M2 Internship Project (leading up to a PhD grant):
Solution of Stochastic Partial Differential Equations on High Performance Computers

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Nowadays, it is very common for scientists and engineers to have access to a supercomputer with thousands or many more processing units. However, hardware only tells part of the story. These supercomputers require algorithms that maintain the same efficiency when the workload grows (scalability), and when the problems become more difficult (robustness). **The objective of this internship is to study state of the art parallel solvers for stochastic partial differential equations on supercomputers, apply them to the simulation of porous media flow and compare them.**

Context: Porous media flow with a stochastic diffusivity field

The dispersion of pollutants in the ground is an important consideration for nuclear waste disposal. In order to account for the partially unknown heterogeneous diffusivity of the terrain, the problem is modelled by a PDE with random coefficients. Formally, let Ω denote a bounded domain, and let (Θ, Σ, P) be a complete probability space, where Θ is the sample space, Σ is the σ-algebra of the subsets and P is a probability measure. The problem is then to find a random function $u(x, \theta) : \Omega \times \Theta \to \mathbb{R}$ that satisfies in the almost surely sense:

$$-\nabla \cdot (\kappa(x, \theta) \nabla u(x, \theta)) = -f(x), \quad x \in \Omega, \quad \theta \in \Theta, \text{ with boundary conditions} \tag{1}$$

where f is a deterministic source term and the diffusion coefficient $\kappa(x, \theta)$ is a log-normal random field. It is still an open question how to most efficiently approximate the expectancy $\mathbb{E}[z(u)]$ of a desired outcome in a non-intrusive manner. A solution is Monte-Carlo sampling:

$$\mathbb{E}[z(u)] \approx \frac{1}{M} \sum_{m=1}^{M} z(u^{(m)}); \text{ where } u^{(m)}\text{random sample of } u(x, \theta) \text{ corresponding to a sampled } \kappa^{(m)}.$$

Computationally:

- M samples of $\kappa(\cdot, \theta)$ are drawn randomly,
- The resulting elliptic PDEs are discretized into M independent linear systems.
- All M linear systems are solved to compute statistical means. Solving the linear systems is the current bottleneck addressed by the internship.

Project: Efficient solution of sets of linear systems by domain decomposition

Domain decomposition methods [4] are state of the art linear solvers for elliptic PDEs with varying coefficients. Methods with spectral coarse spaces are particularly efficient [3]. The objective of this internship is to take advantage of the fact that the set of systems arising from Monte Carlo have similarities to find a solver that is efficient for the whole set of problems. This follows preliminary work by by [2, 1].

Supervising team

- Nicole Spillane (CNRS Junior Research at Centre de Mathématiques Appliquées - CMAP - of école Polytechnique) http://www.cmap.polytechnique.fr/ spillane/
- Olivier Le Maitre (CNRS Senior Researcher at CMAP of école Polytechnique and inria Platon team). https://olemaitre.perso.math.cnrs.fr/

The internship will be partly funded by ANR project DARK and by Inria.

Candidate Skills

We are looking for a final year master student with a strong background in scientific computing and/or numerical analysis for (stochastic) PDEs. The ideal candidate would have some experience with programming in Python or C++. Topics covered by the internship include: Stochastic PDE, Numerical Analysis, Linear solvers, Domain Decomposition, High Performance Computing and Parallel programming (Python and/or C++).

Duration, Grant and Application

Duration: 4 to 6 months starting during first semester of 2025.

Grant: depending on candidate profile ($\geq$ 640 euros per month)

Application: Please send your resume to nicole.spillane@polytechnique.edu and feel free to contact me with any questions.

The internship will take place on the campus of Polytechnique in Palaiseau (also known as Paris-Saclay Campus).

References

[1] A. A. Contreras, P. Mycek, O. P. Le Maître, F. Rizzi, B. Debusschere, and O. M. Knio. Parallel domain decomposition strategies for stochastic elliptic equations. Part B: Accelerated Monte Carlo sampling with local PC expansions. *SIAM J. Sci. Comput.*, 40(4):c547–c580, 2018.

[2] J. F. Reis, O. P. Le Maître, P. M. Congedo, and P. Mycek. Stochastic preconditioning of domain decomposition methods for elliptic equations with random coefficients. *Comput. Methods Appl. Mech. Eng.*, 381:29, 2021. Id/No 113845.

[3] N. Spillane, V. Dolean, P. Hauret, F. Nataf, C. Pechstein, and R. Scheichl. Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. *Numer. Math.*, 126(4):741–770, 2014.

[4] A. Toselli and O. Widlund. *Domain decomposition methods – algorithms and theory.*, volume 34 of *Springer Ser. Comput. Math.* Berlin: Springer, 2005.