + \sum p^{(1)} \mathbf{n} = 0}$$

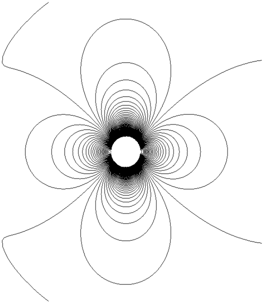
Order $\mathbf{M}^{0}$:

$$d_t \rho^{(0)} + \frac{1}{2} \sum \frac{\Delta p^{(1)}}{\bar{a}^{(0)}} = 0$$

Preconditioning methods



Incompressible exact



Compressible LM-Roe solver

Quad Mesh, $M = 10^{-3}$

- H. Guillard, & C. Viozat, Computers & Fluids. (1999)
- S. Dellacherie, Journal of Computational Physics. (2010)
- F. Rieper, Journal of Computational Physics. (2011)

Results

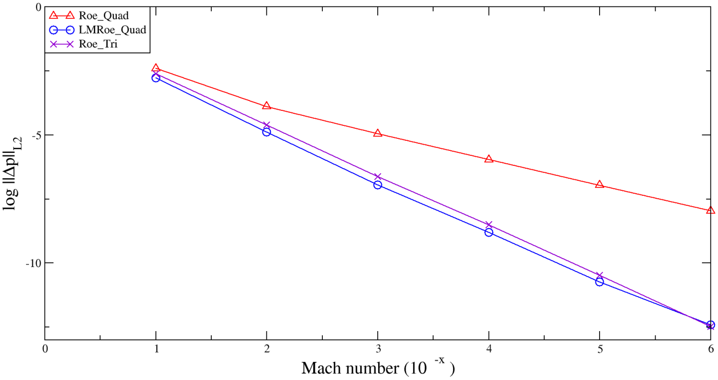
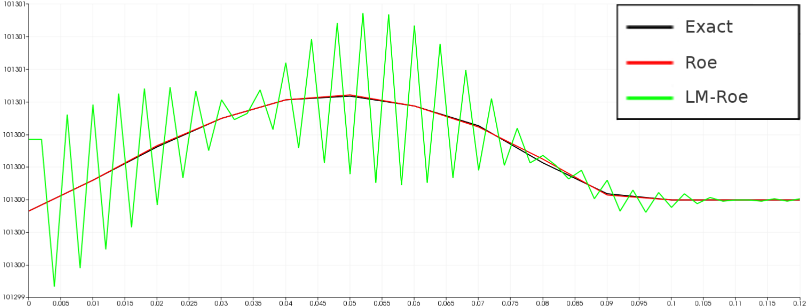


Figure: L2 Error for the pressure in terms of Mach number

What about higher order?

- Any of the higher order schemes work well with the cylinder test.
- Problems occur with e.g. flow around a NACA on quads
  - F. Bassi, C. De Bartolo, R. Hartmann and A. Nigro, A discontinuous Galerkin method for inviscid low Mach number flows, Journal of Computational Physics, 2009.
  - A. Nigro, S. Renda, C. De Bartolo, R. Hartmann and F. Bassi A high-order accurate discontinuous Galerkin finite element method for laminar low Mach number flows International Journal for Numerical Methods in Fluids, 2013.

A unsteady test case



Propagating wave with second order spatial discretisation

Moguen, Y. *et al.* (2013). Pressure-velocity coupling for unsteady low Mach number flow simulations: An improvement of the AUSM+-up scheme Journal of Computational and Applied Mathematics

Two time scales asymptotic development

Let consider a second time scale, the **acoustic time scale** : $\tau = t/M$.
Introducing this scale in the development gives :

$$\mathbf{u}(x, t, \tau; M) = \sum_{n=0}^{N} M^n \mathbf{u}^{(n)}(x, t, \tau) + O(M^N)$$

The time derivative at constant x and M yields :

$$\frac{\partial}{\partial \tilde{t}} = \frac{1}{M} \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t}$$

Continuous two time scales asymptotic development

Order M^{-1} momentum and order M^0 energy equations

$$\begin{cases} \partial_\tau \rho^{(0)}\mathbf{u}^{(0)} + \nabla p^{(1)} = 0 \\ \partial_\tau p^{(1)} + a^{(0)2} div(\rho^{(0)}\mathbf{u}^{(0)}) = -d_t p^{(0)} \end{cases}$$

We will note $\rho^{(0)}\mathbf{u}^{(0)} = \mathbf{u}$, $p^{(1)} = p$ and $a^{(0)} = a$.

First order wave equation

$$\begin{cases} \partial_\tau \mathbf{u} + \nabla p = 0 \\ \partial_\tau p + a^2 div(\mathbf{u}) = -d_t p^{(0)} \end{cases}$$

Discrete two time scales asymptotic development

Roe first order discrete wave equation

$$
\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}
$$

Diffusion given by classical upwind flux for the wave system.
Diffusion is a SPD matrix.

- E. Burman, Alexandre Ern, Miguel Angel Fernandez. Explicit Runge–Kutta schemes and finite elements with symmetric stabilization for first-order linear PDE systems, SIAM Journal on Numerical Analysis, 2010, **48** (6),

Discrete two time scales asymptotic development

Roe first order discrete wave equation

$$\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}$$

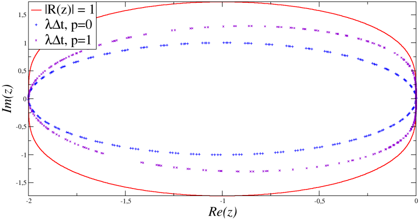
Modified Roe first order discrete wave equation

$$\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + 0 &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}$$

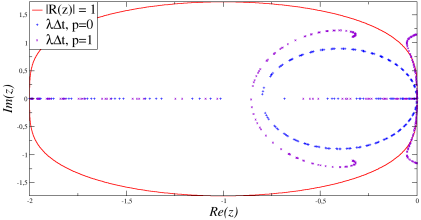
Von-Neumann stability analysis for the wave equation

$$d_\tau \mathbf{U} + A(\mathbf{U}) = 0, \qquad \mathbf{U}_{n+1} = R(A\Delta\tau)\mathbf{U}_n$$

$$z = \lambda_{max}\Delta\tau : |R(z)| \leq 1$$



Roe scheme Eigenvalues

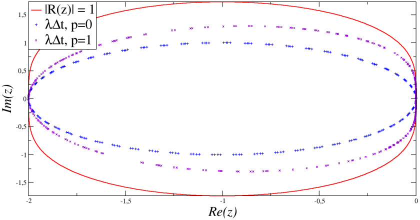


LMRoe scheme Eigenvalues

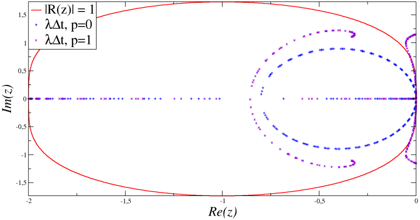
Von-Neumann stability analysis for the wave equation

$$d_\tau \mathbf{U} + A(\mathbf{U}) = 0, \qquad \mathbf{U}_{n+1} = R(A\Delta\tau)\mathbf{U}_n$$

$$z = \lambda_{max}\Delta\tau : |R(z)| \leq 1$$



Roe scheme Eigenvalues



LMRoe scheme Eigenvalues

The LM-Roe scheme is stable for a **CFL condition** of order 10^{-3}.

Modifying dissipation

Roe : not accurate in steady case

$$
\begin{aligned}
&\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} = 0\\
&\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 = 0
\end{aligned}
$$

Modifying dissipation

Roe : not accurate in steady case

$$
\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}
$$

Modified Roe : not stable in unsteady case

$$
\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + 0 &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}
$$

Modifying dissipation

Roe : not accurate in steady case

$$
\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}
$$

Modified Roe : not stable in unsteady case

$$
\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + 0 &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}
$$

A new set of dissipative terms

$$
\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + \frac{1}{2}\sum \Delta_{il} p + 0 &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + -\frac{a^2}{2}\sum \Delta_{il}\mathbf{u} &= 0
\end{aligned}
$$

Stability analysis of our new scheme

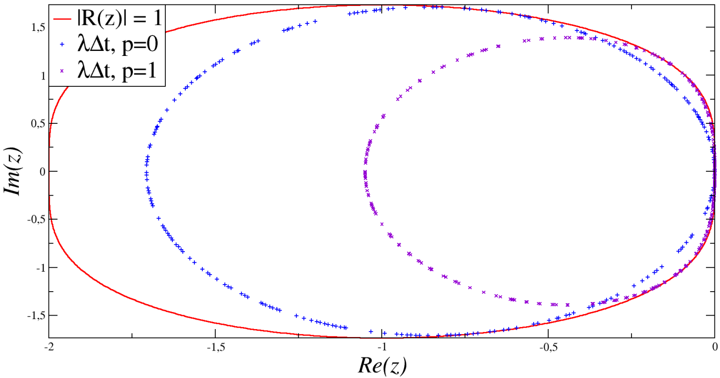


Figure: Stability of the new scheme applied to the first order wave equation

Convergence for the wave equation

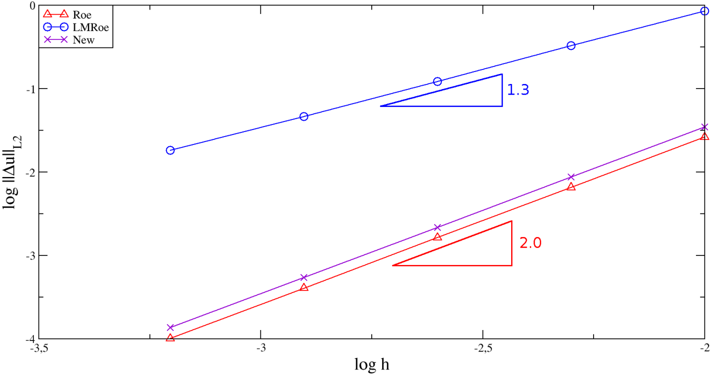


Figure: Convergence for second order spatial discretisation

Back to the Euler system

- Momentum :

$$\frac{1}{2M^2}\sum p\mathbf{n}+$$
$$\frac{1}{2}\sum\left(\rho\mathbf{u}\mathbf{u}.\mathbf{n}+\frac{(\bar{U}\mathbf{n}+\bar{\mathbf{u}})}{\bar{a}}\Delta p+\bar{\rho}\bar{a}\Delta U\mathbf{n}\right)+$$
$$\frac{M}{2}\sum\bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{\mathbf{u}}\Delta U=0$$

- Energy :

$$\frac{1}{2M}\sum\frac{\bar{h}}{\bar{a}}\Delta P+$$
$$\frac{1}{2}\sum((\rho e+p)\mathbf{u}.\mathbf{n})+$$
$$\frac{M}{2}\sum\bar{\rho}\bar{a}\bar{U}\Delta U+\bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{h}\Delta U=0$$

Back to the Euler system

- Momentum :

$$\frac{1}{2M^2}\sum p\mathbf{n}+$$
$$\frac{1}{2}\sum\left(\rho\mathbf{u}\mathbf{u}.\mathbf{n}+\frac{(\bar{U}\mathbf{n}+\bar{\mathbf{u}})}{\bar{a}}\Delta p+\cancel{\bar{\rho}\bar{a}\Delta U\mathbf{n}}\right)+$$
$$\frac{M}{2}\sum\bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{\mathbf{u}}\Delta U=0$$

- Energy :

$$\frac{1}{2M}\sum\frac{\bar{h}}{\bar{a}}\Delta P+$$
$$\frac{1}{2}\sum((\rho e+p)\mathbf{u}.\mathbf{n})+$$
$$\frac{M}{2}\sum\bar{\rho}\bar{a}\bar{U}\Delta U+\bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{h}\Delta U=0$$

Back to the Euler system

- Momentum :

$$\frac{1}{2M^2}\sum(p\mathbf{n}+\Delta P\mathbf{n})+$$
$$\frac{1}{2}\sum\left(\rho\mathbf{u}\mathbf{u}.\mathbf{n}+\frac{(\bar{U}\mathbf{n}+\bar{\mathbf{u}})}{\bar{a}}\Delta p+\cancel{\bar{\rho}\bar{a}\Delta U\mathbf{n}}\right)+$$
$$\frac{M}{2}\sum\bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{\mathbf{u}}\Delta U=0$$

- Energy :

$$\frac{1}{2M}\sum\frac{\bar{h}}{\bar{a}}\Delta P+$$
$$\frac{1}{2}\sum\left((\rho e+p)\mathbf{u}.\mathbf{n}-\bar{a}^2\bar{\rho}\Delta U\right)+$$
$$\frac{M}{2}\sum\bar{\rho}\bar{a}\bar{U}\Delta U+\bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{h}\Delta U=0$$

Steady test case

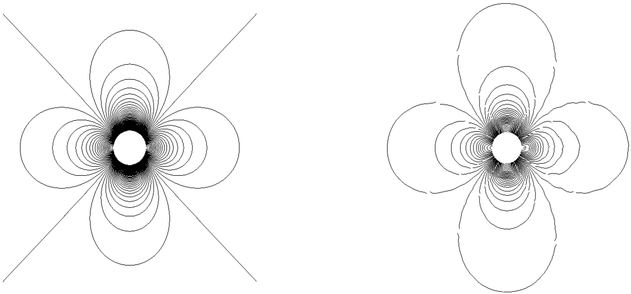


Figure: **Incompressible** solution and **Compressible** new scheme ($M = 10^{-3}$)

Convergence

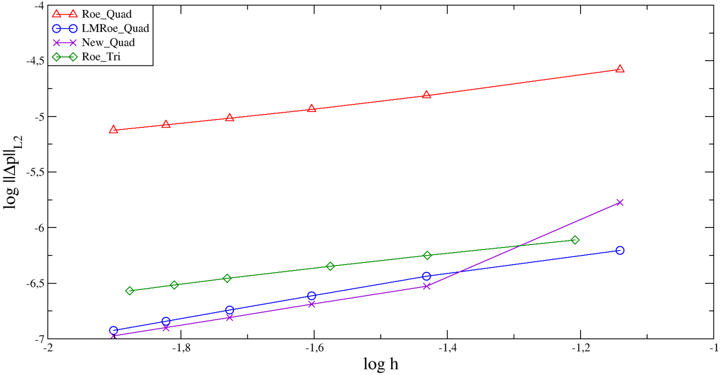


Figure: Convergence for first order spatial discretisation at Mach $= 10^{-3}$

Unsteady test case

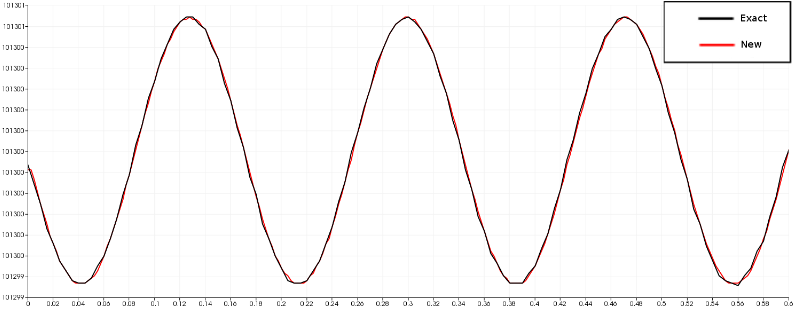


Figure: Propagating wave with second order spatial discretisation

Convergence

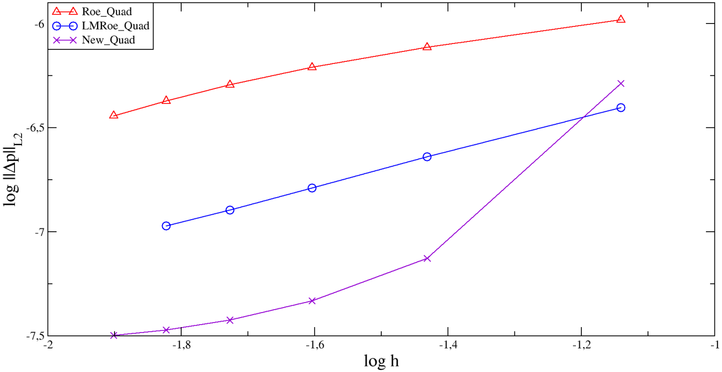


Figure: Convergence for spatial discretisation of order 2 at Mach $= 10^{-3}$

Remark on the wall boundary condition

$$U_l = \begin{pmatrix} \rho \\ \rho \mathbf{u}_l \\ \rho E \end{pmatrix} \qquad U_r = \begin{pmatrix} \rho \\ \rho \mathbf{u}_r \\ \rho E \end{pmatrix}$$

$$\mathbf{u}_r = \mathbf{u}_l - 2(\mathbf{u}_l.\mathbf{n})\mathbf{n} \qquad \Delta\rho = \Delta P = 0$$

With classical schemes

$$F_{wall}(U_l, U_r) = \begin{pmatrix} 0 \\ p^*\mathbf{n} \\ 0 \end{pmatrix} \qquad p^\star = p_l - 2\frac{(\mathbf{u}_l.\mathbf{n})}{\lambda}$$

Remark on the wall boundary condition

$$U_l = \begin{pmatrix} \rho \\ \rho \mathbf{u}_l \\ \rho E \end{pmatrix} \qquad U_r = \begin{pmatrix} \rho \\ \rho \mathbf{u}_r \\ \rho E \end{pmatrix}$$

$$\mathbf{u}_r = \mathbf{u}_l - 2(\mathbf{u}_l.\mathbf{n})\mathbf{n} \qquad \Delta\rho = \Delta P = 0$$

With the new solver: energy equation :

$$\frac{1}{2M^2}\sum p\mathbf{n} + \frac{1}{2}\sum \rho\mathbf{u}\mathbf{u}.\mathbf{n} + \frac{M}{2}\sum \bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{\mathbf{u}}\Delta U = 0$$

$$\frac{1}{2}\sum(\rho e + p)\mathbf{u}.\mathbf{n} - \bar{a}^2\bar{\rho}\Delta U + \frac{M}{2}\sum \bar{\rho}\bar{a}\bar{U}\Delta U + \bar{\rho}\frac{\bar{U}}{\bar{a}}\bar{h}\Delta U = 0$$

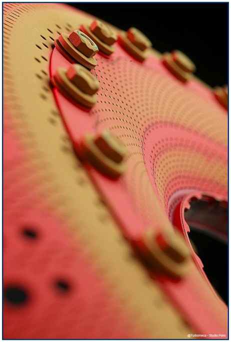
$$F_{wall}(U_l, U_r) = \begin{pmatrix} 0 \\ p^*\mathbf{n} \\ X \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ p^*\mathbf{n} \\ 0 \end{pmatrix}$$

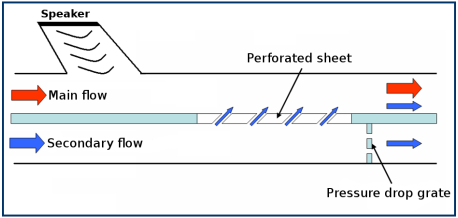
Conclusion on low Mach number flows

- Classical low Mach solver are not stable for high order acoustic calculation.
- A stable and accurate low Mach scheme for both steady and unsteady flow computation.
  - Stabilization of the incompressible system is not achieved by centering the pressure
  - Stabilization of the wave system is not symmetric .
- Higher order: ?
  - E. Burman, A. Ern, I.Mozolevski and B. Stamm. The symmetric discontinuous Galerkin method does not need stabilization in 1D for polynomial orders $p \geq 2$. Comptes Rendus Mathematique, 2007.

Outline

Context

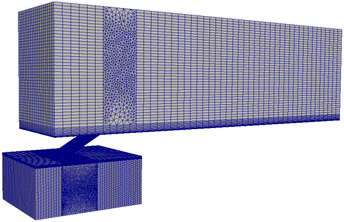




Low (NOT zero) Mach flow, with acoustic

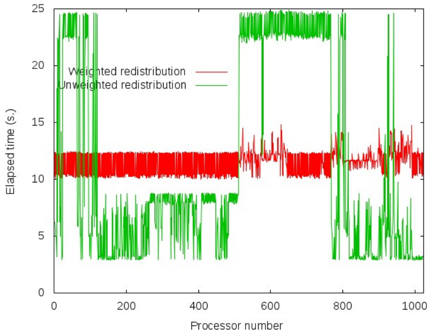
Numerical result

- Around 500 000 degrees of freedom
- Hybrid mesh



Movie

A remark on weighted redistribution on hybrid meshes



Tetrahedron	3	*Pyramid*	9
Prism	10	*Hexahedron*	29

Why I will keep on on discontinuous Galerkin

- A rich topic of numerical analysis
  - Derive stable schemes in different asymptotics
- MARSU: Multigrid AggRegated on unStrUctured meshes
  - Fully matrix-free implementation
  - Derive aggregation strategies depending on the *Re* or *Ma*.
- HONHA: High Order Numerical methods on Heterogeneous Architectures
  - Strong memory locality

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  - F. Bassi, L. Botti, A. Colombo, D. A. Di Pietro, and P. Tesini, On the flexibility of agglomeration based physical space discontinuous Galerkin discretizations
  - Derive aggregation strategies depending on the *Re* or *Ma*.
  - J.J.W. van der Vegt and S. Rhebergen, h-p-multigrid as smoother algorithm for higher order discontinuous Galerkin discretizations of advection dominated flows. Part I. Multilevel Analysis
- HONHA: High Order Numerical methods on Heterogeneous Architectures
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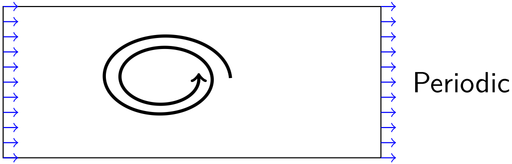
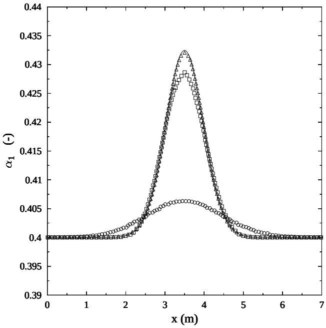
  - Klöckner, A., Warburton, T., Bridge, J., Hesthaven, J. S. Nodal discontinuous Galerkin methods on graphics processors.

Thanks to

- S. Delmas, R. Manceau, P. Bruel
- H. Belkhayat (looking for a PhD position)
- E. Martin, F. Renac from ONERA, the french aerospace lab.

Advection of a vortex





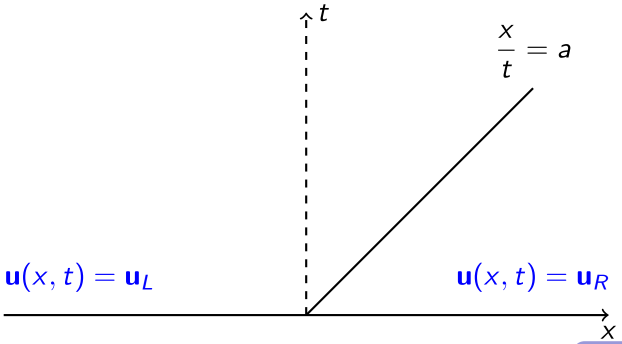
Movie

Back

Riemann problem

$$\partial_t \mathbf{u} + a\partial_x \mathbf{u} = 0$$

$$\mathbf{u}(x,0) = \begin{cases} u_L & \text{if} \quad x < 0 \\ u_R & \text{if} \quad x < 0 \end{cases}$$



Back

Riemann problem

$$\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = 0$$

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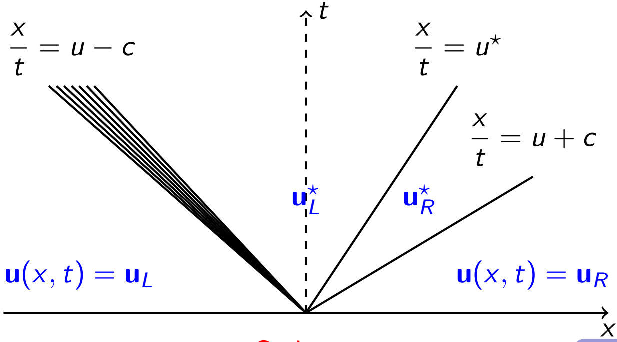

Godunov

Back

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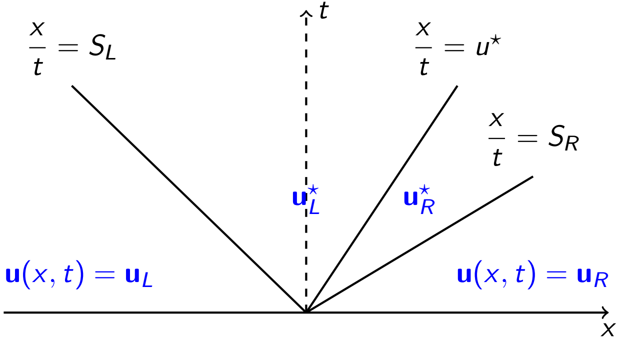

HLLC

Back

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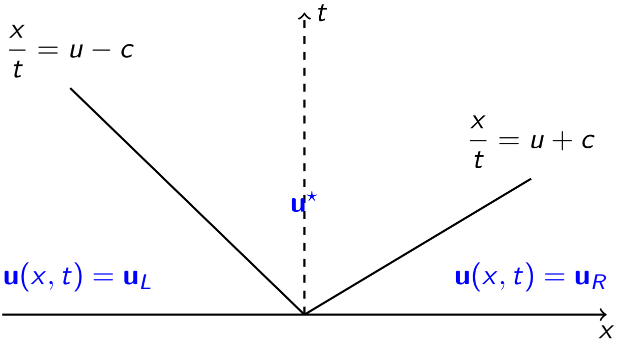

HLL

Back

Riemann problem

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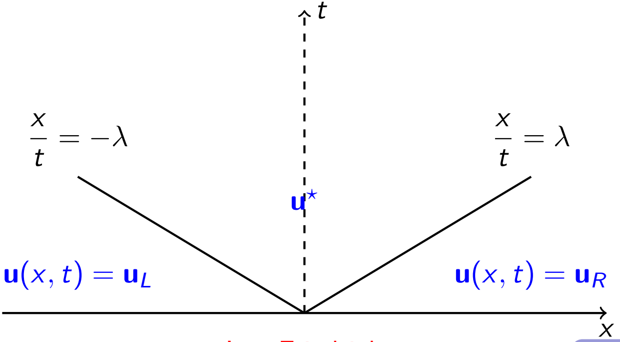

Lax Friedrich

Back